THE FORMATION OF IMAGES IN OPTICAL INSTRUMENTS

GEOMETRICAL INVESTIGATION

OF THE

FORMATION OF IMAGES

IN

OPTICAL INSTRUMENTS.

Embodying the results of scientific researches conducted in German Optical Workshops.

EDITED BY

M. VON ROHR.

(Forming Vol. I of "The Theory of Optical Instruments".)

TRANSLATED BY

R. KANTHACK.

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Prefatory Note.

Soon after the establishment of the Committee of the Privy Council for Scientific and Industrial Research their attention was called by their Advisory Council, who were assisted by the Standing Committee on Glass and Optical Instruments, to the deficiency of books in the English language dealing with geometrical and technical optics.

Accordingly they authorised, in the interests of the scientific and industrial development of optics, translations of Dr. GLEICHEN'S "Theorie der modernen optischen Instrumente" and Dr. VON ROHR'S "Die Theorie der optischen Instrumente." The translation of the former work was first published by the Stationery Office for the Department of Scientific and Industrial Research in 1918.

On the recommendation of the Standing Committee on Glass and Optical Instruments, Mr. R. KANTHACK was invited to translate Dr. VON ROHR'S book.

Based upon the fundamental geometrical laws of light, this work deals with the methods of computation of optical systems. It embodies the results of researches made by eminent men of science, amongst whom British investigators occupy an important place.

A full account of the history and authorship of the original work is given in the preface written for it by the late Dr. CZAPSKI.

In putting forth the English edition of this book, the Committee of Council wish to place on record their appreciation of the great ability and care brought by the translator to the presentation of this subject, and their indebtedness to the Editorial Committee for the pains they have taken in supervising its issue. To Mr. E. B. KNOBEL, in particular, the thanks of the Committee of Council are due for his arduous labours as Chairman of the Editorial Committee.

Department of Scientific and
Industrial Research,
15, Great George Street,
Westminster, S.W. 1.
April, 1920.

(14911) Wt. 14091—599 1000 6/20 H. St.

Translator's Preface.

When first approached on the subject of an English edition of Dr. von ROHR's classical treatise on the Theory of Optical Instruments the Translator accepted the proposition with enthusiasm. It had been his privilege to know Professor ABBE and Dr. CZAPSKI personally, and to witness in the Professor's interesting sociologieal experiments the first effects of his lofty ethical aspirations. From its inception the work with which the Translator was entrusted represented to him far more than the preparation of an English version of a work which may be regarded as the most valuable modern elucidation of the principles of constructive optics. He felt convinced from personal observation, as well as the testimony of others competent to express unbiassed opinions, that the achievements of German workshops and the sympathetic development of the cognate theoretical studies could not be accounted for solely by education and by State assistance. In a very large measure the success of the Germans as opticians is the outcome of a far-seeing policy requiring for its conception something of the imagination and courage of an idealist. The famous Zeiss Works at Jena and its history furnish no more than a somewhat overshadowing example of a widespread and single-minded association of scientific investigation with workshop practice and industrial aims. It is the spirit that prompts a CARL ZEISS to associate with him an ERNST ABBE for the elaboration of an exact system of optical synthesis, in sublime indifference to any immediate material advantage, that is the true source of great achievements. It is not enough to equip workshops with all the best that modern technique and organisation has to offer and to secure the services of the best available scientific brains for carrying out the industrial programme of an establishment, leaving to the savant all scientific effort which has no direct bearing on production. The hope which the translator cherished when he undertook the work is that the book may appeal to the imagination of British opticians and help to stimulate further the rising spirit of scientific enthusiasm which looks beyond the immediate needs of successful production.

The aim of the Translator and of those who helped him has been to ensure a faithful rendering of the authors' intentions. For didactic and terminological reasons this task has been fraught with many difficulties, involving in many cases critical care. Though generously assisted in these difficulties, the Translator must accept responsibility for any shortcomings in the terminology adopted in the book. He is well aware that the book deals with concepts and operations which call for a very precise terminology, such as

might well engage the attention of a competent scientific commission. In these circumstances many of the terms adopted in the book are offered provisionally and it must be understood that no official sanction attaches to any of them.

Much care was needed to express accurately the intentions of the various authors, and in this respect the Translator is particularly indebted to the critical vigilance of Mr. E. B. KNOBEL, F.R.A.S., and to the valuable help of Mr. J. W. FRENCH, B.Sc., and Professor J. W. NICHOLSON, F.R.S.

The book has been divided into numbered paragraphs in accordance with English usage, and this, in conjunction with the copious numbering of equations, has enabled the Translator to assist the reader with references which have been dispensed with in the original.

The figures have been redrawn with such alterations and additions as suggested themselves to the Translator as helpful to the student, and naturally every care has been taken to eliminate errors contained in the original. Considerable modifications and additions will be found in the Bibliography.

The Translator wishes to express his sincere thanks to the gentlemen named, especially Mr. KNOBEL, upon whose shoulders rested an abnormally heavy burden of revisory and typographical work.

R. K.

Preface.

(By the late Dr. S. Czapski.)

Three years ago I was urged to prepare a revised and amplified edition of my work entitled "Theorie der optischen Instrumente nach ABBE."* Owing to the pressure of my ordinary professional work it soon became apparent to me that I was not in a position to undertake the task. On further reflection it occurred to me that the proposed work might be carried out much more efficiently if several contributors could be found to take the place of a single author and reviser, thereby rendering the work less dependent upon the limitations imposed by individual knowledge, judgment and experience. The scientific staff of the Carl Zeiss Works at Jena offered a ready means for such a combined undertaking. Since the members of this staff specialised in varying degrees in different departments of optical manufacture, an excellent opportunity naturally presented itself of selecting suitable contributors for the respective sections of the work in accordance with the subjects upon which they were engaged in their professional capacities. A further obvious advantage of selecting the contributors from the associated members of a single establishment was that it tended to minimise the lack of continuity in consecutive sections of the work from disparities in individual aspect and presentation, which inevitably arises when related subjects are treated by wholly detached authors.

It afforded me great relief to find that this plan met with a kindly reception on the part of all concerned. Professor ABBE was entirely in favour of this scheme from its very inception. The first part of the proposed undertaking, "The Geometrical Investigation of the Formation of Images in Optical Instruments," was accordingly promptly taken in hand by the respective contributors, with the result that we were able to go to press in the summer of this year.

The treatment of the various sections in my older work provided a natural nucleus for the development of the corresponding chapters in the present volume. The original material has been supplemented or extended, and, where necessary, it has been rectified by incorporating the results of the investigations of the respective contributors as well as others which had been published in the meantime. It goes without saying that the nature of the subject and the

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* Originally published by Mr. TREWENDT, 1893, Breslau, and subsequently transferred to Mr. J. A. BARTH, Leipzig.

treatment adopted by different contributors have necessitated changes differing widely in extent and significance. It may, therefore, be of interest to the reader to know beforehand what changes he may expect to encounter. Presuming that he is acquainted with my "Theorie der optischen Instrumente nach Abbe," I shall content myself with a brief review of the changes and expansions which that treatise has experienced.

The ten chapters of which the new treatise is composed may be readily ranged under three groups in this respect. Comparatively small changes will be found in the first chapter on the "Fundamental Principles of Geometrical Optics" (by Dr. H. SIEDENTOPF); in the third chapter dealing with ABBE'S Geometrical Theory of the Formation of Optical Images (by Dr. E. WANDERSLEB); in the fourth chapter on the "Formation of Optical Images" (by Dr. P. CULMANN); and in the eighth chapter on "Prisms and Systems of Prisms" (by Dr. F. LOEWE). It goes without saying that the contributors of the respective sections have throughout made the fullest use of the opportunities afforded them for the removal of imperfections and errors found in my treatise. They have also added more recent contributions as well as references to older investigations which had not received adequate attention in my book; and, finally, they have rendered it more generally available for immediately practical use by discussing the most important cases met with in practice. Thus it will be seen that the discussion of the general theorems of reflection and refraction now includes the "characteristic function" introduced by HAMILTON as far back as 1824. The latter method, though duly considered in English works published some years ago (*e.g.*, R. S. HEATH), has received somewhat belated attention in the German works of M. THIESEN, H. BRUNS, and F. KLEIN and a few others. Respecting the modifications introduced by other contributors the reader is referred to their own remarks on pp. 83, 96, 125 and 416.

Considerably more extensive and fundamentally important changes will be found in Chapters V., VI., and IX. This applies in particular to Chapter V. on the "Theory of Spherical Aberration" (by Drs. A. KOENIG and M. von ROHR), which should not fail to prove helpful to those studying the problems of aberration. It furnishes a complete elaboration of ABBE'S method of invariants as a means of deriving expressions for the whole of SEIDEL'S ten image defects up to the third power of angles, and it may be noted that with respect to the defects due to the second and third powers of the aperture of the pencils (*i.e.*, "coma in a wider sense" and spherical aberration of oblique pencils), these expressions are given for the first time for finite inclinations of the principal rays. SEIDEL'S equations themselves follow from these expressions by the simple suppression of the terms applying to all but small inclinations of the principal ray, or they may be derived independently

by KERBER'S method. The investigation demonstrates in a striking manner the great advantage of ABBE'S method over that of SEIDEL, first, in that it admits of correcting a given defect in a much higher degree than can be accomplished by SEIDEL'S theory; and, secondly, in that it furnishes equations in which every defect is considered separately. ABBE'S method has also been employed to arrive at SEIDEL'S equations as applied to non-spherical surfaces of revolution, though I am aware from a private communication that A. GULLSTRAND had already previously begun to apply ABBE'S theory for this purpose.

I shall have to confine myself to this brief sketch of the contents of this chapter. As already stated, it forms in itself a valuable monograph on the subject. Apart from the great help afforded by KERBER'S published papers it comprises almost entirely original work.

A distinctive feature of the sixth chapter on the "Theory of Chromatic Aberration" (by Dr. A. KOENIG) is that it deals throughout with finite differences of wave-lengths. This is a point of material importance, if we consider that in the spectrometric measurement of dispersion we are always concerned with finite quantities, and the same applies in most cases to the computation of optical instruments.

The ninth chapter on the "Limitation of Rays in Optical Systems" (by Dr. M. von ROHR) furnishes a methodical exposition of the theory of the restriction of aperture for pencils in optical systems which are free from aberration. The chapter discusses in detail the conditions which arise in optical projection instruments both in the case of objects sending out rays in all directions, and in the frequently occurring case of objects radiating within a restricted space under the influence of transmitted light. The conception of a screen plane, already previously introduced by von ROHR, has been made a very simple means of establishing the laws by which optical systems produce on a screen surface pictures, rather than images, of solid objects. The discussion of the condition which arises when the eye operates in conjunction with an optical system yields new aspects, since it has been shown by GULLSTRAND that in certain cases, as when photographs are being viewed, the principal rays, when produced sufficiently far, must meet at the point of rotation of the eye. This point acquires accordingly in the theory of optical instruments a significance comparable to, and in a certain sense surpassing, that of the pupil of the eye.

In addition to these sections, which comprise, as I have already pointed out, more or less extensive modifications and developments of my treatment of the subject, the present volume contains three chapters which present fundamentally new and independent matter. They furnish very important additions to my treatise. These are comprised in Chapter II. on the "Computation of Rays through a

System of Refracting Surfaces" (by Drs. A. KOENIG and M. von ROHR), Chapter VII. on the "Computation of Optical Systems in Accordance with the Theory of Aberration" (by Dr. A. KOENIG), and Chapter X. on the "Intensity of Rays transmitted through Optical Systems" (by Dr. M. von ROHR).

The two first named chapters are mainly intended to satisfy such practical requirements as arise when the useful qualities of any given optical device are to be ascertained by computing the traces of a sufficient number of rays, or when it is required to ascertain whether it is possible with a given type and with the available means to realise certain qualities in the optical performance of a system; for example, a certain degree of freedom from specified defects.

The second chapter gives accordingly a development of the formulæ by the aid of which a ray may be traced through any given system, first when the rays pass through a point on the axis, and next with respect to rays within a meridian plane (ABBE) and rays infinitely near to these. This is followed by a detailed investigation of the general case of skew rays, including the researches of WANACH, KERBER and BRUNS. This chapter contains as a notable feature a treatment of KERBER'S so-called difference formulæ, which are adapted for the direct calculation of the differences in the intercepts of rays proceeding from a point on the axis. With reference to these the writer of the chapter concludes that in all probability SEIDEL was in possession of such formulæ in their most general form.

In the seventh chapter KOENIG applies the practical results of the preceding investigations of spherical and chromatic defects by explaining the means by which several of these defects may be corrected jointly in a given system. After discussing in this way the so-called condition of PETZVAL (relating to the flatness of the image), which is not affected by the otherwise largely employed method of "coflexure," he proceeds to explain the correction of the other image defects of SEIDEL. The influence of the available means (such as coflexure, choice of distance between the component lenses, position of stops), is shown with respect to various cases and the results of calculations are instanced in numerical tables. At the end of the chapter the mode of computing a given type is explained, as well as the influence which the absolute dimensions of a system exercise upon the magnitude of the defects, so as to furnish a foundation for correctly apportioning the duty of an optical system over an objective and eyepiece.

In the concluding chapter Dr. M. von ROHR gives an exposition of the photometrical conditions obtaining in optical images.

Opening the chapter by establishing the fundamental laws of photometry, the author proves, with the aid of the physical theory of light in conjunction with the most recent experimental data, that

the total intensity of the light reflected at mirrors, as well as that transmitted by refraction through the boundary of transparent media, is within wide limits independent of the magnitude of the angle of incidence. This supplies the physical basis, which hitherto was wanting, of ABBE'S two laws of radiation, at least within the limits which enter into practical consideration. These laws are derived from first principles and are applied to the most important species of optical instruments, viz., those designed for projection and as aids to vision.

The laborious task of editing the various sections of the book has devolved upon Dr. von ROHR. It is mainly he to whom we are indebted for a uniforn notation and the bibliographical references; also for the detailed subject and authors' indexes, as well as the reading of the proofs and the choice and preparation of the diagrams. Our sincere thanks are due to the publisher for the lavish production of the work.

I believe that the results of the studies of the various contributors recorded in this book will be welcomed as a valuable addition to the existing literature.

Whether our hope of completing the entire scheme of the work will ever be realised depends unfortunately to a certain extent upon circumstances which no man can foresee or control.*

S. CZAPSKI.

Jena, December, 1903.

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* Thus wrote Czapski at the end of 1903, and on the 29th of June, 1907, his labours were cut short by an untimely death. (R. K.)

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Chapter I.

THE FUNDAMENTAL PRINCIPLES OF GEOMETRICAL OPTICS.

(H. Siedentopf.)

1. INTRODUCTION.

A. Theories of Light.

1. In our present state of scientific knowledge a reasonably complete explanation of all the observed phenomena of light can be furnished by the undulatory theory as propounded by Huygens (*1.*) and subsequently extended by Young (*2.*) and Fresnel (*2.*).

According to this hypothesis light is a manifestation arising from the transverse vibration of an extremely subtle and highly elastic medium which is supposed to pervade all space and to interpenetrate all substances therein, and which is provisionally known as the luminiferous ether.

The electro-magnetic theory of light, first suggested by Clerk Maxwell and further elaborated by Hertz, abandons certain special suppositions respecting the undulatory mechanism and furnishes a fairly complete explanation of nearly all that is known of the phenomena of light and vision.

In the electron theory Lorentz reverts to special hypotheses respecting the luminiferous mechanism, introducing such refinements as are needed to bring them into line with the results of modern research.

Now, there is a very wide field of optical phenomena, including, as it happens, those with which we are more particularly concerned in practical life, all the essential elements of which can be explained without reference to any hypotheses respecting the mechanical, electrical or other nature of the luminous process, and which may be successfully studied as simple deductions from general observed facts pertaining to the propagation of light. The properties of light in relation to its propagation in space are essentially simple, and provide a sufficiently precise basis for all pertinent investiga-

tions. That this is so is amply borne out by the results achieved thereby in the past as well as at the present time.

B. The Geometrical Aspect of the Propagation of Light.

2. The general properties of the propagation of light, in conjunction with a few subsidiary principles established by observation, may be deduced and explained by strictly mathematical reasoning from our fundamental notions of the nature of the propagation of light. On the other hand, there is nothing to prevent us from regarding these properties as empirical fundamentals and from employing them as starting points in our investigations.

These general properties are defined by the following four laws:—

(i) The law of the rectilinear propagation of light;
(ii) The law of the mutual independence of the component parts of a pencil of light;
(iii) The law of the regular reflection of light;
(iv) The law of the regular refraction of light.

These laws, as will be seen, relate solely to the direction in which light travels and involve therefore a purely geometrical property. Their application to natural or artificially produced combinations of the transmission of light constitutes the subject of **Geometrical Optics.**

It must not, however, be thought that the principles of geometrical optics include all phenomena of light which give rise to changes of direction. On the contrary, their application is subject to the condition that the reflecting or transmitting media shall be isotropic and devoid of crystalline structure.

Although these laws provide an adequate foundation for the elaboration of a complete system of investigations enabling us to formulate quantitative statements respecting all that is susceptible of experimental verification, which they do with a considerable degree of precision, and even furnish us with a means of correctly anticipating effects yet unobserved, nevertheless, we should be ill-advised, even when restricting our investigation to the geometrical consideration of optical problems, if we were to set entirely aside our fundamental ideas of the physical nature of light and the inferences to be drawn therefrom. The inadvertent extension of the theorems of geometrical optics beyond limits which are supported by observation, or which have the sanction of physical theory has frequently resulted in serious errors. In particular, any attempt to elaborate a complete theory of optical instruments, or to establish the fundamental principles of meteorological optics, without reference to the foundations of the undulatory theory, is foredoomed to failure.

Indeed, in every investigation of any optical problem it behoves us to ascertain to what extent the results deduced from simple geometrical principles are in harmony with physical theory. Otherwise the subject of geometrical optics, far from being a legitimate method of physical research, runs a serious risk of becoming merely a field for the practice of mathematics.

In accordance with this aspect of the matter, we shall confine ourselves in this book to the study of the general relations which can be derived from the fundamental laws, and to such deductions and applications thereof as are necessary for the better comprehension of important natural phenomena or for the study of optical instruments.

2. THE FUNDAMENTAL LAWS.

A. The Propagation of Light in Straight Lines.

3. The law which states that light within a homogeneous medium proceeds from a luminous particle along rectilinear paths must not be assumed to have been derived from a series of independent and conclusive observations specially devised to this end. Its truth rests entirely on circumstantial evidence, just as that of any other fundamental law of physical science. It cannot even be said that its truth has been, or can be, proved in the strict sense of the word by definite experimental tests. We accept the law as such because of the complete agreement of its applications with established facts of observation. In all phases of human life absolute reliance is placed upon the truth of this law, and in a much more rigorous degree does its validity govern all astronomical measurements and geometrical surveys; and conversely, the accepted certainty that light proceeds along straight paths is taken as a criterion of the straightness of a line joining two points.

In all cases the conclusions deduced from this law have proved to be correct. These innumerable confirmations, many of them of an exceedingly critical order of precision, have endowed the statement with a degree of axiomatic certainty which is scarcely shared by any other physical law.

Nevertheless, it has been known for upwards of a century that this law does not hold unconditionally. Indeed, in the form in which it is generally enunciated it is by no means correct.

When submitted to precise tests by critically conducted experiments, introducing conditions involving the use of elementary portions of a pencil of light, that is of individual rays, the law is no longer strictly valid. If we cause the light to pass through an extremely narrow aperture in a screen we shall find that the direction of propagation of the extremely narrow

pencil transmitted through the minute aperture, becomes more and more ill-defined and complex the more it is endeavoured to get nearer to our ideal conception of an isolated ray, thereby casting increasing doubt on the very reality of such a single ray. For, as the aperture diminishes, the issuing pencil, instead of proceeding along its original rectilinear path, appears to spread out as a pencil of varying intensity. Also, as the size of a screen placed in the path of a pencil of light is diminished, the shadow cast upon a screen placed in the path of the transmitted pencil loses more and more its geometrical similarity to the body casting the shadow; and what we are accustomed to describe as a shadow is replaced in an increasingly pronounced degree by an entirely different appearance. We need not linger over the description of experiments of this kind. They lead us, in fact, to what is an important and widely investigated chapter of physical optics which deals with the phenomenon known as the **diffraction of light.**

Although we thus see that the law of the rectilinear propagation of light is true in a restricted sense only, it loses none of its importance, even when viewed from the critical aspect of the physical theory of light, which readily accounts for the phenomenon described. By that theory rays may be defined according to Kirchhoff (*1.*) as *the wave-normals along which energy is transmitted*, and it shows, in agreement with observation, that under all ordinary conditions, *i.e.*, whenever the pencils are of finite cross sections, these pencils behave very nearly, and in many respects, as if composed of elementary rays proceeding independently of one another along rectilinear paths. The exceptions to this rule are almost exclusively confined to certain cases intimately connected with the phenomena of diffraction and interference. Most of these are of a somewhat elusive order, and critical attention is often needed to appreciate any departure from the rule, notwithstanding that it is never rigidly applicable.

This departure from the fundamental laws which form the basis of geometrical optics, is much more conspicuous at the boundary of pencils of finite cross section. In this case, however, the quantity of anomalous light is negligibly small in comparison with the bulk of the light which follows the law of the rectilinear propagation of light, and is, therefore, in many instances comparatively negligible.

B. The Behaviour of Light at the Interface of Two Different Media.

4. Like the first two fundamental laws, the third and fourth laws can be shown by rigorous analysis to be in strict accordance with physical theory, and it has likewise been proved by critical experiments that they hold good within certain limits, *i.e.*, so long

as the prevailing conditions are such that it is permissible to speak of "rays."

So long as light travels within a single perfectly homogeneous and isotropic medium, it does so, with the reservations already stated, in rectilinear paths. When, however, it reaches the boundary of a medium of a different optical nature, it splits up into two portions which proceed in abruptly changed directions from the point of incidence on the interface of the two adjoining media.

(*a*) Part of the light remains within the first medium. This is known as **reflected light.**

(*b*) The other part passes into the second medium and continues its rectilinear course therein. This is known as **refracted light.**

5. The Nature of the Surface of Separation.—Strictly speaking, we are not justified in making a hard and fast distinction between these two parts of the affected light, as a more critical examination readily discloses. As a matter of fact, that part of the light which returns into the first medium, penetrates to a certain, though very slight, depth into the second medium. Thus the natural colours of bodies derive their origin from the selective absorption which occurs during this process of transmission. We cannot here do more than briefly refer to this subject, a fuller discussion of which would take us too far from our immediate purpose. The extent and depth to which the so-called reflected light enters the second medium depends upon the nature of the adjoining media, and in a very pronounced degree also upon the condition of the surface of separation. For instance, it matters a great deal whether the second substance is a solid or whether it is in a powdered condition, and again in the former case, whether its surface is rough or polished. In polished surfaces this relation of the degree of polish to the wave-length of the light brought to bear upon a substance exercises a marked influence. Thus observation has disclosed that finely ground but unpolished surfaces readily reflect infra-red light, whereas ultra-violet light is reflected and refracted in an imperfect degree unless the surface is polished with exceptional care.

6. Regular and Diffuse Reflection and Refraction.—According to the third fundamental law of geometrical optics, the direction of the reflected light depends, in a manner which will be dealt with more fully later, solely upon the inclination of the incident ray of light with respect to the surface element upon which it impinges. It can readily be shown by experiment that when this surface is polished so as to present the condition of a mathematical plane without the slightest discontinuity, the impinging light travels almost exclusively in directions conforming to that law. The phenomenon is therefore known as **regular reflection.**

The law ceases, however, to be applicable in proportion as the surface between the two media becomes irregular, in the sense that over small areas the elements of the surface change their inclinations frequently by considerable amounts; that is, the more rough or mat the surface the less is the regular reflection of the light in accordance with the fundamental law. Under conditions of this kind the arrangement of the surface elements cannot even approximately be defined in geometrical terms, and hence we lose all data for determining the behaviour of the impinging rays in relation to the law of reflection. Moreover, the dimensions of these promiscuously acting elements are so minute that, from what we have stated, it may be concluded that in cases of this kind the rules of geometrical optics cannot be applied without reservation. Finally, it is not difficult to see that in these cases light which penetrates slightly below the surface of the second medium will necessarily add its effect to that of the light reflected at the surface and thereby influence the general result. For instance, when light meets such a flat surface in a single direction, certain portions of it are scattered in all directions with widely varying intensities. An effect of this kind is termed **diffuse reflection.**

It is precisely because in a diffusely reflecting surface every particle behaves differently with respect to the incident light that such particles, and hence the entire surface, are themselves rendered visible by becoming discrete sources of luminous disturbances. Reflection at perfectly smooth surfaces, on the contrary, as we shall see presently, gives rise exclusively to images of external objects which are either self-luminous or which reflect light by diffusion, whilst the reflecting surfaces themselves remain entirely invisible. In reality the degree of invisibility is more or less modified by the unavoidable presence of scratches, cracks, particles of dust or other blemishes. As a matter of fact, a surface which completely realises one or other of these extreme conditions does not exist, so that in every case the resulting processes of reflection can only be regarded as a more or less complete approximation to an ideal condition.

Refracted light is in a large measure subject to the same rules and reservations as we have found to apply to reflected light. When the surface of separation of the two adjoining media is smooth it can be shown that the direction of the refracted ray, in accordance with the fourth law of geometrical optics, is governed solely by the direction of the incident ray with respect to the elementary surface upon which it impinges and to the properties of the adjoining medium. When, on the other hand, the surface of separation is dull, or rough, the light which enters the second medium undergoes diffuse refraction, much in the same way as light which returns by reflection into the first medium was shown to become scattered or diffused.

Within the second medium the light is susceptible of various modifications. A portion is invariably lost as such, being transmuted into other forms of energy, viz. heat, electricity, chemical energy. Light so lost to vision is said to be absorbed. Frequently light, without being transmuted into some other form of energy, undergoes within the second medium merely a change in kind, *i.e.*, of colour. In this case it presents the interesting phenomenon known as fluorescence.

According to the proportion of light which is transmitted by a stratum of a given thickness, a substance is described as more or less **transparent.** It should be noted that any medium generally absorbs light of different colours in different degrees, and hence its transparency varies in degree according to the colour of the transmitted light.

In any perfectly homogeneous medium light can always be so transmitted that external objects can be seen through it with unimpaired clearness, though with diminished brightness, provided the intervening stratum is sufficiently thin. Media which include within their otherwise homogeneous substance scattered particles of other optical properties, such as milk, blood, porcelain, and moist air, are described as turbid media. Particles held in suspension in these substances give rise to internal diffuse reflection, the nature of which can only be more precisely defined on the basis of the physical theory of light. External objects as seen through media of this kind appear indistinct, and hence these media are called **translucent** substances.

As is well known, there are no substances in nature which can be described as perfectly transparent. The degrees of translucence are, however, so widely different as to justify entirely the division of the more or less diaphanous substances into transparent and translucent media. Thus, it has been shown by the investigations of Siedentopf and Zsigmondy (***1***.) that in the so-called gold-ruby glass the presence of finely divided gold gives rise to a degree of turbidity which is so inappreciably slight that the substance appears quite clear to transmitted light. Conversely, no substances occur in nature which can be described as absolutely opaque, since any medium can be rendered transparent, or at all events translucent, provided it is sufficiently thin. These phenomena, however, no longer concern the behaviour of the light at the interface of the two media, but rather to what occurs within either medium.

In what follows we shall regard all media upon which light impinges as perfectly transparent and homogeneous as well as bounded by perfectly smooth surfaces; in other words, in discussing the laws of geometrical optics and their consequences we shall assume that these ideal conditions are rigorously fulfilled.

In conformity with the character and purpose of geometrical optics we shall take no cognisance of any property of light which does not give rise to changes in the geometrical relations of the optical elements. We shall therefore ignore entirely any question as to whether the light proceeds from a self-luminous body or from one which radiates diffuse light, or again from foci which have been derived from these sources by special optical processes. Neither shall we concern ourselves with considerations as to whether the light is intense or feeble, natural, or in any particular state of polarisation. Questions relating to the intensity of any given pencil of light will concern us only in so far as they have their origin in geometrical conditions.

It is, moreover, entirely in accordance with the principles of geometrical optics, properly regarded as a subsidiary branch of physical optics or indeed of physics itself, to define the focus of a given pencil of rays, not merely as the point of intersection of any rays proceeding from different origins, but rather as the point of confluence of coherent rays, that is to say, of such rays only as have their origin in one identical point of radiation, or, in conformity with the definition of a "ray of light" in the language of the undulatory theory, of such rays only which are normals to the same wave-front. It is desirable to emphasise this restricted definition at this early stage.

C. The Fundamental Laws of Reflection and Refraction.

7. The direction of the regularly reflected ray and of a similarly refracted ray relatively to that of the incident ray is determined by the following laws, which will be referred to as the third and fourth laws of geometrical optics.

8. Definitions.—The angle comprised between the incident ray and the normal to the surface of separation between the two media at the point of incidence is called the **angle of incidence.** The angle which the reflected or refracted ray forms with the normal is called the **angle of reflection** or the **angle of refraction** as the case may be. The plane which contains the incident ray and the normal is called the **plane of incidence.** The laws of reflection and refraction may now be stated as follows:—

The reflected ray and the refracted ray lie in the plane of incidence and on the side of the normal opposite to that of the incident ray.

9. The Fundamental Law of Reflection.—The angle of reflection is equal to the angle of incidence; $APN = - BPN = i$ (Fig. 1).

For convenience of expression we adopt the following convention of signs :—

Angles will be regarded as positive or as negative according as the direction of rotation of the ray towards the normal corresponds with a clockwise or anticlockwise rotation.

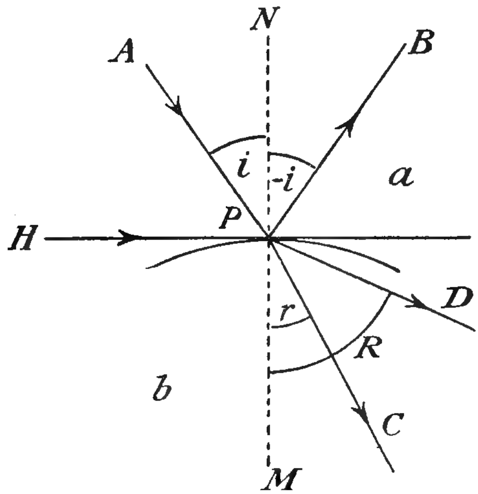

FIG. 1.

Medium a :
i = angle of incidence.
$-i$ = angle of reflection.

Medium b :
r = angle of refraction.
R = angle of total reflection.

10. The Fundamental Law of Refraction.—The sine of the angle of refraction and the sine of the angle of incidence are connected by a relation which depends solely upon the properties of the two adjoining media a and b and the nature, *i.e.*, the wave-length, of the incident light, but which is quite independent of the magnitude of the angle of incidence :—

$$\sin CPM/\sin APN = \sin r/\sin i = n_{ab} = constant .$$

The ratio n_{ab} is called the **refractive index** of the medium a relatively to the medium b with respect to light of a given kind.

Experimental observation is in accord with these necessary inferences from the wave theory, and what was said in our introductory remarks holds equally in this case. Their deduction from the premises of the undulatory theory was first established by Huygens, and subsequently they were developed with greater precision by Fresnel. The most convincing confirmation of the truth of the law of reflection has been furnished by

astronomical observations, in which the altitude of a star is determined on the one hand by direct observation with the transit instrument, and on the other hand by indirect observation of the depression of the star image reflected from an artificial mercury horizon. The depression determined in this way is invariably exactly equal to the altitude as found by direct measurement, whatever its amount. Measurements of this kind are susceptible of an extreme degree of accuracy and thereby furnish an exact means of verifying the law of reflection.

The law of refraction is put to its most rigorous test by its application to the various methods which have been devised, on the assumption of its truth, for determining ratios of refraction, and by the agreement with the calculations of practical results, as achieved in the construction of optical devices.

11. The Principle of the Reversibility of the Path of a Ray.—Concerning the law of refraction we know by observation, as is self-evident in the case of reflection, that the incident and refracted rays are mutually interchangeable. That is to say, if an incident ray in a medium a meeting the surface of separation at an angle i, enters the medium b at an angle of refraction r, then the same ray meeting the surface of separation at an angle of incidence r in the medium b will undergo refraction at an angle i in the first medium. In other words, if $n_{ab} = \sin r/\sin i$ be the refractive index of the medium a with respect to the medium b, it follows that $n_{ba} = \sin i/\sin r = 1/n_{ab}$ is the refractive index of medium b with respect to medium a, so that $n_{ab} = 1/n_{ba}$.

From this we draw the following conclusion:—

A ray, which after any series of reflections or refractions meets a surface in such a manner that it undergoes reflection at right angles to it, will accurately retrace its original course in the reverse direction.

12. The Optical Invariant.—Finally, the measurement of numerous refractive ratios has shown that the relative refractive index n_{ab} of one medium a with respect to another medium b is completely determined when the relative refractive indices n_{ac}, n_{bc} of the media a and b are known with respect to a third medium c, in that $n_{ab} = n_{ac}/n_{bc}$. The relative index of refraction of a medium a with respect to the medium b is thus equal to the ratio of the relative indices of refraction with respect to a third medium c. From this it follows that the number of possible refractive indices which otherwise would have been equal to the number of possible combinations of media, becomes reduced to a single determinate series comprising the refractive indices of all media with respect to a single standard medium. The universal standard medium selected for this purpose is a vacuum. The refractive indices

relative to vacuous space are accordingly termed **absolute refractive indices** or, briefly, **refractive indices**. In accordance with this convention a vacuum itself has the refractive index 1. All other known transparent media have refractive indices exceeding unity, the only exceptions being a few metals which, when experimented on in the form of exceedingly thin prisms, have been found to have refractive indices which are less than 1. Investigations of this kind are due to Kundt (*1.*).

From these invariable relations the equation which expresses the law of refraction can be written in a simple symmetrical form, of which we shall make considerable use in future. According to our original notation $\sin r = n_{ab} \sin i$. But since $n_{ab} = n_a/n_b$ in which n_a, n_b denote the refractive indices of the media a and b with respect to vacuous space, it follows that

$$n_a \sin i = n_b \sin r,$$

which may be translated into the statement that the product of the refractive index and the sine of the angle comprised between the ray and the normal is constant for all refractions. We call this product the **optical invariant.**

It will, moreover, be realised at once that this equation is likewise applicable to the case of reflection. All that is necessary in this case is to substitute for n_a/n_b the special value -1. In what follows we shall confine our attention accordingly to the direct investigation of problems of refraction, and deduce therefrom the proper relation for the corresponding problem of reflection by substituting -1 for n_a/n_b.

D. Dispersion of Light.

13. As has already been stated above, the change of direction by refraction at the interface of two adjoining media, and hence their relative indices of refraction, depends not only upon the properties of these media but likewise upon the colour or wave-length λ of the light transmitted through them. In fact, n is a function of λ.

The relations which we have deduced so far enable us to formulate an expression showing the effect of this multiplicity of refractive indices for differently coloured rays. Let n denote the refractive index for a given colour of a medium relatively to vacuous space (or any other standard medium). Then we have the relation $\sin i = n \sin r$. Now let $n + dn$ be the refractive index with respect to a wave-length which differs infinitesimally from that of the standard ray. Then a ray corresponding to this proximate colour (or wave-length) and having the same angle of

incidence i as the first ray, will be refracted into a direction which forms an angle $r + dr$ with the normal, so that

$$\sin i = (n + dn) \sin (r + dr),$$

and

$$n \sin r = (n + dn) \sin (r + dr).$$

Hence the difference in the two angles of refraction corresponding to two consecutive colours is given by the relation:

$$n (\cos r)\, dr + (\sin r)\, dn = 0$$

or

$$dr = -(dn/n) \tan r.$$

It will thus be seen that rays of different wave-lengths meeting a refracting surface at the same angle of incidence are deflected into different directions by a single refraction and still more so by two appropriately arranged refractions. Newton, by the inverse process of reasoning, was led to attribute the dissimilar deflections of differently coloured rays passing through prisms to the diversity in the refrangibility of light according to its colour, and thence showed that white solar light was composed of light of many colours. The process by which Newton (*3.*) established this fact (in 1666) still ranks as a model of inductive research, and the account of his investigations is one of the most interesting documents of early physical science.

It should be noted that the refractive index of a medium generally increases in the inverse order of the wave-length of the transmitted light, so that it steadily increases within the visible portion of the spectrum from the red to the blue end. There is, however, a class of bodies which forms an exception to this rule and in which, accordingly, the entire visible spectrum or certain portions exhibit the reverse relations between the refractive index and the wave-length. This species of dispersion is distinguished as **anomalous dispersion.**

Formerly it was believed that the magnitude of the refractive index was directly associated with the density of the substance. Although subsequent investigations have shown that this rule does not by any means hold universally, yet it does so in such a great majority of cases that the notion of the **optical density** of a substance has been retained as a conveniently concise term. When, accordingly, we refer to one substance as being optically denser than another it is equivalent to saying that it has a greater refractive index than the other.

E. Total Reflection.

14. The relation between the angle of refraction and the angle of incidence is expressed in a symmetrical form by the equation

$n_a \sin i = n_b \sin r$. If $n_a < n_b$, and therefore $n_{ab} < 1$, this equation will evidently furnish for every value of the angle of incidence i a corresponding value of r. Since the greatest possible angle of incidence $i_{max} = J = \pi/2$, the corresponding greatest angle of refraction is in these circumstances determined by the equation $\sin r_{max} = \sin R = n_a/n_b$. It will thus be seen that all incident rays have corresponding to them refracted rays comprised within a cone of semi-aperture $\sin^{-1}(n_a/n_b)$. If now we consider rays proceeding in the reverse order from medium b into medium a, or which comes to the same thing, if we suppose that $n_a > n_b$ or $n_{ab} > 1$, it is manifest that the equation $n_a \sin i = n_b \sin r$ furnishes real values of the angles of refraction r corresponding with values of the angle of incidence i so long only as $\sin i < n_b/n_a$, or so long as $\sin i < 1/n_{ab}$, which corresponds with the greatest possible angle of refraction $r = \pi/2$. Any ray which impinges at a greater angle than that conforming to this limiting value of the angle of incidence, will cease to undergo refraction, so that the whole of the light will be reflected back into the first medium.

Although accurate observations have shown that in this case also a portion of the light passes into the second medium, it should be noted that this portion penetrates only to a very small depth and then reverses its direction of motion.

This mode of reflection is accordingly known as **total reflection,** and the angle of incidence J which marks the beginning of total reflection in conformity with the equation $\sin J = n_b/n_a = 1/n_{ab} = n_{ba}$, is called the **critical angle** or **angle of total reflection.**

Incidentally it may be noted that in the case of transparent media the proportion of light which is thus reflected increases with the angle of incidence. In fact, it steadily increases from a minimum value corresponding to normal incidence to the maximum value which prevails at total reflection.

The critical angle depends exclusively upon the refractive indices of the two adjoining media. It is therefore in a measure also governed by the wave-length, or the colour, of the incident light. The critical angle of total reflection of a medium bounded by vacuous space furnishes a simple means of obtaining a measure of its absolute refractive index, in that $\sin J = 1/n$.

F. Corollaries.

15. The laws of reflection and refraction furnish by deduction several corollaries, which are capable of useful application in many cases. We therefore enunciate these together with their proofs. From the laws of **reflection** we derive the following corollaries :—

(1) The incident and the reflected rays are equally inclined to any straight line which passes through the point of incidence and is at right angles to the normal.

(2) The projections of the incident and reflected rays upon any plane containing the normal conform likewise to the law of reflection.

The proof of these statements is obvious. We accordingly give only the proofs of the corresponding corollaries as derived from the laws of refraction. Moreover, the corollaries concerning reflection can be deduced from these by simply substituting $-n_b$ for n_a, or $-n'$ for n. The corollaries which hold for **refraction** are :—

(1) The cosines of the angles which the incident and refracted rays form with any straight line which passes through the point of incidence and is at right angles to the latter, *i.e.*, **in the tangential plane of the refracting surface, are likewise related in the inverse ratio of the corresponding indices.**

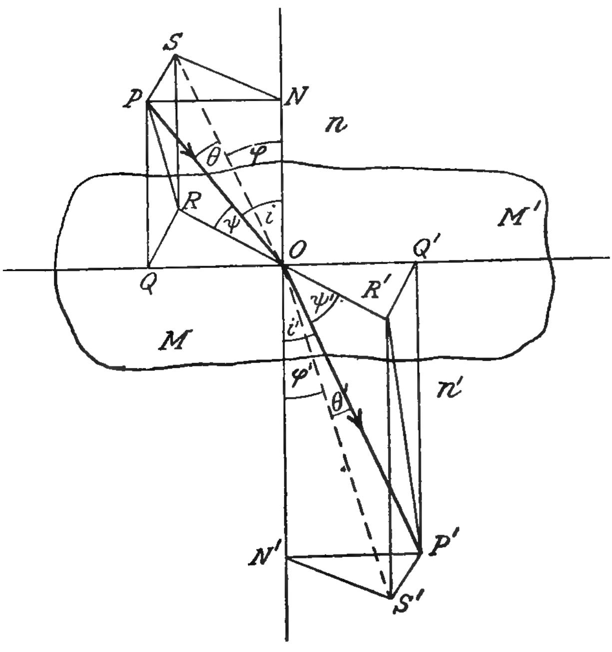

Fig. 2.

Projections of the incident ray PO and the refracted ray OP' upon a straigh line RR' contained in the tangent plane MM' to the refracting surface and upo a plane $SNS'N'$ containing the normal NN'.

(2) The sines of the angles included between these rays an any plane containing the normal to the refracting surface ar related in the same way, *i.e.*, **they are in the inverse ratio o the corresponding refractive indices.**

(3) The projections of the incident and refracted rays upon a plane containing the normal at the point of incidence obey a law of refraction in which the constant ratio of the sines has a value depending upon the inclination of the rays with respect to that plane.

To prove these deductions from the fundamental laws, let ON (Fig. 2), be the normal of incidence; let MM' be the refracting plane or the tangent plane at O to the refracting surface, and let PO, OP' be respectively the incident and refracted rays. We then have the following relation:

$$n \sin PON = n' \sin P'ON', \text{ or } n \sin i = n' \sin i',$$

hence

$$n \cos\left(\frac{\pi}{2} - i\right) = n' \cos\left(\frac{\pi}{2} - i'\right), \text{ or, } n \cos POQ = n' \cos P'OQ'.$$

Now let the lengths of OP and OP' be proportional to n and n' respectively, and let Q, Q' be the points where the perpendiculars let fall from P and P' respectively meet the plane MM'. Then $OQ = OQ'$. In the plane MM' let ROR' be any other straight line passing through O, and let PR, $P'R'$ be perpendiculars from P and P' respectively upon this straight line. Then the lines QR, $Q'R'$ are likewise perpendicular to ROR', and hence $OR = OR'$. Now $OR = OP \cos POR$, $OR' = OP' \cos P'OR'$, whence it follows that $\cos POR/\cos P'OR' = n'/n$, which proves the truth of the first corollary.

Now, retaining the construction, let PS, $P'S'$ be the perpendiculars let fall from the points P, P' upon any plane passing through the normal ON and any straight line RR'. Then $PS = QR$ and $P'S' = Q'R'$, so that $PS = P'S'$, since $QR = Q'R'$. Now $PS = OP \sin POS$, $P'S' = OP' \sin P'OS'$. Therefore

$$\sin POS/\sin P'OS' = OP'/OP = n'/n,$$

which proves the second corollary.

Finally, as will be seen from the figure,

$$SO \sin SON = SN = S'O \sin S'ON' = S'N'.$$

But

$$SO = PO \cos POS, \text{ and } S'O = P'O \cos P'OS',$$

consequently

$$n \sin SON \cos POS = n' \sin S'ON' \cos P'OS',$$

or

$$\sin SON/\sin S'ON' = n' \cos P'OS'/n \cos POS,$$

which proves the third corollary.

Adopting the notation for the angles shown in the figure, we have accordingly the following fundamental equations :

$$n \sin i = n' \sin i', \text{ or } \sin i/\sin i' = n'/n \quad \ldots \quad \ldots \quad \text{(i)}$$

also,

$$\cos \psi/\cos \psi' = n'/n \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

$$\sin \theta/\sin \theta' = n'/n \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

and

$$\sin \phi/\sin \phi' = n'/n\,(\cos \theta'/\cos \theta) = n' \cos \theta'/n \cos \theta \ldots \quad \text{(iv)}$$

G. Analytical Investigation of the Law of Refraction in Terms of Three Co-ordinates.

16. In addition to the above corollaries it is frequently desirable to have a more general expression of the law of refraction, more especially so when tracing through a system rays which are oblique to the axial plane ; such rays will be termed **skew rays.**

To obtain such an expression we assume σ, in Fig. 3, to be the pole of the incident ray on a unit sphere of reference, σ' that of the refracted ray produced backwards, N that of the normal at the point of incidence, and Z that of any straight line drawn through the point of incidence on the refracting surface.

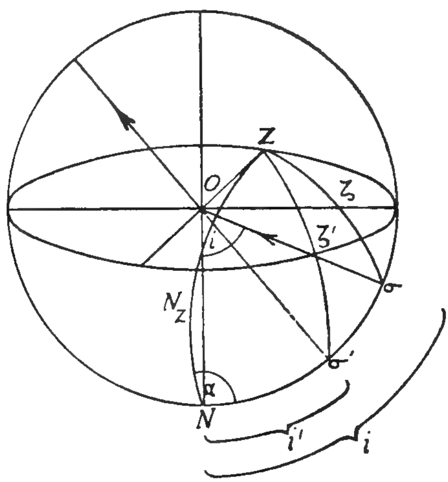

FIG. 3.

σ : Pole of the incident ray ; σ' : Pole of the refracted ray.
N : Pole of the normal of incidence with respect to the pole Z of a chosen straight line through the point of incidence (centre) of the unit sphere.

Since refraction occurs in one plane, according to our fundamental law, it follows that N, σ and σ' are situated on a great circle. Let ζ, ζ' and N_z denote the magnitudes of the angles comprised between σ, σ', N and the arbitrarily chosen axis Z.

We may now express the angle a contained between the plane of refraction and the plane NZ in two ways. For in the two spherical triangles $N\sigma\, Z$ and $N\,\sigma'\, Z$ we have the relations :

$$\cos\ a = \frac{\cos\zeta - \cos N_z \cos i}{\sin N_z \sin i} = \frac{\cos\zeta' - \cos N_z \cos i'}{\sin N_z \sin i'}\ . \qquad \text{(i)}$$

From this we derive the following relation by substituting the fundamental equation $\sin i\,/\sin i' = n'/n$ and by eliminating a :

$$n'\cos\zeta' - n\cos\zeta = \cos N_z\,(n'\cos i' - n\cos i)\,.\ \ldots \qquad \text{(ii)}$$

To abbreviate this expression let the direction cosines of the rays σ and σ' with respect to any three axes X, Y, Z be represented by the letters m, p, q and m', p', q' respectively, and those of the normal to the surface at the point of incidence by λ, μ, ν. Then, substituting J for $n'\cos i' - n\cos i$, we obtain finally the following equations :

$$\left.\begin{aligned} n'm' - nm &= \lambda\, J \\ n'p' - np &= \mu\, J \\ n'q' - nq &= \nu\, J \end{aligned}\right\}, \quad \ldots \quad \ldots \quad \text{(iii)}$$

where it should be noted that

$$\left.\begin{aligned} &J = n'\cos i' - n\cos i\,, \\ &J^2 = n'^2 + n^2 - 2\,nn'\,(mm' + pp' + qq')\,. \\ &\cos\,(i' - i) = mm' + pp' + qq'. \end{aligned}\right\} \qquad \text{(iv)}$$

H. Deviation of a Ray by Reflection and Refraction.

17. It is characteristic of refraction—in contrast to reflection—that the deviation of a ray by refraction becomes greater at an increasing rate as the angle of incidence increases, whilst in the case of reflection it is proportional to the angle of incidence.

In fact, from $n\sin i = n'\sin i'$ it follows with respect to infinitely small increments that $di/\tan i = di'/\tan i'$. If $n' > n$, then $i' < i$, and $\tan i' < \tan i$, and, owing to the identity of the ratios, $di' < di$. Now, the angle of deviation is $\gamma = i - i'$. Since $(di - di') > 0$ it follows that the deviation increases with the magnitude of i.

Further, we find from the above relations

$$\frac{d\gamma}{di} = \frac{di - di'}{di} = 1 - \frac{\tan i'}{\tan i}\,,$$

hence

$$\frac{d^2\gamma}{di^2} = -\frac{d}{di}\left(\frac{\tan i'}{\tan i}\right).$$

Now, $\tan i'/\tan i$ diminishes with i; for

$$\frac{d}{di}\left(\frac{\tan i'}{\tan i}\right) = \frac{d}{di}\left(\frac{\sin i' \cos i}{\sin i \cos i'}\right) = \frac{n}{n'} \cdot \frac{d}{di} \cdot \left(\frac{\cos i}{\cos i'}\right)$$
$$= \frac{\tan i' \,(n^2 - n'^2)}{n'^2 \cos^2 i'} < 0 \,.$$

Hence $\frac{d^2\gamma}{di^2}$ is always positive and $\frac{d\gamma}{di}$ increases as i (or i') increases, that is, the deviation caused by refraction increases at a more and more rapid rate.

If $n' < n$; then γ, $\frac{d\gamma}{di}$ and $\frac{d^2\gamma}{di^2}$ become less than 0. In this case the deviation is towards the other side of the ray away from the normal. In other respects the results are precisely the same as before.

3. GENERAL THEOREMS OF REFLECTION AND REFRACTION.

A. Principle of the Least Path.

18. When a ray of light, undergoing any number of reflections or refractions, travels from one point to another the sum of the products of the refractive index of each of the successive media into the length of the path within each medium, *i.e.* Σnr, has a limiting value, in that it differs from the analogous sum with respect to any other infinitely near or adjacent path by quantities of the second order at most, so that $\delta\Sigma nr = 0$. The product of the path and refractive index is called the **reduced path** or the **optical length** of the ray.

We shall first prove the theorem for a single refraction (which naturally includes reflection as a special case). Let x, y, z be the rectangular co-ordinates of the point of incidence, and let $x + \delta x$, $y + \delta y$, $z + \delta z$ be those of an infinitely near point. If this latter point is likewise situated on the refracting surface, then:

$$\lambda\delta x + \mu\delta y + \nu\delta z = 0\,, \quad \ldots \quad \ldots \quad (1)$$

where λ, μ, ν are the direction cosines of the normal to the surface at the point (x, y, z). By referring equations § 16 (iii) to the same system of co-ordinates, multiplying them respectively by δx, δy, δz, and adding, we obtain the following equation:

$$0 = n'm'\delta x + n'p'\delta y + n'q'\delta z - nm\delta x - np\delta y - nq\delta z \quad (2)$$

If now, on the incident ray, we take any point (X, Y, Z) at a distance r from (x, y, z) and on the refracted ray any point (X', Y', Z') at a distance r' from the surface, measured along the

ray, we shall have for the lengths of these distances the following equations :

$$r^2 = (X - x)^2 + (Y - y)^2 + (Z - z)^2 ,$$
$$r'^2 = (X' - x)^2 + (Y' - y)^2 + (Z' - z)^2 ,$$

whence we derive the equations

$$\left.\begin{aligned} \frac{\partial r}{\partial x} &= - \frac{(X - x)}{r} = - \cos (r, x) = - m \\ \frac{\partial r'}{\partial x} &= + \frac{(X' - x)}{r'} = + \cos (r', x) = + m' \end{aligned}\right\} \qquad (3)$$

and the analogous equations in terms of y and z.

Further,

$$\left.\begin{aligned} \delta r &= \frac{\partial r}{\partial x} \delta x + \frac{\partial r}{\partial y} \delta y + \frac{\partial r}{\partial z} \delta z \\ \delta r' &= \frac{\partial r'}{\partial x} \delta x + \frac{\partial r'}{\partial y} \delta y + \frac{\partial r'}{\partial z} \delta z \end{aligned}\right\} \ldots \quad \ldots (4)$$

Combining the two equations (4) with the factors n and n', and substituting the direction cosines from equation (3) for the partial differentials, we obtain the equation :

$$\begin{aligned} n\delta r + n'\delta r' = - nm\delta r - np\delta y - nq\delta z + n'm'\delta x \\ + n'p'\delta y + n'q'\delta z \end{aligned} \qquad (5)$$

From (2) it follows, however, that this is equal to zero.

Hence

$$\delta (nr + n'r') = \delta \Sigma nr = 0 \quad \ldots \quad \ldots \quad (6)$$

The principle of superimposed differentials at once enables us to apply the equation $\delta \Sigma nr = 0$, which we have obtained for a single reflection or refraction, to any number of reflections or refractions. Also, a continuous change of the refractive index is associated with the relation $\delta \int n dr = 0$.

(The second and higher differentials may be greater or less than zero, or they may be equal to zero. The usual statement of the theorem, according to which their sum is said to be always a minimum, is therefore incorrect and is due to a common error.)

In the case of plane surfaces the reduced path is always a minimum, as was formerly proved with respect to reflection by Hero of Alexandria, and with respect to refraction by Fermat (*I.*). We shall, therefore, for the sake of brevity, retain the same expression for the general case.

It is important to note that this theorem is capable of inversion ; that is to say, for the path of the light between two given points in

a given system of reflecting and refracting surfaces, the condition $\delta \Sigma nr = 0$ cannot be realised except by a process conforming to the laws of reflection and refraction.

Whether in any special case the path is a maximum or minimum, or neither, may be ascertained for any given form of the surface of separation by the following process of reasoning. In Fig. 4 let PQ be a portion of the surface of separation and let PA' be the refracted ray corresponding to the incident ray AP To ascertain whether APA' is a maximum or minimum path connecting A and A', let $nAP + n'PA' = constant$ be represented by a surface, the so-called **Cartesian surface**, of which we will suppose PR to be a portion (*see* § 23). This surface will

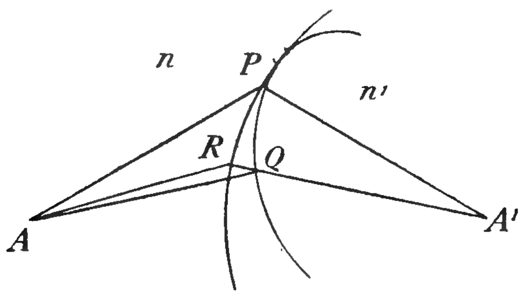

FIG. 4.

The optical path APA' is a minimum or maximum according as the refracting surface PQ at P is less convex or more convex towards the less refracting medium than the Cartesian surface PR.

necessarily touch the refracting surface at P, since at that point $\delta(nAP + n'PA') = 0$, for both surfaces. If now at P the refracting surface is more convex towards the less refracting medium (n) than the Cartesian surface, in that case the reduced path with respect to the refracting surface will be a maximum, whilst in the reverse case it will be a minimum.

In fact, let Q be a point on the refracting surface infinitely near P. Then the optical path $[Q]$ from A to A' through Q, is $[Q] = nAQ + n'QA'$. The reduced path $[R]$ through R on the Cartesian surface, is $[R] = nAR + n'RQ + n'QA'$, where R is the point of intersection of $A'Q$ and the Cartesian surface. Hence,

$$[Q] - [R] = n\,(AQ - AR) - n'RQ.$$

Now, since AQ, AR are the sides of a triangle, it follows that $AQ - AR < RQ$. Therefore, *a fortiori*, when $n < n'$, $n\,(AQ - AR) < n'RQ$, so that the path through Q is shorter than that through R. The latter, however, is equal to that through P, hence, under these conditions, the path through P is a maximum. By analogous reasoning it will readily be seen that when the

refracting surface is less convex towards the less refracting medium than the Cartesian surface the optical path through P is a minimum.

B. The Principle of Least Time.

19. From the experiments of Foucault and from reasoning based upon the undulatory theory of light it follows that the refractive indices of two different media are inversely proportional to the velocities with which light traverses them, so that $n/n' = v'/v$, or, more generally stated, $n = k/v$, $n' = k/v'$, $n'' = k/v''$, &c. Introducing this notation in the equation of the optical path, $\delta\,\Sigma nr = 0$, we may write instead $\delta\,\Sigma\,(r/v) = 0$. But since the velocity is $v = r/t$, where t is the time taken by the light in traversing the distance r, we shall have finally $\delta\Sigma t = 0$ as an expression for the time which the light takes to travel from a given point A through any series of reflections and refractions to another point B, and it will be seen that when a ray proceeds along a path which conforms to the laws of reflection and refraction the time which the ray takes to travel from point to point differs from that which would be required if the ray travelled along any infinitely near path by infinitesimal quantities of the second and higher orders only.

C. The Theorem of Malus.

20. The previous deduction enables us to prove an important theorem enunciated by Malus (*1.*) This theorem is as follows: **A system of rays which at any stage of its existence is normal to a surface remains normal to a surface after any number of reflections or refractions.** Since rays proceeding from a luminous point are normal to any sphere of which this point is the centre, the theorem of Malus is directly applicable to this as a particular case. From the undulatory theory this is self-evident, since, according to this theory, in an isotropic medium rays are nothing more or less than normals to the wave-front. From the point of view of geometrical optics the theorem may be proved by the following argument, which is due to Rayleigh (*1.*) In Fig. 5 let $MABCP$, $M'A'B'C'P'$ be rays which are normal to the surface m at M, M' respectively, and in their further progress let them experience reflections or refractions at points A, A', B, B', C, C' in the surfaces a, b, c respectively. After any one of these successive reflections and refractions, say after those occurring at surface c, we shall be able, in any case, to determine points P, P' so situated on the respective rays that the sum of the reduced paths from M to P, M' to P', &c. may be identical in each case. Then the surface containing the points P, P' is the required orthogonal surface of the rays. To prove the truth of this, join $M'A$ and CP'. Then, if M be sufficiently near to M', $[M'ABCP']$ differs from $[M'A'B'C'P']$ only by an infinitely small quantity of a higher order than MM'.

By supposition, however, $[M'A'B'C'P'] = [MABCP]$. Therefore, subtracting the paths common to both rays and denoting the refractive indices of the first and last medium by n and n' respectively, it follows that ultimately $n \,.\, MA + n' \,.\, CP = n \,.\, M'A + n' \,.\, CP'$. Since the rays are supposed to be originally normal to m, it

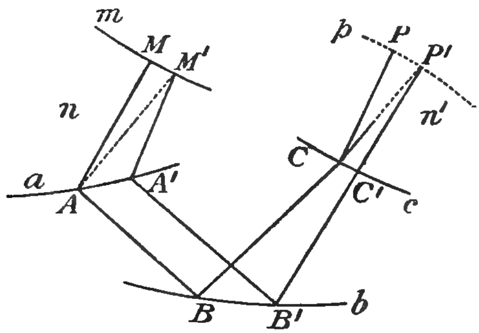

FIG. 5.

Theorem of Malus: A system of rays which at any stage of its existence is normal to a surface remains normal to a surface after any number of reflections and refractions.

follows that the limit of $M'A$ is MA within an infinitesimal amount of the second order at least. The same applies to the limit of CP', which is CP. Hence it will be seen that CP is normal to p at P, and, similarly the other rays $C'P'$, &c.

D. The Characteristic Function.

21. As has been shown above, the reduced path between two points separated by any reflecting or refracting media may be defined as the distance which light would traverse in vacuo in the same time which it actually requires to travel from one given point to another through the given media. If we express this path as a function of the co-ordinates of both points we shall obtain what is known as the **characteristic function,** a conception which was first introduced into mathematical optics by Hamilton (*1.*) in 1824.

In a vacuum the reduced path between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is equal to the distance between them, which is

$$V_{12} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\,.$$

Assuming the point (x_1, y_1, z_1) to be variable in position, we derive from this equation by differentiation

$$\frac{\partial V_{12}}{\partial x_1} = -\frac{x_2 - x_1}{V_{12}} = -\cos(r_{12}, x) = -m_1$$

and analogous equations for the partial differentials with respect to the other co-ordinates. These partial differentials furnish, accordingly, the direction-cosines of the ray corresponding to the point to which they refer. This formula can readily be extended to the general case of any number of refractions and reflections. The characteristic function then assumes the form

$$V_{12} = \int_{x_1 y_1 z_1}^{x_2 y_2 z_2} n_k \, dr_k$$

where n_k is the refractive index at the point (x_k, y_k, z_k) and dr_k a linear element of the optical path between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) in the neighbourhood of the point (x_k, y_k, z_k), noting that the integral is to be regarded as a function of its limits. By subjecting the integral to variation with respect to its upper limit and suppressing the suffixes, we find

$$\delta V = n\delta v = n\,(m\delta x + p\delta y + q\delta z)\,,$$

whence it follows that

$$\frac{\partial V}{\partial x} = nm\,, \quad \frac{\partial V}{\partial y} = np\,, \quad \frac{\partial V}{\partial z} = nq\,,$$

and, further, that the characteristic function V must satisfy the partial differential equation

$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 = n^2.$$

When the characteristic function is known for any optical system all the properties of that system may be deduced theoretically from it. These theoretical deductions, some of which had already been made by Hamilton (*1. . . 4.*), Kummer (*1.*), Maxwell (*3.*), Thiesen (*1.*), Bruns (*1.*), involve, however, the use of an elaborate analytical equipment, particular use being made of the theory of curved surfaces and that of pencils. Whilst it will thus be seen that the efforts of these few investigators have already gone far to erect upon the theory of these general systems an intricate analytical structure, which is only accessible to those endowed with very extensive mathematical resources, the difficulties to which these theories lead become insuperable as we proceed to their practical application. Hitherto the practical embodiment of the characteristic function has proved realisable in the very simplest cases only. Unfortunately, cases of this kind either have no practical significance or have already been solved for practical purposes by simple specialised methods. Scarcely less are the difficulties which are encountered if one proceeds to secure practical working formulae by applying Fermat's principle as a means of eliminating the intermediate variables associated with the points where the rays intersect the successive surfaces (see Bruns (*1, 403.*). Moreover, the extremely general nature of such analytical investigations does not at all provide the practical optician with that which interests him most, viz., standard forms for the computation of optical combinations.

E. The Optical Length between Conjugate Foci.

22. Fermat's principle in conjunction with the theorem of Malus furnishes the following important conclusion :—

In any system of rays contained within a given solid angle in which all component rays issuing from a given point are made to meet homocentrically in another point, in consequence of any number of reflections and refractions, the reduced path from the point of issue to the point of confluence is the same for all the rays. Since for any adjacent rays $\delta \Sigma\ nr=0$, it follows that $\Sigma\ nr=constant$ within the given solid angle. Rays, or elementary waves, which issue in similar phase from a given point or surface element are in the same phase when they meet homocentrically in the new focal point. The truth of this conclusion is not prejudiced by disturbance of phase, such as may be occasioned by reflection or refraction; for these affect all rays equally, assuming, of course, that they do not depend upon the angle of incidence. From the undulatory theory it follows moreover, that at these foci the rays have a mutually intensifying effect, creating thus within the given solid angle a new centre of disturbance, which in its turn is endowed with all the qualities of a luminous point. This, indeed, is the phenomenon which makes possible the formation of images by optical devices.

F. Cartesian Surfaces.*

23. From the preceding investigation of the principle of the reduced path, it will be seen that a reflecting or refracting surface which causes all rays proceeding from a luminous point to re-converge to a single point should be such that $nr + n'r'$ shall have the same value with respect to all points of that surface, r being measured from the luminous point and r' from the corresponding point of confluence. The solution of this problem does not present any particular difficulties and its treatment has already repeatedly been given elsewhere. Since, on the other hand, it does not present any special points of physical or practical interest it will be sufficient to summarise a few results and to point out that infinitesimals of the second and higher orders of the expression Σnr for the reduced path vanish in these cases, as indeed, in all cases where we are dealing with homocentric pencils of rays.

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* By many writers we find these surfaces frequently referred to as *aplanatic surfaces*, or surfaces *free from aberration*. Since, however, we shall have occasion to use the terms *aplanatic* and *free from aberration* in a more strictly defined sense, in which the Cartesian surfaces are not necessarily aplanatic or free from aberration, we shall in this investigation ascribe the term *Cartesian* to those surfaces of the fourth degree which cause one point to be the homocentric image of another.

In the case of a single reflection the expression $r \pm r' = constant$ represents the bi-polar equation of an ellipsoid or hyperboloid of revolution whose foci correspond to the positions of the luminous point and its conjugate focus. This follows geometrically from the properties of the ellipsoid and hyperboloid of revolution, in that their radii vectores are similarly inclined to the normal at the surface point under consideration, *i.e.* are in conformity with the law of reflection. If one of the points recedes to infinity the surface becomes a paraboloid of revolution. According as the reflecting surface is the concave side of a paraboloid or of an ellipsoid of revolution, or the convex side of a paraboloid or of a hyperboloid of revolution, the focus of the incident rays after reflection is real or virtual.

In the case of a single refraction the application of the equation $nr + n'r' = constant$ produces the form of a surface of revolution of the fourth degree, whose meridian curves in the case of real foci are **Cartesian ovals**, properly so-called. In the special case when one of the points, for instance the luminous point, is situated at infinity, the surface becomes a surface of revolution of the second degree, one of its foci being likewise the focus of the rays. The refracting surface is then a hyperboloid of revolution when the point at infinity is situated in the optically denser medium, and it is an ellipsoid of revolution when the point lies in the less refracting medium. The numerical eccentricity of the surfaces then becomes equal to the ratio of the refractive indices of the media separated by these surfaces.

Any given Cartesian surface can be made to produce two simultaneous images at P' and P'' of a given point P situated on its axis in respect of two particular values of the relative indices of refraction. Both image points are either real or conjugately imaginary. The latter case, as a physical conception, corresponds to a particular species of what is known as **confusion of rays.**

In another special position of the two points relatively to the surface, with which we shall deal later on, the surface is a sphere.

Quite analogous conditions result if, in place of a single surface, several surfaces are employed to bring the rays again to a focus.

Theoretically, the Cartesian surfaces may be employed to determine for a given position of two points and for a surface of given form separating two media, those points of the latter traversed by the ray in passing from the one given point to the other as the result of refraction or reflection. According as the two points lie in the same or in different media, *i.e.*, on the same side or on different sides of the surface, this may be done by describing about them the respective families of surfaces conforming to the conditions $r + r' = constant$ or $nr + n'r' = constant$. All points of the

surface of separation touched by any one of these Cartesian surfaces have the required property, in that at these points $\delta \Sigma nr = 0$.

Of all the Cartesian surfaces it is only the paraboloids of revolution which have received any practical application as reflectors for searchlights. In practice, the accuracy with which paraboloidal surfaces can be made has not reached anything like that degree of precision which is demanded, and attained, in the case of spherical surfaces. Actually, however, the requirements as regards the accuracy of parabolic mirrors for searchlights are generally by no means great.

G. Non-spherical or Figured Surfaces.

24. Apart from spherical and Cartesian surfaces we have others of a more general kind, comprised under the collective term of non-spherical or figured surfaces, which possess certain special properties of particular value in the realisation of specific optical effects.

Two roads are open to us for their theoretical investigation. We may attempt to ascertain by analysis the law of the surfaces whereby the required optical result may be attained. An analytical investigation of this kind involves, however, such great difficulties that nothing has been hitherto accomplished in this way. According to the other method, initiated by Abbe (*8. 9.*), we may start from a given optical combination of spherical surfaces and then proceed to calculate by expansion into a power series the amount of deviation from the spherical form of one or more of the surfaces which will produce the required modification in the disposition of the homocentric pencils. If we confine ourselves to surfaces of revolution we may write the equation to the meridian curve of such a surface in terms of polar co-ordinates; thus

$$r = f(\phi) = r_0 + \sigma ,$$

r_0 being the radius of curvature at the vertex and σ the radial deviation from the osculating sphere at the vertex. By the expansion of σ in terms of powers of the arc $l = r_0 \phi$ we may write

$$\sigma = \kappa l^4 + \lambda l^6 + \nu l^8 + \quad . \; . \; . \; .$$

These powers are all of an even degree, since the meridian curve is symmetrical with respect to the axis of revolution, and there cannot be any powers of less than the fourth order since the radius of curvature r_0 at the vertex approximates to the curve to within the second order. The **coefficients of non-sphericity** κ, λ, ν may be computed from the required deviations in the positions of the foci from those obtained in the spherical combination.

The systematic application of theoretically determined non-spherical surfaces has only quite recently been initiated at the Optical Works of Carl Zeiss at Jena.

In the past, intentional deviations from spherical contours have only been attempted in the production of astronomical reflecting and refracting telescopes as well as large photographic lenses. In these cases the deviations from the strictly spherical form are exceedingly small; and, moreover, it is to be noted that these refinements are practically applied by a process of pure trial and error, known as the method of "figuring" or "local retouching."

H. General Theory of Optical Rays. Caustics.

25. In our introductory remarks we were careful to emphasize that the essence of an optical ray is that it shall ultimately be traceable to a self-luminous point, or centre of oscillation. It follows, then, from the theory of Malus that such a pencil has always a system of orthogonal surfaces, which are no other than wave-fronts in the same phase of vibration. "Any optical pencil, whatever its genesis, has accordingly the properties of the normals to continuously curved surfaces. Now, from the theory of these surfaces we can make the following deductions:—If we consider a plane containing any ray a and intersecting the given surface along a curve, and if we suppose this plane to rotate about the ray, the curve of intersection will generally have different curvatures at the points of intersection with the ray a, and the plane of greatest curvature of the curve of intersection will be at right angles to the plane of its smallest curvature. Of all the normals to the wave-front which are infinitely near to the ray a, and which are accordingly *adjacent* rays, those meeting the wave-fronts on the lines of greatest or least curvature intersect the ray a at the centre of the greatest or smallest circle of curvature respectively, whereas others do not meet the ray a at all. On every ray there are accordingly in general two foci where it is intersected by adjacent rays. These points correspond to the centres of the greatest and least curvatures at the point where the ray meets the wave-front. When the curvature of the wave-front at the foot of the ray is the same in all directions, in this case only does the ray meet all infinitely near rays at a single point." (Helmholtz).

The rays whose feet are situated in a line of curvature of the wave-front, when that line is finite, intersect successively at various points. They are, therefore, the envelope of a curve consisting exclusively of foci of infinitely near rays.

This line is accordingly called a **focal curve** or **caustic**. The successive lines of curvature of a wave-front yield in the aggregate a surface, the so-called **caustic surface** of the system of rays. Since there are two families of lines of curvature it follows that,

generally speaking, there will be two focal surfaces to every orthogonal surface, and every ray is a common tangent to both surfaces.

The case is particularly simple when the wave-front is a surface of revolution. In this case one of the families of lines of curvature is made up of the meridian curves of the surface, and the corresponding caustic is a surface which is generated by the rotation of the evolute of the meridian curve about the axis of symmetry. The other family of lines of curvature corresponds to the circles of latitude of the globe, being, accordingly, parallel circles having their centres on the axis of symmetry. The corresponding normals form a set of right circular cones having their vertices each at a point on the axis. The second caustic surface reduces accordingly to a portion of the axis. Fig. 6 shows a meridian section of such a caustic surface. In this figure RQS is the generating curve of one of the caustic surfaces, whilst it will be seen that the other caustic surface reduces in this case to a straight line PQ.

We shall not here occupy ourselves with the forms of the caustics in special cases, since whatever interest attaches to them is mathematical rather than physical or practical.

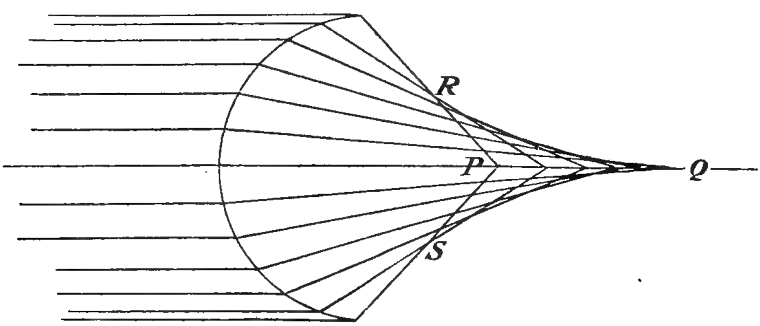

FIG. 6.

Refraction of parallel rays at a sphere of refractive index 1.5. RQS is the generating curve of one of the focal surfaces, whilst PQ is the straight line into which the other focal surface degenerates.

It should be noted that investigations relating to the intensity which obtains at different points of a caustic and especially those dealing with its distribution at planes intersecting the caustic are of very restricted value, and in most cases even quite fictitious, if treated purely in terms of geometrical optics. The foci formed by caustics, it should be remembered, are those of infinitely thin pencils, and, considered by themselves, these cannot be said to obey the rules of geometrical optics in view of the pronounced effects of diffraction which are inseparable from such an extremely narrow pencil of rays.

In adjacent foci the light arrives in continuously varying phases, and the intensity at points of the caustic itself as well as at points in an intersecting plane, depends very largely upon the phase of the wave elements meeting them. Whilst, therefore, a caustic, ascertained in terms of the geometrical laws of reflection and refraction, does exhibit, more or less correctly, the places of greatest brightness, in so far as it is made up of real foci, it requires the aid of the undulatory theory, and the principle of interference in particular, to investigate more fully the distribution of light within a pencil. Investigations conducted on these lines, notably by Airy (*4.*), with special reference to the rainbow, account, moreover, for the less intense repetitions of the caustic in its vicinity (the so-called arcs of higher orders), for which geometrical optics offers no explanation whatever.

I. General Theory of the Constitution of Infinitely Thin Optical Pencils.

26. It has already been stated that, in general, a ray of any pencil meets only those infinitely adjacent rays which are contained in two definite planes, viz. in the planes of greatest and least curvature of its orthogonal surfaces; in other words, those rays only whose feet constitute the elements of the circles of curvature themselves. We shall now endeavour to obtain a more concrete notion of the relative position and direction of the rays constituting an infinitely thin pencil. To this end let P be a point on the wave-

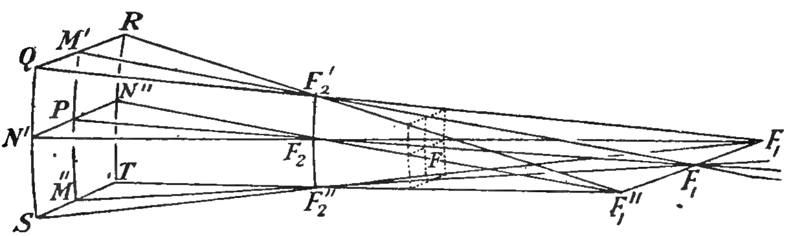

FIG. 7.

Elementary astigmatic pencil with two focal lines F_2', F_2''; F_1', F_1'' at right angles to each other and to the principal ray PF_2FF_1.

front and let $M'M''$ and $N'N''$, intersecting at right angles at P, be respectively the arcs of greatest and least curvature. Let F_1, F_2 be the centres of these arcs, and hence also the focal points of the ray which passes through P, and, for the sake of brevity, let them be called the **first** and **second focus** respectively. We shall call the ray which passes through P the **principal ray** of the entire pencil. It should be noted that it is also the principal ray of the plane partial pencils $M'F_1M''$ and $N'F_2N''$. The arcs of greatest and least curvature at a point adjacent to P, say M', are likewise at right angles to each other. Owing to the smallness of the

elements of the wave-front which we are considering, in other words, the smallness of the arc $M'P$, the arc $N'PN''$ is parallel to the corresponding arc QR of the second principal curvature at M', if we neglect infinitesimals of higher orders, whilst the arc of the first principal curvature which passes through M' coincides with $M'M''$ to a similar degree of approximation. Clearly the focus of the principal pencil $QM'R$ will be situated on its principal ray, *i.e.*, on the ray which passes through M'. The latter, being a ray of the pencil $M'M''$, passes through F_1. From this it follows that the plane pencil which we are considering is wholly contained within the plane QRF_1; and, similarly, the rays which pass through other arcs at right angles to $M'M''$ are wholly contained within the planes which pass through F_1 and these arcs. Whatever the positions of the foci F_2', F_2'', etc., since all these planes of rays contain F_1 and are at right angles to $F_1M'M''$, it follows that they intersect on the line $F_1'F_1''$ which passes through F_1 at right angles to the plane $F_1M'M''$ and therefore also to the principal ray PF_1. We shall call this line $F_1'F_1''$ the **first focal line.**

The same reasoning applies to those arcs of the first principal curvatures QS, RT which are parallel to $M'M''$. The whole of the principal rays of these plane partial pencils, being rays of the partial pencil $N'F_2N''$, pass through its focus F_2. All these plane partial pencils are, moreover, at right angles to the plane of the second principal curvature $F_2N'N''$. They intersect accordingly on a line $F_2'F_2''$ which is at right angles to $F_2N'N''$ and therefore also to the principal ray PF_2. We shall call this line $F_2'\ F_2''$ the **second focal line.** If now we consider jointly these two systems of plane partial pencils it is clear that the second focal line $F_2'F_2''$ is nothing more or less than the aggregate of the foci of the partial pencils which we considered first, and, similarly, that the first focal line $F_1'F_1''$ is the aggregate of the foci of the second set of partial pencils.

These considerations may be summarised as follows:—**All the rays constituting an infinitely thin optical pencil intersect in two small straight lines. These lines—the focal lines—are at right angles to the principal ray at the two foci and lie in two mutually perpendicular planes—the principal planes of curvature to the given wave-front.** The focal line which passes through the first focus contains the foci of the plane partial pencils which are parallel to the first principal plane curvature, *i.e.*, of those of the first kind; and similarly, with respect to the second focus. This is known as **Sturm's Theorem** (*I. 2.*).

It will thus be seen that the entire pencil may be constructed from the two focal lines and the principal ray. All that is required is to join every point of one focal line to every point of the other, confining ourselves in so doing to operating throughout with rays which are infinitely near to the principal ray.

Moreover, the constitution of the pencil is determined by any four of its rays, not more than two of which should pass through any one point of either focal line.

Planes at right angles to the principal ray intersect the pencil in contours of various forms, according to the place of the intersection and also according to the contour of the pencil at the refracting surface, or wave-front. At a certain readily determinable place F, situated between the two foci F_1 and F_2, the cross section of the pencil is similar to its original boundary and is called the **section of least confusion.**

In the preceding investigation it has been assumed, as an essential condition, that like principal planes of curvature at adjacent points of an infinitely small element of the surface will be very approximately parallel to one another. Strictly speaking, this is not the case; for, in general, there are discrepancies from the second order downwards. The relations which we have established on the assumption here made hold accordingly only within a similar degree of approximation. A more rigorous procedure will enable us to determine for each elementary wave-front a surface element which attains a closer degree of tangency than the surface at the vertex of an elliptic paraboloid, as was implied before; and in a more rigorous analysis it would, therefore, become necessary to proceed from the conditions postulated by such a higher degree of osculation. The plane partial pencils, say those of the first principal curvature, will then cut the plane of the second principal curvature corresponding to the principal ray successively along different lines, such that the angles comprised between these lines and the principal ray assume all kinds of magnitudes; and the same applies, in general, to the partial pencil of the second principal curvature with respect to the plane of the first curvature. If we consider small quantities of the second order it will be seen that the two foci of the principal ray are not traversed by focal lines but rather by focal surface elements. Under special conditions the case may arise that these focal surface elements degenerate into lines, which frequently are far from normal to the principal ray. An instance of this is furnished by the oblique refraction of a homocentric pencil at a surface of revolution. When, in consequence of the special geometrical conditions, a pencil of finite dimensions is essentially similar in its constitution to an infinitely thin pencil—which may easily occur—we shall be able to demonstrate the deviation from the typical case which we have just considered. It will be readily appreciated that these qualifications do not invalidate Sturm's theorem as such, since the latter does not claim to be applicable beyond the limits of accuracy clearly indicated by its propounder.

In general, it should be remembered that the intersection of rays at foci or focal lines does not exceed the first order of approximation.

Manifestly, theorems relating to the general properties of optical pencils of rays cannot include in their statement relations involving infinitesimals of the second order; and, conversely, theorems which take cognisance of such infinitesimals cannot embrace such a general statement as that furnished by Sturm's theorem.

Even in those cases where more rigorous analysis shows that the pencils give rise to focal lines which are inclined at an acute

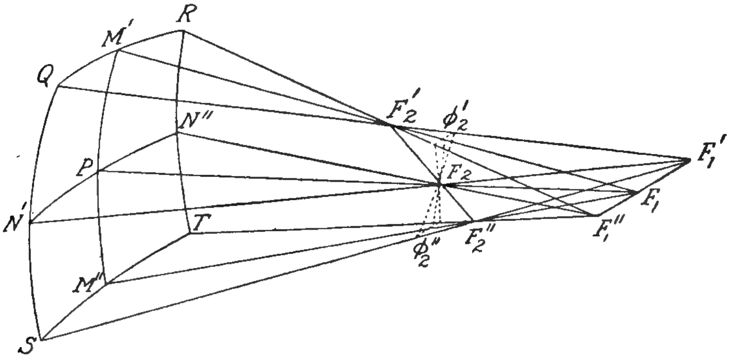

FIG. 8.

Elementary astigmatic pencil with focal lines inclined at an acute angle to the principal ray.

angle to the principal ray, neglecting infinitesimals of a higher order than the second, a simple consideration will show that a plane through one of the foci at right angles to the principal ray intercepts nowhere a pencil of a smaller cross section than one of the second order as compared with the width of the portion of the wave-front in question, which shows that in this case Sturm's theorem is still true. Fig. 8 illustrates a case in which the conditions are somewhat extreme. Let a perpendicular section through F_2 have its greatest cross intercept where it intersects the outermost plane partial pencils QRF_2' and STF_2'' (at a place situated between F_2 and Φ_2', assuming the elementary portion of the wave-front to have a circular boundary). Now, the length of the intercept is q, and if we make $F_2'\,\Phi_2' = dr$, *i.e.*, equal to the difference of the principal radii of curvature of the second kind at M' and P, and $N'N'' = ds$, $N'N''$ being the mean length of any of the principal arcs of curvature of the second kind, we obtain the following expression for the length of the greatest cross section of the pencil, *viz.*, $q = \frac{ds \,.\, dr}{r}$, from which it will be seen that, in general, it is infinitely small with respect to ds.

The infinitely thin pencil of rays which we have here considered is a particular species of the general rectilinear pencil, which no longer has the property of a system of orthogonal surfaces. In

nature, an instance of this type of pencil is furnished by the irregular refraction of rays passing through doubly refracting media. The investigation of the properties of this general pencil naturally serves to elucidate the case of the special pencil in that it supplies a criterion as to which of its properties arise from the simple principle of the rectilinear propagation of light and as to which should be attributed to special conditions. The general properties of pencils have been investigated extensively by Hamilton (*1.* to *4.*) and subsequently by Kummer (*1.*), Moebius (*5.*), Meibauer (*1.*), and others.

From the preceding deductions we may thus define the purpose of the dioptric investigation of infinitely thin pencils in its most general aspect: Having given the refractive indices n and n' of two media, the form of the surface $f(x, y, z)$ separating them, as well as the direction of an incident ray and the position of the foci and focal lines on this ray; to determine the last three particulars with respect to the corresponding refracted or reflected ray, as the case may be. Neumann (*2.*) has given a general solution of this problem, proceeding from the assumed truth of Sturm's theorem, whilst Matthiessen (*10.*) does so without reference to this theorem. We have here only briefly referred to these investigations, deferring the discussion of the problem itself until we come to deal with a number of particular cases.

4. DEFINITIONS.

27. Before proceeding further with our investigations we shall give a few definitions, so as to obviate any ambiguity of terms.

We shall suppose the light which issues from any surface, no matter whether self-luminous or indirectly luminous, to be made up of components proceeding separately from each element of the surface. The light emanating from such a surface element is said to form a **physical pencil of light.**

The physical pencil of smallest dimensions, as regards its angular aperture and cross section, which in practice can be realised by detachment from the neighbouring light and which when isolated can be made to undergo further changes, is called a **physical ray** (Newton).

In purely geometrical optics we start from the assumption that light emanates from the individual points of the luminous surface. Such a point is called a **focus** of the pencil of light or briefly the **luminous point.** When the solid angle and the cross section of the pencil are infinitely small the pencil is called an **elementary pencil** (in the absence of any supposition to the contrary, we assume the form of the elementary pencil to be that of a right circular cone or cylinder). A pencil may thus be supposed to be an aggregate of rays of light, the latter being treated as if they were mathematical straight lines.

The **optical axis** or **principal ray** of a finite pencil, and frequently also of infinitely narrow pencils, is that ray which coincides with the centre line of mass of the pencil, if we imagine the latter to be a homogeneous body, so that in the case of conical and cylindrical pencils it is the geometrical axis, or axis of symmetry, of the cone or cylinder. The axis determines accordingly the direction of a pencil.

Every ray of an elementary pencil may therefore be regarded as the axis.

A pencil is called **divergent** when viewed in the direction of the propagation of light from a point behind its focus, and **convergent** when viewed in the same direction from a point in front of its focus. The degree of convergence or divergence of pencils is measured by their **angular aperture**; in the case of elementary pencils by the ratio of their plane or solid angles of aperture.

The space within which the light travels, whether occupied by ponderable matter or not, is called the optical **medium.** The focus of a pencil is always considered to be situated in that medium through which the pencil actually passes.

Whenever rays which have issued from a luminous point are re-united by any device, so as to meet at a point, the latter is called the **optical image** of the former, which in its turn is known as the **object-point.** An optical device by which this is accomplished is referred to as an **image-forming system.** Rays thus united by this means may constitute a finite or an infinitely thin (or elementary) pencil, or merely a plane partial pencil.

One difference between a self-luminous object and an optical image is that the former is visible from all directions, whilst, in general, the latter can only be seen within restricted boundaries.

By reason of the principle of the reversibility of the optical path, the object-point and the corresponding image-point may be made to interchange their functions: in other words, any image-point regarded as an object-point from which rays proceed backwards in the direction in which they originally converged upon it, has its image formed at exactly that point which originally was the object-point. Instead of saying that a point is the image of another point we may refer to both jointly as **conjugate points** with respect to the transforming media.

What holds good for a single point applies equally to several points which go to form a more or less extensive object or image.

In an image-forming system an object is **real** or **virtual** according as its position, measured in the direction of the motion of light, precedes or succeeds the system, whilst the converse applies to the image.

CHAPTER II.

COMPUTATION OF RAYS THROUGH A SYSTEM OF REFRACTING SURFACES.

(A. KOENIG and M. VON ROHR.)

1. TRACING OF RAYS PASSING THROUGH A POINT ON THE AXIS.

A. Rays Inclined at Finite Angles to the Axis.

28. Formulæ referred to the Axis of the System—Let it be required to trace the path of a ray BO (Fig. 9) meeting at O the axis SC at an angle u in its passage through a spherical surface SB of radius $r = SC$ separating two media having

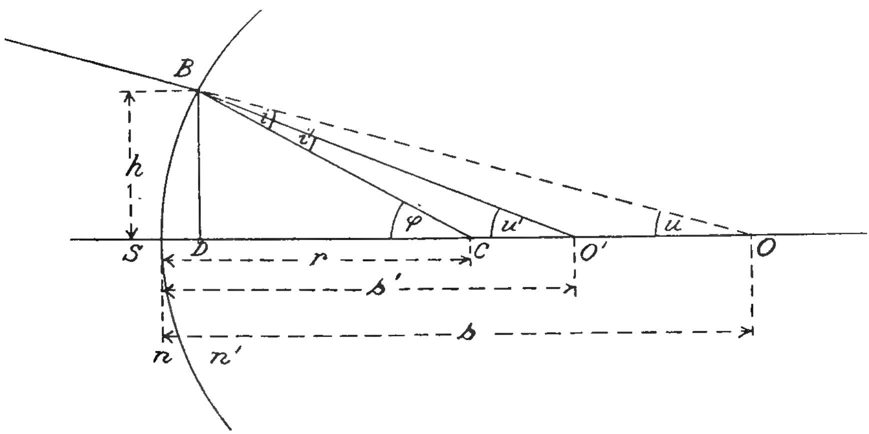

FIG. 9.

$SO =\quad$; $SO' = s'$; $SC = r$; $DB = h$.

Refraction through a spherical surface of a ray of finite aperture meeting the axis at O.

refractive indices n, n' respectively. Let the position and the direction of the ray be defined in terms of the two quantities $SO = s$ and u, s being in our case the chosen **intercept** on the axis measured from the vertex and u the angular aperture. We shall

adopt the following convention of signs:—**All such distances as SO will be reckoned positive if S and O are ranged in the order of the direction of propagation of the light (which we shall always suppose to proceed from left to right); and they will be considered negative if S and O are ranged in the opposite sense. The angles will be reckoned positive if rays situated above the axis converge, and negative if they diverge.***

From the triangle COB we derive the following sine relation:

$$\frac{\sin i}{\sin u} = \frac{CO}{BC} = \frac{CS + SO}{BC} = \frac{s - r}{r}, \quad \ldots \quad \text{(i)}$$

from which it will be seen that the angle of incidence i has the same sign as the angular aperture u, when, as in the present case, $s - r$ and r have the same sign.

It will be seen that this convention respecting the signs of $\sin i$ conforms to that given in § 9.

By the law of refraction we now find

$$\sin i' = \frac{n}{n'} \sin i, \quad \ldots \quad \ldots \quad \text{(ii)}$$

where the signs of i and i' are always in agreement.

From the figure it will be seen that the angle comprised between the axis and the normal is

$$\phi = u + i = u' + i', \quad \ldots \quad \ldots \quad \text{(iii)}$$

whence

$$u' = u + (i - i'). \quad \ldots \quad \ldots \quad \text{(iv)}$$

Similarly, we find with respect to the ray after refraction

$$s' - r = \frac{r \sin i'}{\sin u'}, \quad \ldots \quad \ldots \quad \text{(v)}$$

whence the conjugate distance s' may at once be found by the addition of r.

This formula cannot be used when r becomes infinitely great, which is the case when a spherical surface having its centre on the axis is replaced by a plane surface at right angles to the axis.

* This convention of signs for the angle u agrees on the whole with that adopted by Steinheil and Voit (*3. 41.*) and Kerber (*7. 5.*). It differs, however, from that proposed by Wanach (*1. 164.*).

In this case, as represented in Fig. 10, the relation becomes

$$s' \tan u' = s \tan u \qquad \dots \qquad \dots \qquad \text{(vi)}$$

Since in this case $\phi = 0$, we have the following relations,

$$i = -u\,,$$

$$\sin i' = \frac{n}{n'} \sin i\,,$$

$$u' = -i'\,.$$

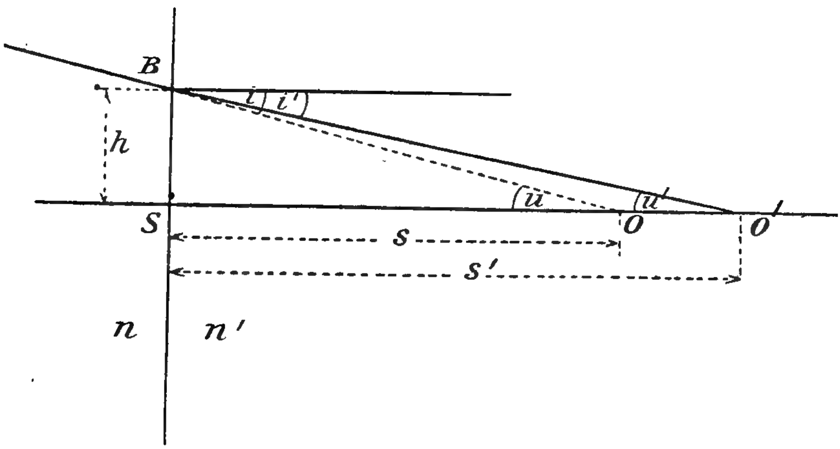

FIG. 10.

$SO = s$; $SO' = s'$.

Refraction at a plane surface of a ray of finite aperture and cutting the axis at O.

When the radius of the sphere, though finite, is very great, in which case we shall write R for r, it is not advisable to calculate the intercept s in the usual way from the value of $(s' - R) + R$, as its trigonometrical evaluation under these conditions is liable to lead to excessive inaccuracy. It is then better to proceed as in the case of a plane surface by putting

$$s' = \frac{h}{\tan u'} + 2\,R \sin^2 \frac{\phi}{2} = R \sin \phi \cot u' + 2\,R \sin^2 \frac{\phi}{2}\ ;$$

or we may, after the manner suggested by Abbe, transform the relation $\sin i' = \dfrac{s' - r}{r} \sin u'$ into the expression

$$\frac{s'}{r} \sin u' = \sin u' + \sin i' = 2 \sin \frac{\phi}{2} \cos \left(\frac{i' - u'}{2}\right),$$

whence

$$s' = \frac{2\,R \sin \frac{\phi}{2} \cos \left(\frac{i' - u'}{2}\right)}{\sin u'} \quad \ldots \quad \ldots \quad \text{(vii)}$$

This formula for radii of any magnitude, may be used as a check upon results obtained by other means ; an identical expression may also be obtained by the transformation of the other formula.

When the distance of the object-point from the vertex is infinite, a plane surface at right angles to the axis has no effect upon the direction of rays which meet the surface in a direction parallel to the axis. In the case of rays parallel to the axis and meeting the refracting spherical surface at a distance h from the axis, so that $u = 0$,

$$\sin i = \frac{h}{r}.$$

The transition from any refracting surface of radius r_v to the next surface of radius r_{v+1}, separated by an axial distance, or **thickness** d_v, will give us the transformed co-ordinates of the ray, *viz.*:

$$s_{v+1} = s_v' - d_v \,; \text{ and } u_{v+1} = u_v'. \quad \ldots \quad \text{(viii)}$$

In this way we may trace a ray inclined at an angle u to the axis (or at a distance h from the axis when the ray is parallel to the latter) through a system of centred spherical surfaces. We have now to find expressions for other values, which may be important in the computation of optical combinations.

The **incidence height** $h = D\,B$.

$$h = r \sin \phi\,, \quad \ldots \quad \ldots \quad \ldots \quad \text{(ix)}$$

where

$$\phi = i + u = i' + u'.$$

The **versed sine** SD is

$$SD = SC - DC = r\,(1 - \cos \phi) = 2\,r \sin^2 \frac{\phi}{2} \quad \text{(x)}$$

The **intercepts** $BO = p$ and $BO' = p'$ are

$$p = \frac{h}{\sin u} \text{ and } p' = \frac{h}{\sin u'}\,;$$

hence

$$\frac{p'}{p} = \frac{\sin u}{\sin u'}. \quad \ldots \quad \ldots \quad \ldots \quad \text{(xi)}$$

In the case of two successive surfaces it is occasionally interesting (*e.g.*, for the purpose of determining the absorbing power of a substance) to know the path of a ray between them, or, as we shall call it, the **oblique thickness** of the medium. The magnitude $\boldsymbol{d}_v$ of this oblique thickness follows from the relation :

$$\boldsymbol{p}_{v+1} = \boldsymbol{p}_v' - \boldsymbol{d}_v . \quad \ldots \quad \ldots \quad \text{(xii)}$$

More precise values may be obtained by reference to the versed sines of the bounding spherical surfaces. We shall then find for $\boldsymbol{d}_v$

$$\boldsymbol{d}_v = \frac{d_v - 2\left(r_v \sin^2 \frac{\phi_v}{2} - r_{v+1} \sin^2 \frac{\phi_{v+1}}{2}\right)}{\cos u_v'} . \quad \text{(xiii)}$$

It may here be noted that we shall apply these formulæ both to rays which proceed from the object-point, as has been here assumed, and to rays which originate from the centre of a diaphragm (or stop). Rays of this kind will in future be referred to as **principal rays.** To distinguish them at once from object-rays of finite aperture we shall adopt the following notation :

$\boldsymbol{s}, u, i, \phi, h, \boldsymbol{p},$ for rays of finite aperture proceeding from object-points on the axis.

and

$\boldsymbol{x}, w, j, \Phi, y, \boldsymbol{q},$ for rays proceeding from the centre of a diaphragm.

29. Formulæ in Terms of the Intercepts on the Normal through the Centre (the Centre Normal). Instead of adopting a system of co-ordinates involving the length of the intercept SO we may formulate expressions in terms of the ordinate of the point where the ray cuts the perpendicular to the axis at the centre of the spherical surface. Thus $CH = \boldsymbol{U}$ becomes one of the quantities by which the ray is defined.

We shall then have

$$\boldsymbol{U} = (\boldsymbol{s} - r) \tan u ,$$

and since

$$\frac{\sin i}{\sin u} = \frac{\boldsymbol{s} - r}{r} ,$$

$$\sin i = \frac{\boldsymbol{U} \cos u}{r} ,$$

and, as before,

$$\sin i' = \frac{n}{n'} \sin i ,$$

$$u' = u + (i - i').$$

also

$$\sin i' = \frac{U' \cos u'}{r} ,$$

therefore

$$U' = U \frac{n}{n'} \cdot \frac{\cos u}{\cos u'} . \quad \dots \quad \dots \quad \text{(i)}$$

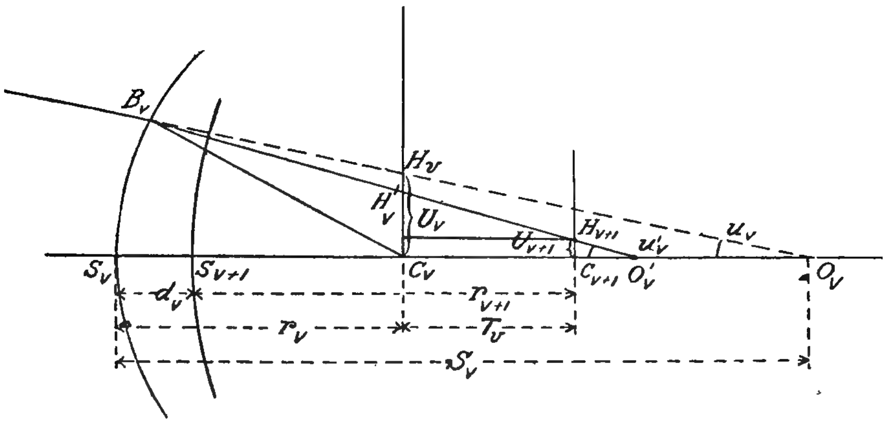

FIG. 11.

$U_v = C_v H_v$; $U'_v = C_v H'_v$; $U_{v+1} = C_{v+1} H_{v+1}$; $S_v S_{v+1} = d_v$; $S_v C_v = r_v$; $S_{v+1} C_{v+1} = r_{v+1}$.

Diagram showing the notation of Seidel-Wanach's formulæ of refraction and transition.

With the aid of the angles of incidence we have thus established as before, an expression for U' and u', in terms of U and u.

For the transition from one surface to another we require to know the axial distance c_v between the centres of two successive surfaces. From Fig. 11 it is clear that

$$C_v C_{v+1} = C_v S_v + S_v S_{v+1} + S_{v+1} C_{v+1} ,$$

or

$$c_v = - r_v + d_v + r_{v+1} ,$$

whence

$$U_{v+1} = U_v' - c_v \tan u_v' , \quad \dots \quad \dots \quad \text{(ii)}$$

and, as before,

$$u_{v+1} = u'_v . \quad \dots \quad \dots \quad \dots \quad \text{(iii)}$$

These formulæ have been deduced by Wanach (*I.*) from the scheme introduced by Seidel for the computation of skew rays, as we shall see later.

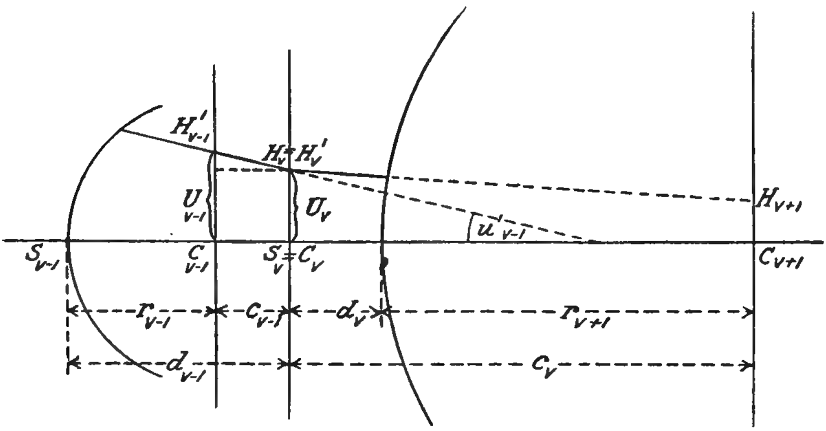

FIG. 12

Diagram of notation for Seidel-Wanach's transition to and from a plane surface.

In the case of plane surfaces matters are simplified by taking the plane itself, at right angles to the axis, as the normal of reference. The notation in Fig. 12 will then give us the following sequence of equations :

$$C_{v-1}\, C_v = C_{v-1}\, S_{v-1} + S_v \quad C_v$$

$$c_{v-1} = -\, r_{v-1} + d_v \quad ,$$

$$U_v = U'_{v-1} - c_{v-1} \tan u'_{v-1} \, ,$$

$$u'_{v-1} = -\, i'_v$$

$$U'_v = U_v$$

$$c_v = d_v + r_{v+1}$$

$$-\, i'_v = u_{v+1} \, .$$

The remaining formulæ for determining the point of incidence, the versed sine, and the length of path are immediately ascertainable from the previous formulæ in terms of this notation.

B. Rays in the Neighbourhood of the Axis (Paraxial Rays).

30. Specialised Trigonometrical Formulæ. When it is required to calculate the axial positions of rays which are infinitely near to the axis we shall always put s, s' as well as du, di, du', di' in the place of $\mathbf{s}$, $\mathbf{s'}$, u, i, u', i'.

We shall then be at liberty, as has in fact been proposed by Steinheil and Voit (*3. 81.*), to use the following sequence of equations similar to that of the trigonometrical scheme adopted with respect to rays of finite inclination.

$$di = \frac{s-r}{r}\, du\,, \quad \ldots \quad \ldots \quad \text{(i)}$$

$$di' = \frac{n}{n'}\, di\,, \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

$$du' = du + di - di' \quad \ldots \quad \ldots \quad \text{(iii)}$$

$$s' - r = r\,\frac{di'}{du'}\,. \quad \ldots \quad \ldots \quad \ldots \quad \text{(iv)}$$

Since the infinitely small value du is a factor of all the quantities to be determined, we may give it any value we choose, and we may treat it accordingly as a unit factor.

31. Introduction of the Invariants of the Paraxial Rays.—By eliminating the small angles in the above set of formulæ by successively substituting their values in terms of the other quantities and forming the value of $1/s'$ we obtain the expression

$$(s' - r)\left(\frac{s}{r} - \frac{n}{n'}\cdot\frac{s-r}{r}\right) = \frac{n}{n'}\,(s - r)$$

and thence the **zero invariant** of the refraction

$$Q_s = n'\left(\frac{1}{r} - \frac{1}{s'}\right) = n\left(\frac{1}{r} - \frac{1}{s}\right), \quad \ldots \quad \text{(i)}$$

from which we readily derive the working formula

$$\frac{n'}{s'} = \frac{n}{s} + \frac{n' - n}{r}\,. \quad \ldots \quad \ldots \quad \text{(ii)}$$

which involves no reference to the small angles.

This purely algebraical formula has the advantage that it offers a certain check upon fundamental errors of calculation when used in conjunction with the computation of zero and trigonometrical rays. The formulæ previously given cannot be checked in this way since in their application the procedure is much the same as in the computation of trigonometrical rays.

2. RAYS IN THE MERIDIAN PLANE.

A. Rays inclined at a Finite Angle to the Principal Ray.

32. The Usual Method of Successive Calculation of Rays proceeding from Points on the Axis.—If we regard every ray contained in the meridian plane, and hence intersecting the axis, as proceeding from the point of intersection with the axis, we shall have for each ray as a final result the values of s' and u'.

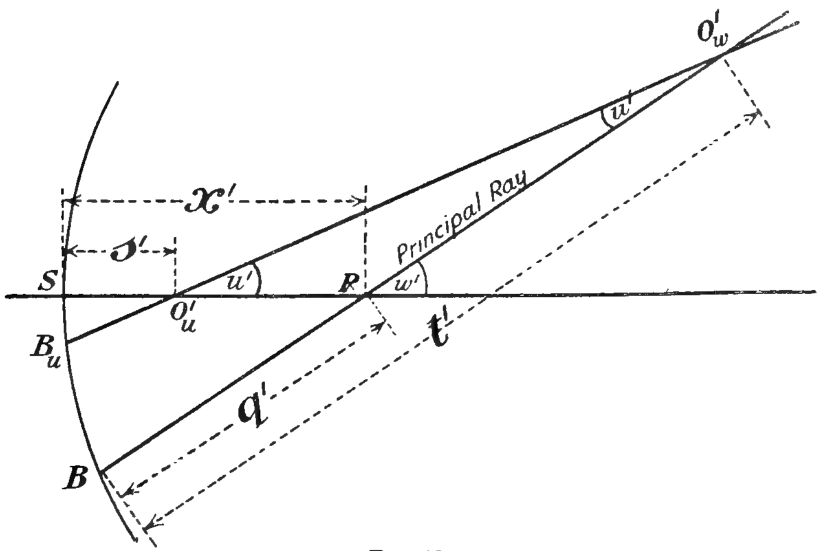

FIG. 13.

$t' = BO'_w$; $s' = SO'_u$; $x' = SP$; $q' = BP$.

Diagram showing the notation for the usual method of computing tangential pencils of finite aperture.

In general, the object of the calculation will then be to determine the intercept t' on the principal ray through the centre of the diaphragm, of any given ray whose co-ordinates, after refraction, are s' and u'. If now we define the principal ray by the co-ordinates x', w' in accordance with the convention adopted at the end of § 28, we shall obtain on reference to Fig. 13 the following relation:

$$\frac{\sin \mathrm{u}'}{\sin u'} = \frac{\sin (u' - w')}{\sin u'} = \frac{x' - s'}{q' - t'},$$

where q' may be determined in the way shown above. The quantities t', u' may thus consistently be regarded as the co-ordinates of the oblique rays with reference to the principal ray.

When it is required to calculate several rays through a system of centred spherical surfaces it will often be necessary to go through this supplementary operation after refraction at each surface in rotation, so as to check the calculation from a consideration of the resulting intercepts.

33. Direct Method of Abbe.—The object of this method is as before to determine the intercepts of any oblique ray in a meridian plane on the principal ray passing through the object-point

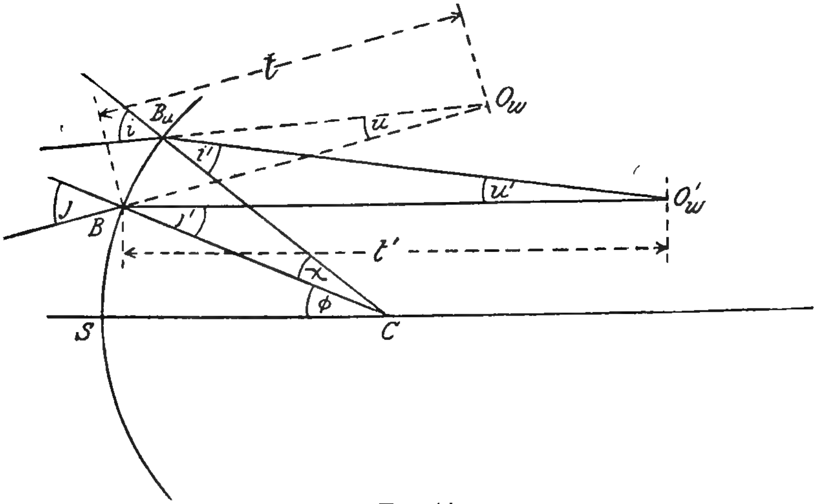

FIG. 14.

$t = BO_w$; $t' = BO'_w$.

Diagram showing the notation for Abbe's method of computing tangential pencils of finite aperture.

and the centre of the diaphragm, and also the corresponding angular aperture u′. The particulars of the principal ray, including its angles of incidence j, j', are supposed to be known.

As before, let u, u′ denote the angular aperture of the oblique ray, and let χ be the increment of the angle ϕ subtended by the arc between the points of incidence. From the figure it will be seen that

$$i + \mathrm{u} = j + \chi,$$

$$i' + \mathrm{u}' = j' + \chi,$$

$$\text{therefore} \quad \mathrm{u}' = \mathrm{u} + (i - i') - (j - j'),$$

which gives us an expression for u′ in terms of i and i',

For the determination of t' it will be necessary to establish an additional relation. To this end the circle, of which a portion only is indicated in Fig. 14, should be completed, and the intercept t produced to D, where it meets the circumference. Let $BD = T$ and join the second point of intersection D with the point of incidence B_u and let $B_uD = \theta$. From Fig. 15 it will be seen that

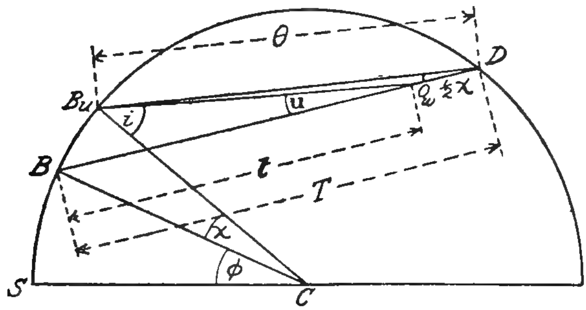

FIG. 15.

$t = BO_w$; $T = BD$; $\theta = B_u D$.

Diagram showing the notation for Abbe's method of computing tangential pencils of finite aperture.

$$T = 2r \cos j, \ldots \qquad \ldots \qquad \ldots \qquad \text{(i)}$$

$$\theta = 2r \cos\left(j + \frac{\chi}{2}\right), \qquad \ldots \qquad \ldots \qquad \text{(ii)}$$

and

$$\frac{\theta}{T-t} = \frac{\sin \mathrm{u}}{\sin\left(\mathrm{u} - \frac{\chi}{2}\right)}. \qquad \ldots \qquad \ldots \qquad \text{(iii)}$$

Introducing in this last equation the equivalents of θ and T, we have

$$t \sin \mathrm{u} = 2r\left[\cos j \sin \mathrm{u} - \cos\left(j + \frac{\chi}{2}\right) \sin\left(\mathrm{u} - \frac{\chi}{2}\right)\right].$$

Expressing the two products within the bracket in terms of the sum of the sines, which will cause $\sin (j + \mathrm{u})$ to cancel out, we obtain the simplified equation :

$$\frac{t \sin \mathrm{u}}{r} = \sin i - \sin (j - \mathrm{u}), \qquad \ldots \qquad \text{(iv)}$$

where

$$i = j + \chi - \mathrm{u}.$$

Similarly, we shall have the following relations after refraction.

$$\frac{t \sin u'}{r} = \sin i' - \sin (j' - u'); \ \text{ and } i' = j' + \chi - u'.$$

Introducing the auxiliary angular value η, such that

$$\sin \eta = \frac{t}{r} \sin u, \qquad \ldots \qquad \ldots \qquad (\text{v})$$

the above equation assumes the form

$$\sin i = 2 \sin \frac{\eta + j - u}{2} \cdot \cos \frac{j - u - \eta}{2},$$

and also

$$\sin i' = \frac{n}{n'} \sin i.$$

This, as has already been seen, determines also the value of u′, and we have accordingly

$$\frac{t'}{t} = \frac{\sin u}{\sin u'} \cdot \frac{\sin i' - \sin (j' - u')}{\sin i - \sin (j - u)}$$

$$= \frac{\sin u}{\sin u'} \cdot \frac{\sin \frac{i' + u' - j'}{2} \cos \frac{i' + j' - u'}{2}}{\sin \frac{i + u - j}{2} \cos \frac{i + j - u}{2}}$$

and since

$$i' + u' - j' = i + u - j = \chi$$

$$\frac{t'}{t} = \frac{\sin u}{\sin u'} \cdot \frac{\cos \frac{i' + j' - u'}{2}}{\cos \frac{i + j - u}{2}}. \qquad \ldots \qquad (\text{vi})$$

B. Rays in the neighbourhood of the Principal Ray.

34. Special Forms of Abbe's Formulæ.—Let the value of the angular aperture u be so small that even infinitesimals of the second order may be neglected in comparison with those of the first order.

In this case we will denote the angle by du, which will enable us to simplify the preceding formulæ for u very considerably; for we shall now have

$$i = j + di \text{ and } \chi = d\phi ,$$

and, since

$$j + di + du = j + d\phi$$

$$di + du = d\phi$$

and

$$di' + du' = d\phi ;$$

also,

$$i + j - u \text{ becomes } 2j + di - du .$$

Substituting these transformed quantities in the formula of the intercept due to finite values of u, as given at the end of § 33,

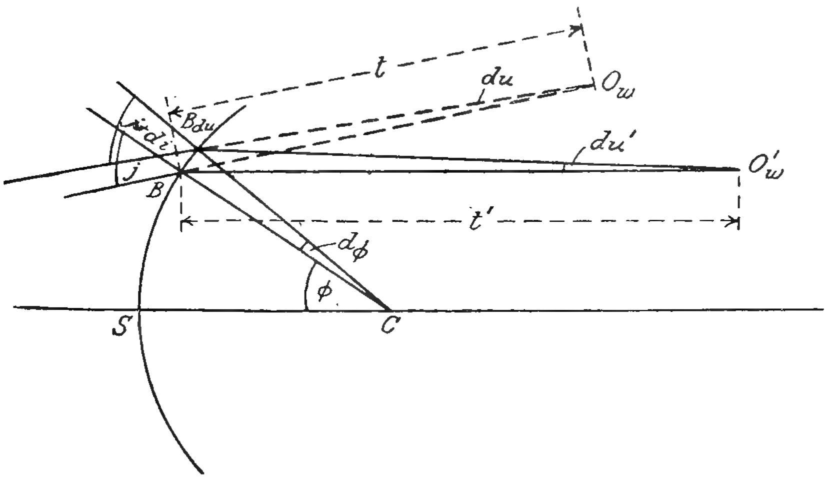

FIG. 15A.

$t = BO_w$; $t' = BO_w'$.

Abbe's method applied to tangential pencils of infinitesimal apertures of the first order.

and confining ourselves to magnitudes of the first order, we shall have the relation

$$\frac{t'}{t} = \frac{du \cos j'}{du' \cos j} , \quad \dots \quad \dots \quad \dots \quad \text{(i)}$$

where t, t', in conformity with the notation adopted in the previous case of finite angles, denote the intercepts of infinitely near rays.

This formula becomes suitable for the calculation of t' if we replace $\frac{du}{du'}$ by quantities relating to the unrefracted ray or containing elements of the principal ray only.

From the equation

$$di + du = di' + du',$$

and differentiating the equation of the law of refraction, we find

$$du' - du = di\left(1 - \frac{\tan i'}{\tan i}\right). \quad \ldots \quad \ldots \quad \text{(ii)}$$

Also if we neglect quantities smaller than those of the first order the relation established in § 33 (iv)

$$\frac{t}{r}\sin u = \sin i - \sin(j - u),$$

becomes

$$\frac{t\,du}{r} = \sin(j + di) - \sin(j - du) = (du + di)\cos j; \quad \ldots \quad \text{(iii)}$$

hence

$$di = \left(\frac{t}{r\cos j} - 1\right) du.$$

This relation in conjunction with equation (ii) for $d\,u' - d\,u$ gives us finally the required expression:

$$\frac{d\,u'}{d\,u} = 1 + \left(1 - \frac{\tan j'}{\tan j}\right)\left(\frac{t}{r\cos j} - 1\right)$$

$$= \frac{t}{r\cos j}\left(1 - \frac{\tan j'}{\tan j}\right) + \frac{\tan j'}{\tan j},$$

and by equation (i)

$$\frac{t}{t'} = \frac{\cos j}{\cos j'}\left[\frac{t}{r\cos j}\left(1 - \frac{\tan j'}{\tan j}\right) + \frac{\tan j'}{\tan j}\right]. \quad \ldots \quad \text{(iv)}$$

This formula is not applicable when $t = \infty$ causes the object-point on the principal ray to be at infinity. When this is the case du vanishes, and accordingly we may replace the small angular aperture by the small linear magnitude $t\,du = r\cos j\,di$. If now in the equation

$$t' = \frac{t\,du}{du'} \cdot \frac{\cos j'}{\cos j}$$

we determine the corresponding values of the first factor, we shall have

$$\left(\frac{tdu}{du'}\right)_{du=0.} = \frac{r\cos j}{1-\tan j'/\tan j} = \frac{n'r\cos j\cos j'}{n'\cos j' - n\cos j};$$

hence

$$(t')_{t=\infty} = r\,\frac{n'\cos^2 j'}{n'\cos j' - n\cos j}. \quad \ldots \quad \text{(v)}$$

Working out the general formula (iv) for t/t', we obtain the following formula for the intercepts within the tangential plane, viz. :

$$\frac{n'\cos^2 j'}{t'} - \frac{n\cos^2 j}{t} = \frac{1}{r}(n'\cos j' - n\cos j). \ldots \quad \text{(vi)}$$

The term on the right side of the equation, the **astigmatic constant,** is capable of further reduction, which renders it more convenient for operation with logarithms, thus :

$$n'\cos j' - n\cos j = \frac{n}{\sin j'}\sin(j-j') = \frac{n'}{\sin j}\sin(j-j').\text{(vii)}$$

The transition from t' to the intercept t of the next surface is accomplished by the introduction of the oblique thickness $\boldsymbol{d}$, for which we have already obtained in § 28, (viii) the expression

$$t_{v+1} = t_v' - d_v$$

When at the outset of our computation we are given the distance from the vertex of the object-plane at right angles to the axis and the inclination w of the principal ray, we may proceed to determine the axial intercept

$$t_1 = \frac{1}{\cos w_1}\left(s_1 - 2r_1\sin^2\frac{\varphi_1}{2}\right), \quad \ldots \quad \text{(viii)}$$

and, conversely, after carrying the calculation through k surfaces, we can find the abscissa $\bar{s}_k'$ of the tangential image-point on the principal ray, reckoned from the last surface The value of this abscissa, which varies in general with the magnitude of w_1, is then

$$\bar{s}_k' = t'_k\cos w'_k + 2r_k\sin^2\frac{\phi_k}{2}.$$

35. Derivation of Wanach's Formulæ.—Proceeding from the formulæ for a ray issuing from a point on the axis in terms of the co-ordinates U and u, as given in § 29, Wanach (*1.*) has derived a series of formulæ for determining the intercepts of narrow tangential pencils. His procedure was as follows. He determined the variations which occur in the co-ordinates when investigating, conjointly with the principal ray W, another ray w, whose inclination to the axis has the value $w + du$.

The two rays under consideration will accordingly include the infinitely small angular aperture du. Differentiating the equation $\sin j = \frac{W \cos w}{r}$, which follows from $\sin i = \frac{U \cos u}{r}$ in § 29 and indicating by the notation that we are dealing with the co-ordinates of the principal ray, we obtain the following relation with respect to any surface :

$$du' - du = di\left(1 - \frac{\tan j'}{\tan j}\right)$$

since $du - du' = di - di'$,

$$di = \frac{\cos w\, dW - W \sin w\, du}{r} \tan j .$$

where, owing to the relation $\frac{\sin j}{\sin w} = \frac{s - r}{r}$,

we find $r = \frac{W \cos w}{\sin j} = \frac{W' \cos w'}{\sin j'}$,

whence

$$du' = du + (\tan j - \tan j')\left[\frac{dW}{W} - \tan w\, du\right], \qquad \text{(i)}$$

and, similarly,

$$dW' = W'\left[\frac{dW}{W} - \tan w\, du + \tan w'\, du'\right]. \qquad \text{(ii)}$$

For the transition from the v^{th} to the $(v + 1)^{\text{th}}$ surface

$$du_{v+1} = du'_v \quad \ldots \quad \ldots \quad \ldots \qquad \text{(iii)}$$

$$dW_{v+1} = dW'_v - c_v \frac{du'_v}{\cos^2 w'_v} . \quad \ldots \qquad \text{(iv)}$$

In the event of the μ^{th} surface being plane these relations assume the following modified forms :

$$dW_\mu = W'_{\mu-1} - c_{\mu-1} \frac{du'_{\mu-1}}{\cos^2 w'_{\mu-1}}$$

$$c_{\mu-1} = d_{\mu-1} - r_{\mu-1}$$

$$du'_\mu = \frac{\tan w'_\mu}{\tan w_\mu} du_\mu ,$$

$$dW'_\mu = dW_\mu$$

$$dW_{\mu+1} = dW'_\mu - c_\mu \frac{du'_\mu}{\cos^2 w'_\mu}$$

and

$$c_\mu = d_\mu + r_{\mu+1}.$$

It only remains now to establish, by means of the given quantities, the relation between the initial quantities du_1 and dW_1 prevailing before the first refraction. Wanach proceeds by assuming the position of an object-point in a plane at right angles to the axis and at a distance s_1 from the vertex to be given by the position of the point where the plane is intersected by a principal ray inclined at an angle w_1 to the axis.

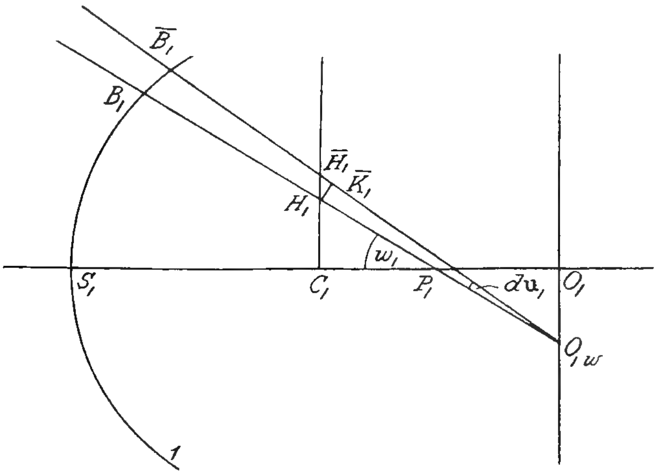

FIG. 16.

$C_1H_1 = W_1$; $H_1\overline{H}_1 = dW_1 = \varepsilon$; $H_1\overline{K}_1 = \mu$; $H_1O_{1w} = \tau_1$.

Diagram showing notation in Wanach's formulæ for the computation of narrow tangential pencils.

In Fig. 16, if we assume $dW_1 = \varepsilon$ to be an arbitrary infinitesimal quantity, and if at H_1 we let fall upon H_1O_{1w} a normal $H_1\overline{K}_1 = \mu$, we shall have

$$du_1 = \frac{\mu}{\tau_1},$$

where, on reference to the figure, the numerator and denominator are seen to be equivalent to

$$\mu = dW_1 \cos w_1 = \varepsilon \cos w_1$$

$$\tau_1 = \frac{s_1 - r_1}{\cos w_1},$$

whence, finally,

$$du_1 = \varepsilon \frac{\cos^2 w_1}{s_1 - r_1} \quad \ldots \quad \ldots \quad \ldots \quad (\text{v})$$

Since the two quantities dW_1 and du_1 are both proportional to ε, the same will apply to all analogous quantities dW_v and du_v which

arise in the course of the investigation. We may therefore introduce in the place of the small quantity ε multiples thereof, and accordingly put $\varepsilon = 1$.

When $t_1 = \infty$ it is only necessary to put $du_1 = 0$. After carrying the computation through the last surface it is again desirable to determine the abscissæ of the last tangential point of intersection. Similarly, as in the case of the first surface, if we suppose the plane at right angles to the axis to pass through this point, we shall have

$$du'_k = \frac{d\,W'_k \cos^2 w'_k}{\bar{s}'_k - r_k},$$

hence

$$\bar{s}'_k = r_k + \frac{d\,W'_k \cos^2 w'_k}{du'_k} \ldots \qquad \ldots \qquad \text{(vi)}$$

3. REFRACTION OF SKEW PENCILS.

36. Each of these various methods for the computation of rays inclined at finite angles to the meridian plane may be regarded as being made up of two operations. The first of these consists in tracing the ray within its plane of incidence in accordance with the sine law of refraction. The other embraces the process of transforming the data resulting after any one refraction into the new conditions for the next refraction.

A. Transformation Formulæ for the Transition from one Surface to another and the Computation of the Initial Data.

37. We shall first deal with the operation of transforming the successive results of the calculation. It is obvious that this will assume different forms according to the manner in which we define the position and direction of a ray. There are four mutually independent parts by which any ray may be completely defined. We select for this purpose, in the first place, the co-ordinates of the points where the ray intersects two appropriately chosen planes.

Now, the choice of a suitable system of co-ordinates is not an arbitrary matter. One of the co-ordinate axes, at any rate, should coincide with the axis of the optical system. As one of the planes of reference we should obviously choose the **meridian plane,** *i.e.*, the **vertical** or **longitudinal plane,** as represented by the plane of the paper, which we may readily assume to contain the given object-point without prejudice to the general purpose of our investigation. A plane at right angles to the axis of the system and passing through the centre of the refracting surface under consideration

furnishes the second plane, whilst the **horizontal** or **latitudinal plane,** *i.e.*, the **sagittal plane,** containing the axis of the system provides the third plane of reference. As will be seen from Fig. 17, we refer to the axis of the system as the axis of x; to the axis at right angles to it (and hence lying in the plane of the paper) as the axis of y; and to the horizontal axis at right angles to the plane of the paper as the axis of z.

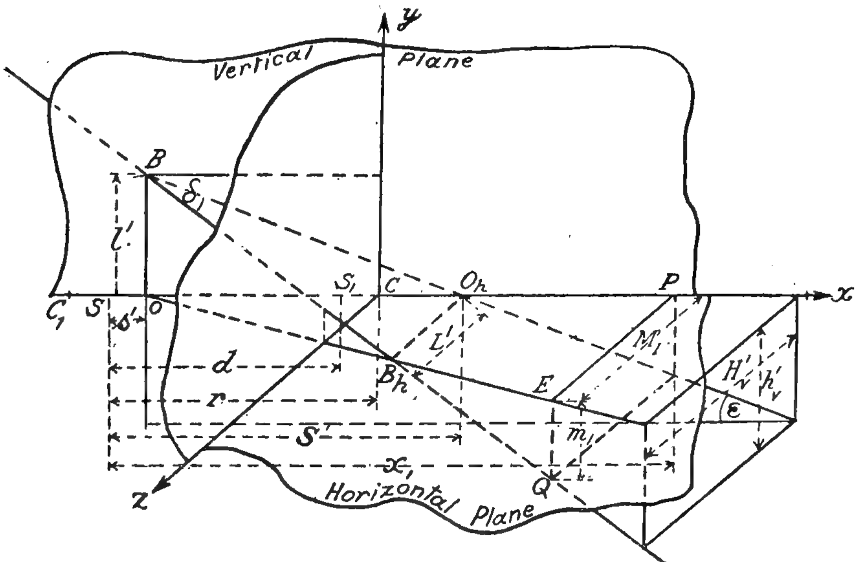

FIG. 17.

$SO = s'$; $SO_h = S'$; $OB = l'$; $O_h B_h = L'$; $SC = r$; $S_1C_1 = r_1$; $SS_1 = d$.

Diagram showing the notation for Kerber's transition from one refracting surface to the next.

38. Kerber's Transition Formulæ.—In this case the parts by which the rays are defined are the points where the ray intersects the vertical and the horizontal planes. Let the co-ordinates of the point of penetration in the first plane be denoted by small letters, that in the second plane by capital letters, and let s' and S' be the axial distances of these points from the vertices of the surfaces; then the co-ordinates with reference to a system (x, y, z) having its origin at the centre of the spherical surface are, according to the notation adopted by Kerber (*8.*).

$$s' - r, \; l', \; 0; \qquad S' - r, \; 0, \; L'.$$

The transition to the next surface r_1 causes merely a change in the s, S co-ordinates, since the system of co-ordinates undergoes as a whole a displacement along the axis of x parallel to itself by

an amount which is equal to the distance $d - r + r_1$ between the centres of two successive spherical surfaces. Also, since by reason of this displacement

$$s_1 = s' - d \text{ and } S_1 = S' - d,$$

we have further

$$s_1 - r_1 = s' - r - (d - r + r_1)$$

and

$$S_1 - r_1 = S' - r - (d - r + r_1).$$

The other two co-ordinates do not change in any way, seeing that the ray itself does not experience any modification as a result of the axial displacement. We have, therefore,

$$l_1 = l' \text{ and } L_1 = L'.$$

In the event of the μth spherical surface being replaced by a plane at right angles to the axis we shall choose this plane surface as the yz plane, which will make the displacement before refraction equal to

$$c_{\mu-1} = d_{\mu-1} - r_{\mu-1}$$

and after refraction

$$c_{\mu} = d_{\mu} + r_{\mu+1}.$$

The initial data which are available before the first refraction generally include, in addition to the *unaccented* co-ordinates of the object-point (s_1, l_1, 0), the rectangular co-ordinates m_1, M_1 of the points in the plane of the diaphragm on the object side of the skew ray proceeding from the object-point. Let x_1 denote the distance of the diaphragm from the first spherical surface. In Fig. 17 let the centre of the plane of the diaphragm be represented by the letter P, and let Q be the point where a skew ray BB_h intersects the plane of the diaphragm. The various lengths in Fig. 17 will then, with suitable and readily intelligible modifications, have the following notations

$$SO = s_1\,; \quad SO_h = S_1\,; \quad OB = l_1\,; \quad O_hB_h = L_1$$
$$SP = x_1\,; \quad EQ = m_1\,; \quad PE = M_1\,.$$

From the ratios

$$\frac{OB}{EQ} = \frac{OB_h}{EB_h} = \frac{OO_h}{PO_h} = \frac{OS + SO_h}{PS + SO_h}$$

or,

$$l_1 / m_1 = (S_1 - s_1) / (S_1 - x_1)$$

we derive the following value of S_1;

$$S_1 = \frac{l_1 x_1 - m_1 s_1}{l_1 - m_1}. \quad \dots \quad \dots \quad \text{(i)}$$

The corresponding value of L_1 follows from the ratios

$$\frac{O_h B_h}{PE} = \frac{O O_h}{OP} = \frac{OS + SO_h}{OS + SP}$$

or

$$L_1 / M_1 = (S_1 - s_1) / (x_1 - s_1),$$

whence

$$L_1 = M_1 \frac{S_1 - s_1}{x_1 - s_1} = M_1 \frac{l_1}{l_1 - m_1}. \quad \ldots \qquad \text{(ii)}$$

When the object-point recedes into infinity, so that $l = \infty = s_1$, this formula becomes indeterminate, but we still have the relation for the inclination of the principal ray, viz. :

$$\tan w_1 = - \frac{l_1}{s_1}, \qquad \ldots \qquad \ldots \qquad \text{(iii)}$$

and from

$$S_1 = \frac{x_1 - m_1 \dfrac{s_1}{l_1}}{1 - \dfrac{m_1}{l_1}}$$

we find in the limit

$$S_1 = x_1 + m_1 \cot w_1 \qquad \ldots \qquad \ldots \qquad \text{(iv)}$$

and similarly,

$$L_1 = M_1 . \quad \ldots \qquad \ldots \qquad \ldots \qquad \text{(v)}$$

Returning to the formulæ for objects at a finite distance, we are now able to obtain corresponding expressions for finding the co-ordinates of the points of intersection in a transverse plane at an axial distance s_v' from the vertex.

If we denote these co-ordinates by h'_v, H'_v we must replace the terms x_1, m_1, M_1 respectively by s'_v, h'_v, H'_v and thence determine h'_v and H'_v thus :

$$h'_v = l'_v \frac{s'_v - S'_v}{s'_v - S'_v}; \quad H'_v = L'_v \frac{s'_v - s'_v}{S'_v - s'_v}. \quad \ldots \qquad \text{(vi)}$$

It does not make any material difference whether we define the direction of the ray in terms of the point of intersection (S, L) in the horizontal plane, or whether we do so by appropriately chosen angular quantities. For this purpose we shall here take the angle ε comprised between the axis of x and the projection of the ray on the plane of x, y, as well as the angle of inclination δ of the ray with respect to its projection. None of these angular quantities are

affected by the parallel displacement of the co-ordinate system. We likewise retain the identity $l_1 = l'$, so that the only requisite change is

$$s_1 - r_1 = s' - r - (d - r + r_1)$$

As before, to operate in terms of angular quantities, it will be necessary to set up formulæ for the initial data, assuming these to contain the co-ordinates of the plane of the aperture. We shall in this case have

$$\cot \varepsilon_1 = \frac{x_1 - s_1}{l_1 - m_1}\,.$$

$$\tan \delta_1 = -\frac{M_1 \cos \varepsilon_1}{x_1 - s_1}\,, \quad \ldots \quad \ldots \quad \text{(vii)}$$

The co-ordinates of the points of intersection in a transverse plane at an axial distance s'_v from the vertex will then be

$$h'_v = l'_v + (s'_v - s'_v) \tan \varepsilon'_v$$

and

$$H'_v = \frac{\tan \delta'_v}{\cos \varepsilon'_v}(s'_v - s'_v)\,. \quad \ldots \quad \ldots \quad \text{(viii)}$$

39. Seidel's Transition Formulæ.—If for the planes of reference we choose the horizontal plane and that passing through the centre of the sphere the co-ordinates of the points B_h and A where the ray intersects the planes of reference will be

$$S' - r,\ 0,\ L' \text{ and } 0,\ v',\ V'.$$

When the system is displaced by an amount $c = d - r + r_1$

$$S_1 - r_1 = S' - r - (d - r + r_1)\,.$$

Only $L_1 = L'$ does not change, whilst v_1, V_1 assume the values

$$v_1 = v'\frac{S_1 - r_1}{S' - r}\,;\ \ V_1 = L' + (V' - L')\frac{S_1 - r_1}{S' - r}\,.$$

These quantities are so connected as to enable us to formulate the following relations in terms of invariants:

$$\frac{v_1\,L_1}{V_1 - L_1} = \frac{v'\,L'}{V' - L'}\,.$$

These transition formulæ are not directly available for use with Seidel's method of computation, since he did not define his ray in

terms of the co-ordinates of the point of intersection. However, his transition formulæ, derived from geometrical considerations, can readily be deduced analytically.

The point of intersection in the first transverse plane is defined by Seidel in terms of the polar co-ordinates U', ζ', which are accordingly connected with the quantities introduced above by the relations

$$v' = U' \cos \zeta' \; ; \; V' = U' \sin \zeta',$$

where ζ' denotes the angle of inclination with respect to the zero direction.

Seidel (*6.*) defines the direction of the ray by the two angles τ' and π', the angle comprised between the direction of the ray and that of the axis of the system being denoted by τ'; whilst π' is the angle comprised between the projection of the ray upon the transverse plane and the zero direction therein.

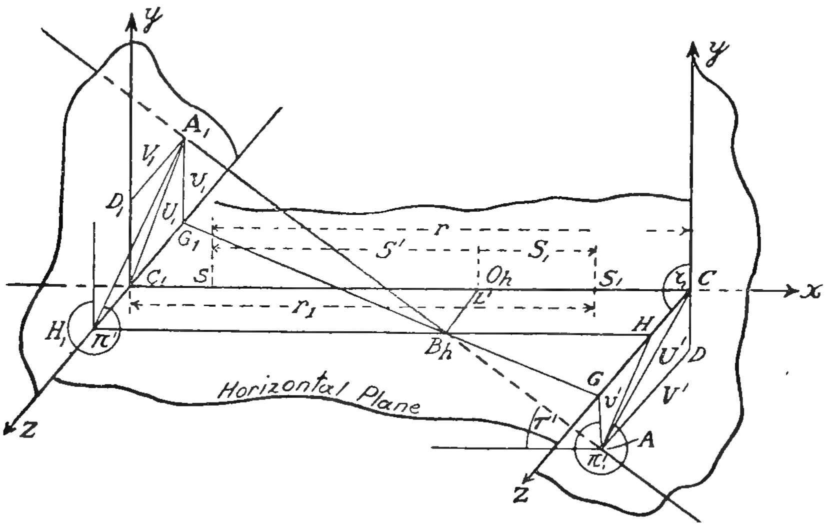

FIG. 18.

$CO_h = S' - r$; $O_h B_h = L'$; $GA = v'$; $DA = V'$; $CA = U'$;
$C_1O_h = S_1 - r_1$; $CH = C_1H_1 = L'$; $G_1A_1 = v_1$; $D_1A_1 = V_1$; $C_1A_1 = U_1$.

Diagram illustrating Seidel's transition formulæ from one refracting surface to the next.

From Fig. 18 it will be seen that

$$\tan \pi' = \frac{V' - L'}{v'} \; ; \; \tan \tau' = \frac{\sqrt{v'^2 + (V' - L')^2}}{S' - r} = \frac{v'}{S' - r} \cdot \frac{1}{\cos \pi'} \, .$$

If now we pass on to the transverse plane through the centre of the second sphere of radius r_1, there will be no change in the direction of the ray, that is of π' and τ', and accordingly

$$\tan \pi' = \frac{V_1 - L'}{v_1}\ ; \ \tan \tau' = \frac{v_1}{S_1 - r_1} \cdot \frac{1}{\cos \pi'} = \frac{v_1}{S' - r - c} \cdot \frac{1}{\cos \pi'},$$

and also

$$v_1 = U_1 \cos \zeta_1, \quad V_1 = U_1 \sin \zeta_1 .$$

We can now obtain Seidel's transition formulæ with the aid of the relation found above, viz.:

$$\frac{V' - L'}{v'} = \frac{V_1 - L_1}{v_1}$$

$$\tan \pi' = \frac{V_1 - V'}{v_1 - v'}$$

or

$$v_1 \sin \pi' - V_1 \cos \pi' = v' \sin \pi' - V' \cos \pi'$$

$$U_1 \cos \zeta_1 \sin \pi' - U_1 \sin \zeta_1 \cos \pi' = U' \cos \zeta' \sin \pi' - U' \sin \zeta' \cos \pi'$$

and finally

$$U_1 \sin (\pi' - \zeta_1) = U' \sin (\pi' - \zeta') . \quad \ldots \quad \text{(i)}$$

This is **Seidel's first Transition Formula.**

From the relation established at the beginning of this article we have

$$\frac{v'}{S' - r} = \frac{v_1}{S' - r - c}$$

and hence for the value of τ'

$$\tan \tau' = \frac{v' - v_1}{c} \cdot \frac{1}{\cos \pi'}$$

or

$$(v_1 - v') \sin^2 \pi' + (v_1 - v') \cos^2 \pi' = - c \tan \tau' \cos \pi',$$

and, by introducing the value of π' from the last equation for $\tan \pi'$,

$$(V_1 - V') \sin \pi' + (v_1 - v') \cos \pi' = - c \tan \tau',$$

whence we obtain by substitution

$$U_1 \sin \zeta_1 \sin \pi' + U_1 \cos \zeta_1 \cos \pi' = U' \sin \zeta' \sin \pi' + U' \cos \zeta' \cos \pi' - c \tan \tau',$$

or

$$U_1 \cos (\pi' - \zeta_1) = U' \cos (\pi' - \zeta') - c \tan \tau' . \quad \ldots \quad \text{(ii)}$$

This equation embodies **Seidel's second Transition Formula.**

In the case of plane refracting surfaces we have to introduce the modifications occasioned by the transition from $C_{\mu-1}$ to C_μ, in precisely the same way as in the previous method of transformation.

This case disposes of one of the possible ways in which the initial data may appear.

Let the object plane be at an axial distance s_1 from the surface of reference, and in that plane let U_0, ξ be the co-ordinates of the object-point (where as a rule $\xi = 0$ or $180°$, since in most cases the object-point is taken to be in the meridian plane), and, finally, let the direction of the incident ray be defined by τ and π. We shall in this case have the following equations for U and ζ in the first transverse plane of reference :

$$U \sin (\pi - \zeta) = U_0 \sin (\pi - \xi)$$
$$U \cos (\pi - \zeta) = U_0 \cos (\pi - \xi) - (-s_1 + r_1) \tan \tau .$$

When, however, the object-point is situated at infinity, though $U_0 = \infty = s_1$, the angular distance from the axis can again be defined as before, by the magnitude of τ, whilst $\pi = 0$ or $180°$. The points where the ray intersects the first transverse plane follow from the given positions of the points where the aperture plane is required to be intersected by the skew rays. Assuming m_1 and M_1 to be the given co-ordinates of these points in the aperture plane, and further supposing that the distance between the latter and the first transverse plane is $x_1 - r_1$ and that the principal ray is inclined at an angle w_1 to the axis, then the principal ray will intersect this transverse plane at a point whose ordinate is

$$v_1 = (x_1 - r_1) \tan w_1 ,$$

and the co-ordinates of the points of intersection of extra-axial rays will accordingly be

$$\left.\begin{aligned} v_1 &= U \cos \zeta = v_1 + m_1 \\ V_1 &= U \sin \zeta = M_1 \end{aligned}\right\} \qquad \ldots \qquad \text{(iii)}$$

An image plane at a distance s' from the surface of reference is likewise treated as a transverse plane at right angles to the axis. As in the case of an object-point at a finite distance, we shall then obtain the co-ordinates of the points of intersection

$$Y = U \cos \xi , \quad Z = U \sin \xi$$

from the values U', ζ', τ', π', resulting at the last refraction, with the aid of the relations

$$U \sin (\pi' - \xi) = U' \sin (\pi' - \zeta')$$
$$U \cos (\pi' - \xi) = U' \cos (\pi' - \zeta') - (s' - r_k) \tan \tau' .$$

In the case of finite positions of the object-point it is not usual for the initial data τ, π to be given in an explicit form. In fact, in most cases, as in the preceding instance, the given data will consist of the co-ordinates $(s_1, l_1, 0)$ of the object-point and (x_1, m_1, M_1) of the point where the ray intersects the aperture plane.

When the data are as here suggested, the co-ordinates S_1 and L_1 are furnished by the formulæ given in § 38, and we have then the case from which we proceeded, where we assumed the points of intersection to be given in the horizontal and transverse planes. By a displacement of the planes of reference from the aperture plane to the centre of the sphere we shall finally obtain the desired values of v_1 and V_1, viz.:

$$v_1 = m_1 \frac{S_1 - r_1}{S_1 - x_1} = \frac{l_1 (x_1 - r_1) - m_1 (s_1 - r_1)}{x_1 - s_1} \qquad \text{(iv)}$$

$$V_1 = - M_1 \frac{s_1 - r_1}{x_1 - s_1}. \quad \ldots \quad \ldots \qquad \text{(v)}$$

We are now able, with the aid of the formulæ given above, to determine π and τ, and accordingly we have

$$\tan \pi = \frac{M_1}{m_1 - l_1} \quad \ldots \quad \ldots \qquad \text{(vi)}$$

$$\tan \tau = \frac{\sqrt{(m_1 - l_1)^2 + M_1^2}}{x_1 - s_1}. \quad \ldots \qquad \text{(vii)}$$

40. Bruns' Transition Formulæ.—There is yet another possible method of defining the position and direction of the ray, viz., by stating the co-ordinates of the points in which it intersects two planes at right angles to the axis. One of these two planes is, as before, the transverse plane at the centre of the sphere. We again denote the co-ordinates of the points in these planes by v and V. The other plane contains, as we shall see later, the point of incidence of the ray on the spherical surface. Let X_1, Y_1, Z_1* be the co-ordinates of this point.

This change in the choice of the planes of reference has the advantage that it defines at once in a most direct manner the inclination and position of the ray which passes through both points, in that the direction cosines are in this case

$$C_{,} = \frac{X_1}{R}; \quad C_{,,} = \frac{Y_1 - v}{R}; \quad C_{;} = \frac{Z_1 - V}{R};$$

where

$$R = \sqrt{X^2_1 + (Y_1 - v)^2 + (Z_1 - V)^2}.$$

* The majority of the formulæ which will be given in the succeeding pages, like those given in § 43, have been derived from an unpublished exposition of Bruns' method by Wandersleb.

It is evident that these direction cosines, in conjunction with v, V, suffice to define completely the ray. In the scheme of computation, which we shall discuss in subsequent articles, we shall give preference to this method of defining a ray.

To establish a set of formulæ for the transition from one refracting surface to the next, let v', V' be the known co-ordinates in the transverse plane. It is now required to determine the new co-ordinates v_1, V_1 in terms of C'_x, C'_y, C'_z, v', V' occasioned by a displacement of magnitude c. This displacement does not in any

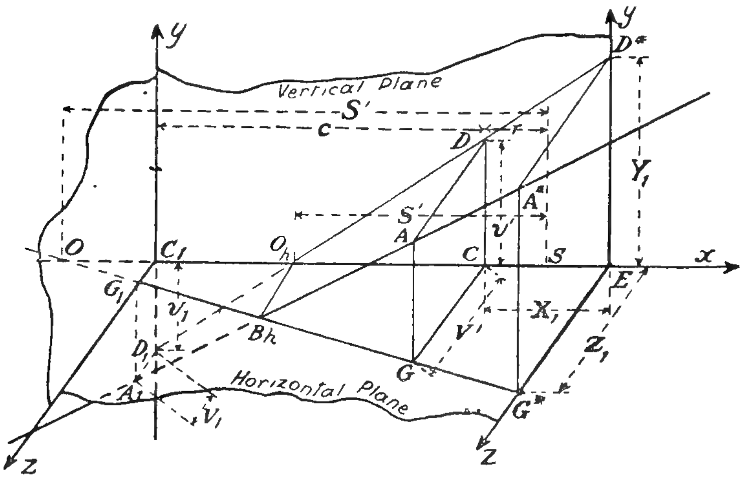

FIG. 19.

$CE = X_1$; $ED^* = Y_1$; $EG^* = Z_1$; $CD = v'$; $CG = V'$; $C_1D_1 = v_1$; $C_1G_1 = V_1$.

Diagram of Bruns' formulæ for the transition from one refracting surface to the next.

way affect the direction of the ray, so that the quantities C'_x, C'_y, C'_z retain their original values; but the following changes take place in the co-ordinates of the point of intersection at the transverse plane, as will be readily seen on reference to Fig. 19

$$CD/C_1D_1 = CO_h/C_1O_h \,; \quad CG/C_1G_1 = CO/C_1O.$$

$$v'/v_1 = (S' - r)/(S' - r - c) \,; \quad V'/V_1 = (s' - r)/(s' - r - c).$$

$$v_1 = v' - \frac{c\,v'}{S' - r} \,; \quad V_1 = V' - \frac{c\,V'}{s' - r}.$$

From Fig. 19 we obtain the equations:

$$\frac{v'}{S'-r} = \frac{Y_1 - v'}{-X_1} = -\frac{C'_y}{C'_x}; \quad \frac{V'}{s'-r} = \frac{Z_1 - V'}{-X_1} = -\frac{C_z'}{C_x'},$$

whence finally
$$v_1 = v' + \frac{C_y'}{C_x'}c; \quad V_1 = V' + \frac{C_z'}{C_x'}c. \quad \ldots \quad \text{(i)}$$

The distance from the centre of one refracting sphere to the next is, as before,

$$c = d - r + r_1.$$

When the refracting surface takes the special form of a plane at right angles to the axis the displacements which have to be introduced before and after refraction are given by the expressions already found in § 38 for the transition from $C_{\mu-1}$ to C_μ.

When the object is at a finite distance the normal initial values $(s_1, l_1, 0)$ and (x_1, m_1, M_1) give us at once the direction cosines

$$C_{x1} = \frac{x_1 - s_1}{R_1}; \quad C_{y1} = \frac{m_1 - l_1}{R_1}; \quad C_{z1} = \frac{M_1}{R_1}, \quad \ldots \quad \text{(ii)}$$

where

$$R_1 = \sqrt{(x_1 - s_1)^2 + (m_1 - l_1)^2 + M_1^2}, \quad \ldots \quad \text{(iii)}$$

and, if $r_1 - x_1$ is the distance between the aperture plane and the transverse plane, v_1 and V_1 assume the following forms:

$$v_1 = m_1 + \frac{C_{y1}}{C_{x1}}(r_1 - x_1); \quad V_1 = M_1 + \frac{C_{z1}}{C_{x1}}(r_1 - x_1). \quad \text{(iv)}$$

When the object-point lies in the meridian plane at an infinite distance, let its angular distance be w_1. We shall then have

$$C_{x1} = \cos w_1; \quad C_{y1} = \sin w_1; \quad C_{z1} = 0$$

and

$$v_1 = m_1 + (r_1 - x_1)\tan w_1; \quad V_1 = M_1. \quad \ldots \quad \text{(v)}$$

By our previous reasoning we may now obtain the following co-ordinates of the points of intersection in a plane at right angles to the axis and at a distance s' from the refracting surface—viz.:

$$Y = v' - \frac{C_y'}{C_x'}(r_k - s'); \quad Z = V' - \frac{C_z'}{C_x'}(r_k - s'). \quad \text{(vi)}$$

B.—Formulæ for the Transition from one Medium to another.

41. Kerber's Refraction Formulæ.—In addition to the quantities which already occur in the transition formulæ we must now introduce a few others. Let the meridian and the horizontal planes intersect the sphere of radius r along two great circles SA

and SA_h, and let A and A_h be the points where the lines passing through CB and CB_h meet the sphere. AC and A_hC may then be regarded as the **secondary axes** of refraction with respect to the two axial object-points B and B_h, as well as the aperture angles $v = ABI$ and $V = A_hB_hI$, which will enable us, by the method of Kerber (*8.*) to make immediate use of the formulæ established in § 28.

Let ($\boldsymbol{s}$, $\boldsymbol{l}$, 0) be the co-ordinates of B, and ($\boldsymbol{S}$, 0, $\boldsymbol{L}$) those of B_h. It is then required to determine after refraction the co-ordinates of the corresponding points B' and B_h'.

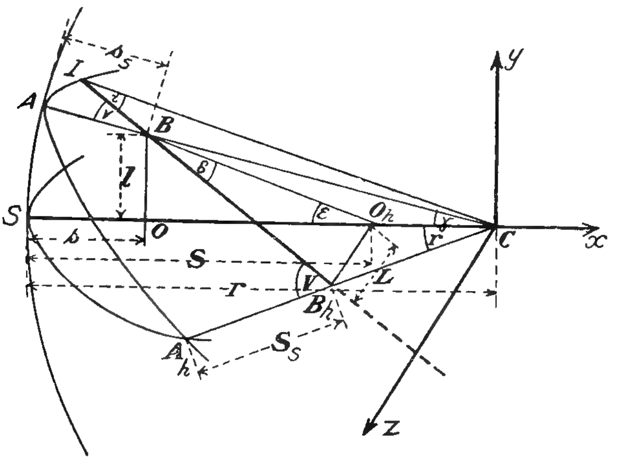

FIG. 20.

$SO = \boldsymbol{s}$; $SO_h = \boldsymbol{S}$; $OB = \boldsymbol{l}$; $O_hB_h = \boldsymbol{L}$; $AB = \boldsymbol{s}_s$; $A_hB_h = \boldsymbol{S}_s$.

Diagram showing notation for Kerber's transition formulæ.

From Fig. 20 we obtain at once expressions for the angles ε and δ, as defined above, viz.:

$$\tan \varepsilon = \frac{OB}{OO_h} = \frac{OB}{OS + SO_h} = \frac{\boldsymbol{l}}{\boldsymbol{S} - \boldsymbol{s}}$$

$$\tan \delta = \frac{O_hB_h}{O_hB} = \frac{O_hB_h \cos \varepsilon}{OO_h} = -\frac{\boldsymbol{L} \cos \varepsilon}{\boldsymbol{S} - \boldsymbol{s}},$$

whereas the angles γ and Γ between the axis of the system and the two secondary axes respectively, can be expressed by the following relations:

$$\tan \gamma = \frac{\boldsymbol{l}}{r - \boldsymbol{s}} \text{ and } \tan \Gamma = \frac{\boldsymbol{L}}{r - \boldsymbol{S}}.$$

The two distances BC and $B_{\prime\prime}C$, from the centre of the sphere, are equivalent to

$$r - s_s = \frac{r - s}{\cos\gamma}\,; \qquad\qquad r - S_s = \frac{r - S}{\cos\Gamma},$$

and the aperture angles with respect to the secondary axes are

$$\cos \mathrm{v} = \cos(\varepsilon - \gamma)\cos\delta\,; \text{ and } \sin \mathrm{V} = \frac{r - s_s}{r - S_s}\sin \mathrm{v}\,.$$

This supplies us with all the elements which are necessary for the application of the formulæ given in § 28. Accordingly,

$$\sin i = \frac{s_s - r}{r}\sin \mathrm{v} \qquad \ldots \qquad \ldots \qquad \text{(i)}$$

$$\sin i' = \frac{n}{n'}\sin i \quad \ldots \qquad \ldots \qquad \ldots \qquad \text{(ii)}$$

$$\mathrm{v}' = \mathrm{v} + i - i'\,; \quad \mathrm{V}' = \mathrm{V} + i - i' \qquad \ldots \qquad \text{(iii)}$$

$$s'_s - r = \frac{r\sin i'}{\sin \mathrm{v}'}\,; \quad S'_s - r = \frac{r\sin i'}{\sin \mathrm{V}'}\,. \qquad \ldots \qquad \text{(iv)}$$

Reverting to the primary axes of the system, we have

$$\begin{aligned} r - s' &= (r - s'_s)\cos\gamma\,; & r - S' &= (r - S'_s)\cos\Gamma \\ l' &= (r - s')\tan\gamma\,; & L' &= (r - S')\tan\Gamma \qquad \text{(v)} \end{aligned}$$

When a radius is infinitely great the co-ordinate system coincides with S and the secondary axes are parallel to the primary axis. Hence we have in this case

$$\left.\begin{array}{c} l' = l\,; \quad L' = L\,; \\ \gamma = 0\,; \quad \Gamma = 0\,; \\ \cos \mathrm{v} = \cos\varepsilon\cos\delta\,; \\ s_s = s\,; \quad S_s = S\,; \quad \mathrm{V} = \mathrm{v}\,; \\ \sin \mathrm{v}' = \dfrac{n}{n'}\sin \mathrm{v}\,; \\ s'_s = s_s\dfrac{\tan \mathrm{v}}{\tan \mathrm{v}'}\,; \quad \mathrm{V}' = \mathrm{v}'\,; \quad S'_s = S_s\dfrac{\tan \mathrm{v}}{\tan \mathrm{v}'} \\ s' = s'_s\,; \quad S' = S'_s\,. \end{array}\right\} \qquad \text{(vi)}$$

When the object-point is at infinity S and L are expressed in terms of the initial data, as was done in § 38. Since, however, we must take the object-point in the meridian plane, we have

$$\left.\begin{array}{c} \delta = 0\,;\ \gamma = \varepsilon \\ \tan \Gamma = \dfrac{L}{r - S} \\ s_s = \infty\,;\quad r - S_s = \dfrac{r - S}{\cos \Gamma}\,;\quad \mathrm{v} = 0\,. \end{array}\right\} \quad \ldots \quad \text{(vii)}$$

From the right-angled spherical triangle at B_h

$$\cos \mathrm{V} = \cos \varepsilon \cos \Gamma\,, \quad \ldots \quad \ldots \quad \text{(viii)}$$

and finally we have with respect to the secondary axis

$$\sin i = \frac{S_s - r}{r} \sin \mathrm{V}\,, \quad \ldots \quad \ldots \quad \text{(ix)}$$

which brings the calculation into a known form.

When the values of v are small the relation

$$\cos \mathrm{v} = \cos (\varepsilon - \gamma) \cos \delta$$

is not adapted for numerical operations. Greater accuracy is in this case obtainable by determining the invariant angle η between the plane of incidence and the vertical plane from the formula

$$\tan \eta = \frac{\tan \delta}{\sin (\varepsilon - \gamma)}\,,$$

which gives us

$$\sin \mathrm{v} = \frac{\sin \delta}{\sin \eta}\,.$$

If now, in accordance with Kerber's formula, we proceed to calculate in terms of

$$l,\ s,\ \delta,\ \varepsilon\,,$$

we may altogether dispense with the calculation of

$$\Gamma,\ S_s\,,\ \mathrm{V}.\ \mathrm{V}',\ S'_s\,,\ S',\ L',$$

and accordingly, we may confine ourselves to the quantities denoted

by small letters. For the determination of δ' we may then use the alternative formulæ given above for small values of v,

Thus

$$\sin \delta' = \frac{\sin v'}{\sin v} \sin \delta ,$$

whilst ε' may be calculated from

$$\cos(\varepsilon' - \gamma) = \frac{\cos v'}{\cos \delta'}$$

or from

$$\sin(\varepsilon' - \gamma) = \frac{\tan \delta'}{\tan \eta} .$$

This method may be shortened still further by omitting the calculation of l_1' ; for since $l_2 = l_1'$, we can write

$$\tan \gamma_2 = \frac{s_1' - r_1}{s_2 - r_2} \tan \gamma_1 .$$

Only when the object-point is at an infinite distance, is it necessary in this scheme of computation to determine Γ.

In this case

$$\delta = 0 ; \quad \gamma = \varepsilon$$

$$\tan \Gamma = \frac{M}{r - (x + m \cot w_1)}$$

$$\cos V = \cos \gamma \cos \Gamma$$

$$\sin i = \frac{M}{r \sin \Gamma} \sin V ,$$

and, naturally,

$$v' = i - i'.$$

Since before refraction

$$\sin \eta = \frac{\sin \Gamma}{\sin V} ,$$

we derive from the general equation

$$\sin \eta = \frac{\sin \delta'}{\sin v'}$$

the formula

$$\sin \delta' = \frac{\sin \Gamma}{\sin V} \sin (i - i') ,$$

which brings us back to our standard method.

42. Seidel's Refraction Formulæ.—In Fig. 21, which embodies several new terms introduced by Seidel (*6*), let PQ be a skew ray, P being the point where it meets the refracting spherical surface. In the first place, it is required to compute the angle of incidence i = QPM. Let λ be the angle PQM. About Q as centre let the spherical triangle RST be described. Then it follows by the theorem of cosines that

$$\cos (180^\circ - \lambda) = \cos \tau \cos 90^\circ + \sin \tau \sin 90^\circ \cos (\pi - \zeta) ; \quad \text{(i)}$$
$$- \cos \lambda = \sin \tau \cos (\pi - \zeta) .$$

When the radius is negative in sign, λ should be replaced by its supplementary angle $180^\circ - \lambda$, in which case the equation will be

$$\cos \lambda = \sin \tau \cos (\pi - \zeta).$$

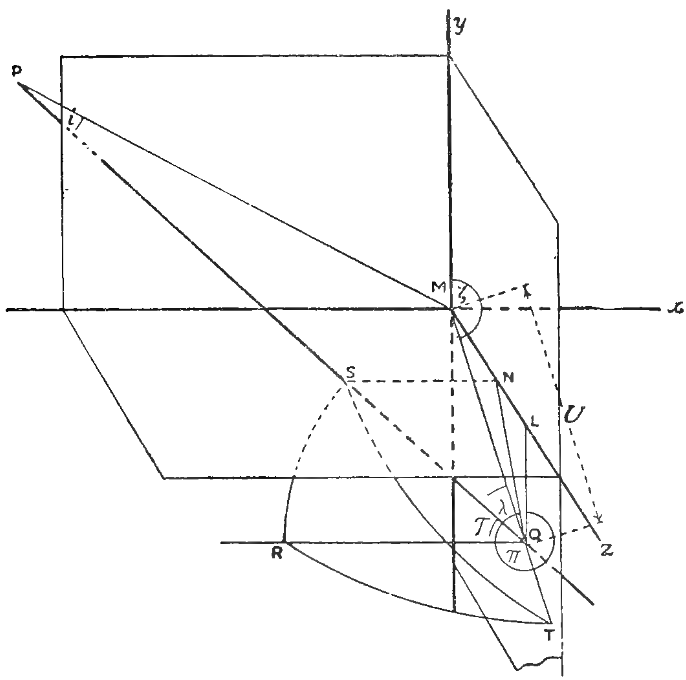

FIG. 21.

N = projection of S on MZ ; L = projection of Q on MZ ; QR = normal to the plane of yz ; therefore RQT = 90° ;

λ = PQM
$\pi = 360^\circ$ − NQL
$\zeta = y$MQ = LQT
} in plane of incidence PMQTS; $\pi - \zeta = 360^\circ$ − NQL − LQT = NQT = SRT ; MQ = U

Diagram for Seidel's refraction formulæ.

Applying the law of sines to the triangle PMQ,

$$\sin i = \frac{U}{r} \sin \lambda \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

$$\sin i' = \frac{n}{n'} \sin i \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

$$\lambda' = \lambda + (i - i') \quad \ldots \quad \ldots \quad \text{(iv)}$$

$$U' = \frac{r \sin i'}{\sin \lambda'} = U \frac{n \sin \lambda}{n' \sin \lambda'} \,. \quad \ldots \quad \ldots \quad \text{(v)}$$

Similarly, we may describe a spherical triangle R′S′T′ with respect to the point Q′, whose position is defined by the refracted ray PQ′ within the plane of incidence. This triangle and the triangle RST have similar angles at T, T′, since the two planes comprising this angle, viz., the plane of yz containing the axis and U and the plane of incidence, do not change in consequence of refraction. Applying the law of sines to both spherical triangles, we have with respect to the equal angles at T and T′

$$\frac{\sin \tau \sin (\pi - \zeta)}{\sin \lambda} = \frac{\sin \tau' \sin (\pi' - \zeta)}{\sin \lambda'}, \quad \ldots \quad \text{(vi)}$$

which, in conjunction with the relation

$$- \cos \lambda' = \sin \tau' \cos (\pi' - \zeta)$$

enables us to calculate π' and τ'.

In the triangles RST, R′S′T′ we have likewise for the angles at T, T′

$$\frac{\cos \tau}{\sin \lambda} = \frac{\cos \tau'}{\sin \lambda'}.$$

This relation is ill adapted for numerical calculation; but when taken in conjunction with the first of the two preceding equations we obtain the expression

$$\tan \tau' = \tan \tau \frac{\sin (\pi - \zeta)}{\sin (\pi' - \zeta)}, \quad \ldots \quad \ldots \quad \text{(vii)}$$

which is particularly well adapted for the calculation of small angles τ'.

This gives us all the new co-ordinates of Seidel's system, viz.:

$$U', \zeta, \pi', \tau',$$

and we are now in a position to apply the transformation formulæ given above.

Seidel gives a number of control equations, which we will briefly state without giving their proofs.

$$\frac{\sin(i-i')}{\sin(\pi-\pi')}=\frac{\sin\lambda\sin\tau'}{\sin(\pi-\zeta)}=\frac{\sin\lambda'\sin\tau}{\sin(\pi'-\zeta)}. \quad \ldots \quad \text{(viii)}$$

Putting

$$\frac{n}{n'}=\tan\omega\,,$$

this will give us yet another control equation of the form

$$\frac{\sin(i-i')\sin(i+i')}{\sin i\sin i'}=2\cot 2\,\omega. \quad \ldots \quad \text{(ix)}$$

Finally, we have the following identities for checking the transformation formulæ given above:

$$\frac{c\tan\tau'}{\sin(\zeta-\zeta_1)}=\frac{U_1}{\sin(\pi'-\zeta)}=\frac{U'}{\sin(\pi'-\zeta_1)}. \quad \ldots \quad \text{(x)}$$

If one of the refracting surfaces happens to be a plane surface, let the latter, as before, be the transverse plane of reference. In this case

$$U'=U$$

and

$$i=-\tau\,;\quad \tau'=-i'\,;$$

hence

$$\sin\tau'=\frac{n}{n'}\sin\tau$$

and also

$$\pi'=\pi\,,$$

since the plane of incidence is at right angles to the refracting plane surface before, as well as after, refraction.

The computation of a skew ray proceeding from an object-point at a very great distance does not occasion any difficulty, if the co-ordinates of the ray are determined from the initial data in accordance with our previous investigation.

43. Bruns' Refraction Formulae.—When a straight line is defined by its direction cosines C_x, C_y, C_z and the co-ordinates $\bar{X}$, $\bar{Y}$, $\bar{Z}$ of any point through which it passes, its general equation will be

$$\frac{X-\bar{X}}{C_x}=\frac{Y-\bar{Y}}{C_y}=\frac{Z-\bar{Z}}{C_z}=l$$

In our special case, since we are concerned with the co-ordinates of the point where the ray intersects the plane of $y\,z$, these equations assume the forms

$$\frac{X}{C_x}=\frac{Y-v}{C_y}=\frac{Z-V}{C_z}=l.$$

If now we let the parameter l assume increasing values from $-\infty$ to $+\infty$, it follows that the numerator of the first term will likewise increase so long as C_x is positive. Now C_x must be positive as a necessary and sufficient condition that the straight line may correspond with a ray proceeding in the direction of the light, which we must suppose to be always from left to right.

If now we determine the point of intersection (X_1, Y_1, Z_1) of this straight line with a sphere of radius r about the centre of the refracting surface, whose equation is $X^2 + Y^2 + Z^2 = r^2$, we find the following expression for $l = l_1$:

$$l_1 = -(C_y v + C_z V) \pm \sqrt{r^2 - (v^2 + V^2) + (C_y v + C_z V)^2},$$

which likewise serves for determining the values of X_1, Y_1, Z_1. It will be readily seen on reference to Fig. 22 that l_1 represents geometrically the distance AI.

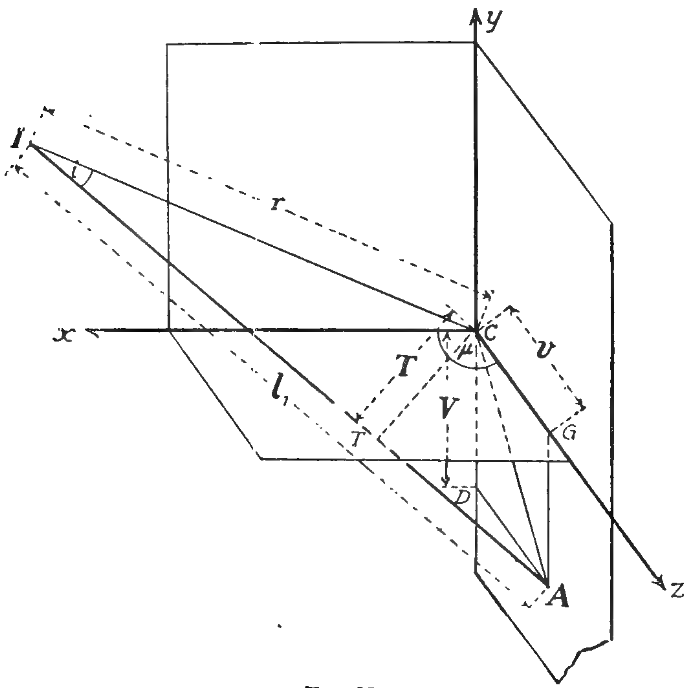

FIG. 22.

$CG = v$; $CD = V$; $CT = T$; $AI = l_1$; $IC = r$.

Diagram for Bruns' refraction formulae.

Of the two roots of the above equation one only is applicable to the optical problem with which we are dealing. In fact, the choice of the root is determined by the following consideration :—

When the radius of the refracting surface is positive (remembering that by our convention of signs the positive direction of light is that from left to right), we have only to consider the smaller value of l_1, which defines the position of the first point of intersection, since at this point the ray changes its direction. When the radius is positive the lower sign of the root expression must be chosen.

When the sign of the radius of the sphere is negative we have only to consider the greater value of l_1, which defines the second point of intersection, and hence in this case we must choose the upper sign of the square root.

Let CT be a perpendicular let fall from C upon the ray IA. Then its length $CT = T$ will be

$$T = \pm \sqrt{(v^2 + V^2) - (C_y v + C_z V)^2}\,;$$

also

$$\sin i = \frac{T}{r}\,.$$

The sign of T is immaterial in this scheme of computation, since in what follows the acute angles i, i' occur in terms of cosine functions only.

With the aid of this relation we may now write

$$l_1 = -\,(C_y v + C_z V) \pm r \cos i\,,$$

or with due regard to the proper choice of the sign, as just explained,

$$l_1 = -\,(C_y v + C_z V) - r \cos i\,.$$

The direction cosines of IC, the normal at the point of incidence, are

$$\cos(r, X) = -\frac{X_1}{r}\,;\ \cos(r, Y) = -\frac{Y_1}{r}\,;\ \cos(r, Z) = -\frac{Z_1}{r}\,,$$

whence

$$\cos(r, X) = -\frac{C_x l_1}{r}\,;\quad \cos(r, Y) = -\frac{C_y l_1 + v}{r}\,;$$

$$\cos(r, Z) = -\frac{C_z l_1 + V}{r}\,.$$

Also, from the analysis developed in § 16, we have the following equations

$$n' C'_x - n C_x = A \cos(r, X)$$

$$n' C'_y - n C_y = A \cos(r, Y)$$

$$n' C'_z - n C_z = A \cos(r, Z)\,,$$

where

$$A = n' \cos i' - n \cos i\,.$$

From these sets of equations we obtain finally

$$\frac{C_x'}{C_x} = \frac{1}{n'}\left(n - A\,\frac{l_1}{r}\right)$$

$$C'_y = \frac{C_x'}{C_x}\,C_y - v\,\frac{A}{n'r}$$

$$C_z' = \frac{C_x'}{C_x}\,C_z - V\,\frac{A}{n'r},$$

where for A, instead of the expression just given, we may write the alternative form of equation § 34 (vii), viz.:

$$A = \frac{n'}{\sin i}\sin(i - i').$$

Let μ be the angle ICA subtended by the skew ray IA which is not changed by refraction. Then

$$\frac{\sin i}{\sin \mu} = \frac{CA}{l_1},$$

therefore also

$$\frac{\sin i'}{\sin \mu} = \frac{CA'}{l_1'},$$

and hence

$$\frac{\sin i'}{\sin i} = \frac{n}{n'} = \frac{l_1}{l_1'}\,\frac{CA'}{CA}.$$

Now

$$\frac{CA'}{CA} = \frac{v'}{v} = \frac{V'}{V},$$

and also

$$C_x'\,l_1' = C_x\,l_1,$$

since the projections of l_1 and l_1' upon the axis of x are of equal length.

This establishes the equations

$$v' = \frac{n}{n'}\,\frac{C_x}{C_x'}\,v\,,\ \text{and}\ V' = \frac{n}{n'}\,\frac{C_x}{C_x'}\,V,$$

whereby it will be seen that from the data obtaining before refraction, viz.:

$$C_x,\ C_y,\ C_z,\ v,\ V,$$

we have derived the corresponding quantities after refraction, viz.:

$$C_x',\ C_y',\ C_z',\ v',\ V'.$$

When the radius is infinitely great the refracting surface at right angles to the axis becomes the transverse plane and the normal of incidence will be parallel to the axis of x, so that

$$n'C_x' - nC_x = A\ ;\quad n'C_y' - nC_y = 0\ ;\quad n'C_z' - nC_z = 0\,.$$

Moreover,

$$C_x = \cos(-i) \text{ and } C_x' = \cos(-i')\,;$$

also

$$C_y' = \frac{n}{n'}C_y\ ;\quad C_z' = \frac{n}{n'}C_z\ ;$$

and finally,

$$v' = v \text{ and } V' = V\,.$$

C. Rays Inclined to the Meridian Plane at an Infinitely Small Angle.

44. Derivation of the Formulæ for Sagittal Rays, based on Kerber's Method.—When it is required to determine

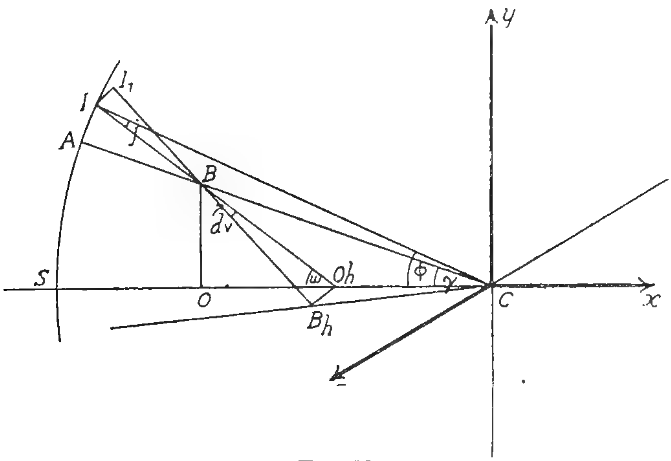

Fig. 23.

$IB = f\ ;\ O_h B_h = dL\ ;\ AB = s_s\,.$

Diagram of notation for the computation of sagittal rays by Kerber's method

sagittal rays corresponding to the previously computed principal ray whose co-ordinates are x, w, let the latter be represented in Fig. 20 by BO_h, so that now $\varepsilon = w$, and the ordinate L will be replaced by a small quantity dL; hence it will also be necessary for us to treat the angular aperture of the sagittal pencil as an infinitely small quantity $dv = \delta$.

Since the positions of the two points I and I_1, where the refracting surface is intersected by the rays, are not affected by the refraction, it follows that the expression for the arc between them remains likewise unchanged when the unaccented terms are replaced by accented quantities. In Fig. 23 let IB, the distance of the sagittal object-point from the point where the ray intersects the sphere, be denoted by f. Then the magnitude of the arc II_1 corresponding to infinitely small values $d\mathrm{v}$ will be

$$\widehat{II_1} = f\,d\mathrm{v} = f'\,d\mathrm{v}' \qquad \ldots \qquad \ldots \qquad \text{(i)}$$

In the relations established in § 41, (i, ii, iv), if we replace v by $d\mathrm{v}$, we may now write the ratio of the sines of δ' and δ as follows

$$\frac{\sin\delta'}{\sin\delta} = \frac{d\mathrm{v}'}{d\mathrm{v}} = \frac{\sin v'}{\sin v} = \frac{\sin j'}{\sin j}\cdot\frac{s_s - r}{s'_s - r} = \frac{n}{n'}\cdot\frac{s_s - r}{s'_s - r} \qquad \text{(ii)}$$

It will now be necessary to establish a relation between s_s and f. It should be noted that the intercepts with which we are now concerned are all in the plane of the paper; also, that the angle ICO is the angle ϕ subtended at the centre of the sphere, which is not changed by refraction. We have accordingly the following relations between the angles:

$$IAB = 90^\circ - \frac{\phi - \gamma}{2}; \quad AIB = 90^\circ - \frac{\phi - \gamma}{2} - j,$$

so that in the triangle IAB we shall have the following relation between the sides and angles:

$$f \,/\, s_s = \cos\frac{\phi - \gamma}{2} \,/\, \cos\left(\frac{\phi - \gamma}{2} + j\right),$$

whence

$$s_s \doteq f\cos j - f\tan\frac{\phi - \gamma}{2}\sin j,$$

and similarly,

$$s'_s = f'\cos j' - f'\tan\frac{\phi - \gamma}{2}\sin j'.$$

Substituting these relations in the above expressions for $\frac{d\mathrm{v}'}{d\mathrm{v}}$ in terms of s_s, we obtain the equation

$$n'd\mathrm{v}'\,(f'\cos j' - r) - \tan\frac{\phi - \gamma}{2}\,n'\sin j' f'\,d\mathrm{v}'$$

$$= n\,d\mathrm{v}\,(f\cos j - r) - \tan\frac{\phi - \gamma}{2}\,n\sin j\,f\,d\mathrm{v}.$$

In this equation the second terms on either side cancel out, since, as we have shown above, the arc $\widehat{II_1} = f' dv' = f dv$, so that finally we have

$$\frac{dv'}{dv} = \frac{f}{f'} = \frac{n\ (f \cos j \ - r)}{n' (f' \cos j' - r)} \quad . \quad \ldots \quad \text{(iii)}$$

By a simple transformation this gives us the formula for the sagittal intercepts measured on the principal ray, thus :

$$\frac{n'}{f'} = \frac{n}{f} + \frac{n' \cos j' - n \cos j}{r} \ . \ \ldots \quad \ldots \quad \text{(iv)}$$

Here the astigmatic constant defined in § 34 (vii), which again occurs, can be written as before, thus :

$$n' \cos j' - n \cos j = \frac{n}{\sin j'} \sin (j - j') = \frac{n'}{\sin j} \sin (j - j') \ . \quad \text{(v)}$$

No difficulties arise here when the object-point is at infinity.

The transition from f' to the f of the succeeding surface is made by introducing the oblique thickness as in the case of tangential pencils.

Since the object may be supposed to be free from astigmatism, f_1 can be equated to the value of t_1 as given in § 34 (viii) ; and, finally, after computing the ray through k surfaces, we shall obtain an analogous expression for the abscissa of the sagittal image-point on the principal ray, reckoned from the vertex of the last surface, viz. :

$$\bar{\bar{s}}_k' = f_k' \cos w_k' + 2\ r_k \sin^2 \frac{\phi_k}{2} \ . \quad \ldots \quad \text{(vi)}$$

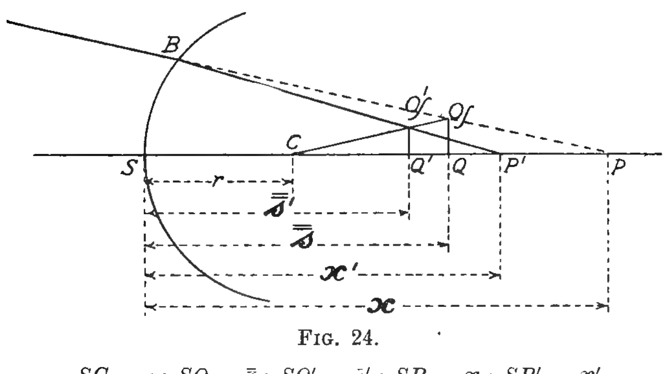

FIG. 24.

$SC = r$; $SQ = \bar{\bar{s}}$; $SQ' = \bar{\bar{s}}'$; $SP = x$; $SP' = x'$.

Diagram of notation for the direct computation of the axial position of the image-receiving plane.

It is also possible from the abscissa $\overline{\overline{s}}$ of the perpendicular plane containing the object-point, to arrive directly at an expression for s', the abscissa of the receiving plane of the image, by availing ourselves of a principle which will be proved in Chapter IV., where it will be shown that C is the perspective centre of the conjugate sagittal points.

If we express the cotangent of the invariant angle $O_f CQ$ both in terms of the accented and unaccented quantities s, x and w, it will be seen from Fig. 24 that

$$\frac{\overline{s}' - r}{x' - \overline{\overline{s}}'} \cot w' = \frac{\overline{\overline{s}} - r}{x - \overline{\overline{s}}} \cot w$$

or

$$\frac{x' - r}{x' - \overline{\overline{s}}'} \cot w' = \frac{x - r}{x - \overline{\overline{s}}} \cot w + \cot w' - \cot w .$$

With the aid of the relation $w - w' = -(j - j')$ this expression becomes

$$\frac{x' - r}{x' - \overline{\overline{s}}'} \cot w' = \frac{x - r}{x - \overline{\overline{s}}} \cot w - \frac{\sin (j - j')}{\sin w \sin w'} \ldots \quad \text{(vii)}$$

The advantage of this formula lies in the fact that for the transition from the v^{th} to the $(v + 1)^{\text{th}}$ surface we have the identity

$$x_v' - \overline{\overline{s}}_v' = x_{v+1} - \overline{\overline{s}}_{v+1} . \qquad \ldots \quad \text{(viii)}$$

45. Derivation of Wanach's Formulæ for Sagittal Rays, by Seidel's Method.—A sagittal ray and the principal ray to which it is adjacent do not differ as regards their respective values of U, u and W, w. The $d\pi$ and $d\zeta$ values of the sagittal ray are small quantities which may be determined in the course of calculation in the following manner.

The formulæ established in § 42 (vii, x) for skew rays of any position and direction, viz. :

$$\tan w' = \frac{\sin (\pi - \zeta)}{\sin (\pi' - \zeta)} \tan w \; ; \quad W_1 \sin (\pi' - \zeta_1) = W' \sin (\pi' - \zeta)$$

now assume this form :

$$d\pi' - d\zeta = \frac{\tan w}{\tan w'} (d\pi - d\zeta) \; ; \quad d\pi' - d\zeta_1 = \frac{W'}{W_1} (d\pi' - d\zeta), \quad \text{(i)}$$

and if one of the radii is infinitely great, in that case $d\pi' = d\pi$ for this plane surface.

It remains now to establish relations between the initial and final values and the quantities $d\pi$ and $d\zeta$ obtaining before the first and after the last refraction (Fig 25).

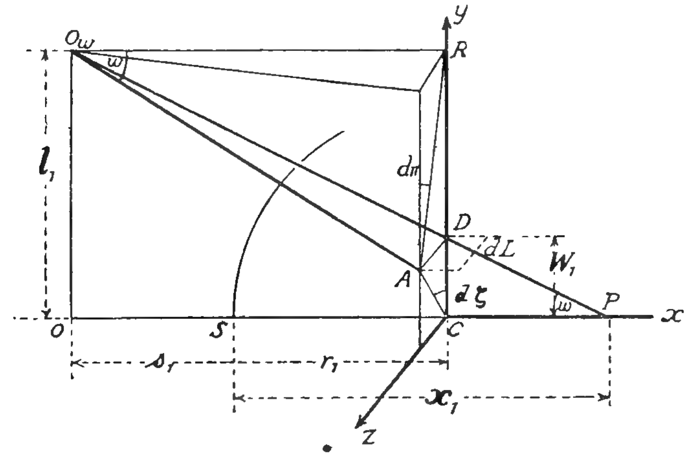

FIG. 25.

$OO_w = l_1$; $SO = s_1$; $SC = r_1$; $SP = x_1$; $CD = W_1$; $DA = dL$.

Wanach's derivation of the formulæ for sagittal rays by Seidel's method

Accordingly, let a given object-plane at right angles to the axis and at a distance $SO = s_1$ from the refracting surface be intersected by the principal ray at a point O_w whose ordinate is

$$l_1 = (x_1 - s_1) \tan w_1 .$$

Equating by Wanach's method, the two values of dL in terms of a side and an angle in the triangles ARD, CAD, and substituting

$$l_1 - W_1 = (r_1 - s_1) \tan w_1 ,$$

we obtain the equation:

$$d\pi_1 = - \frac{W_1 d\zeta_1}{(r_1 - s_1) \tan w_1} . \quad \dots \quad \dots \text{(ii)}$$

In consequence of this relation all differences of the type

$$d\pi_v{}' - d\,\zeta_v ; \quad d\pi_v{}' - d\,\zeta_{v+1}$$

determined in accordance with the present scheme, are proportional to $d\,\zeta_1$, which may accordingly be replaced by a finite multiple ε.

When the object-point is at an infinite distance, $d\pi_1$ vanishes owing to s_1 being infinite.

By analogous reasoning the distance of the last sagittal image-point from the vertex to the last surface is

$$\bar{\bar{s}}_k{}' = r_k + \frac{W_k{}'\,d\zeta_k}{\tan w_k{}'\,d\pi_k{}'} . \quad \dots \quad \dots \text{(iii)}$$

4. KERBER'S FORMULÆ FOR DETERMINING THE DIFFERENCES OF THE INTERCEPTS OF RAYS PROCEEDING FROM A POINT ON THE AXIS.

46. In all the methods discussed in the preceding articles the co-ordinates of the axial point of intersection and the point where the computed ray intersects a surface of reference were obtained directly. By subtraction their differences could be determined for suitably chosen normal values. These differences constitute appropriate measures of what we are accustomed to regard as **defects** in the union of rays. Adopting the method of Kerber (**7.**), we shall now formulate expressions for determining these differences for rays proceeding from a point on the axis.

Let e, e' be the lengths of the normals let fall from the vertex of the surface S upon the direction of the ray. We thus obtain the following expressions which will be used later, namely :

$$e' = s' \sin u' ; \qquad e = s \sin u . \quad \ldots \qquad \ldots \quad \text{(i)}$$

In the case of paraxial rays, where the tangents and sines of the small angles are mutually interchangeable, these quantities become equal to one another and also equal to the ordinate of the point of incidence, so that $e' = e = h = sdu = s'du'$.

The identity

$$\sin u' = \sin u + \sin i - \sin i' + D, \qquad \ldots \quad \text{(ii)}$$

and hence

$$D = -\sin i + \sin i' - \sin u + \sin u' \qquad \ldots \text{(iii)}$$

gives us a quantity by the aid of which the difference formula may be very elegantly derived. There will be given later a logarithmic expression which renders it convenient for practical calculations.

From the formulæ given in § 28 we see that

$$\frac{1}{s' - r} = \frac{n' \sin u'}{n \sin u} \frac{1}{s - r} .$$

If now in the expression just given we eliminate $\sin u'$, noting also that

$$\sin i - \sin i' = \frac{n' - n}{n'} \frac{s - r}{r} \sin u ,$$

we have

$$\frac{1}{s' - r} = \frac{n'}{n} \left(\frac{1}{s - r} + \frac{n' - n}{n'} \cdot \frac{1}{r} + \frac{D}{(s - r) \sin u} \right) ;$$

further, by the use of the formula established for the intercepts of the paraxial rays, it will be seen that

$$\frac{1}{s' - r} = \frac{n'}{n}\left(\frac{1}{s - r} + \frac{n' - n}{n'} \cdot \frac{1}{r}\right).$$

By subtraction of the two equations we find

$$\frac{\delta s'}{(s' - r)\,(\boldsymbol{s}' - r)} = \frac{n'}{n}\,\frac{\delta s}{(s - r)\,(\boldsymbol{s} - r)} - \frac{n'}{n}\,\frac{D}{(\boldsymbol{s} - r)\sin u},$$

where $\delta s'$, δs denote the finite differences

$$\delta s' = \boldsymbol{s}' - s'\,; \quad \delta s = \boldsymbol{s} - s\,.$$

Re-introducing the angles by means of the equation just employed, viz.,

$$n'\,(\boldsymbol{s}' - r)\sin u' = n\,(\boldsymbol{s} - r)\sin u\,,$$

and finally applying the zero invariant defined in § 31 and introducing the incidence height h of the paraxial rays, we have

$$n'\,\delta s' \sin u'\,\frac{h}{s'} - n\,\delta s \sin u\,\frac{h}{s} = - D\,r\,h\,Q_s\,. \qquad \text{(iv)}$$

We must now provide these quantities with indices to indicate that they relate to successive surfaces of a centred system, and thus obtain the following set of equations;

$$n_1'\,\delta s_1' \sin u_1'\,\frac{h_1}{s_1'} - n_1\,\delta s_1 \sin u_1\,\frac{h_1}{s_1} = - D_1\,r_1\,h_1\,Q_{1s}\,,$$

$$n_2'\,\delta s_2' \sin u_2'\,\frac{h_2}{s_2'} - n_2\,\delta s_2 \sin u_2\,\frac{h_2}{s_2} = - D_2\,r_2\,h_2\,Q_{2s}$$

. . . .

$$n_k'\,\delta s_k' \sin u_k'\,\frac{h_k}{s_k'} - n_k\,\delta s_k \sin u_k\,\frac{h_k}{s_k} = - D_k\,r_k\,h_k\,Q_{ks}\,.$$

By addition of this series of formulæ the first term of each row on the left side will cancel the second term of the next row, since, as a brief consideration will show, all paraxial rays are connected by the identity

$$\frac{h_v}{s_v'} = \frac{h_{v+1}}{s_{v+1}}\,. \qquad \ldots \qquad \ldots \qquad \ldots \qquad \text{(v)}$$

Since we may assume that before the first refraction $\delta s_1 = 0$, we shall have finally on the left only the first term of the last row, so that the resulting equation will be

$$\delta s_k' = -\frac{s_k' \sum\limits_{v=1}^{k} D_v r_v h_v Q_{vs}}{n_k' h_k \sin u_k'} = -\frac{s_k'}{n_k' \sin u_k'} \frac{h_1}{h_k} \sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right) D_v r_v Q_{vs}. \text{ (vi)}$$

Since in this expression we are concerned only with small quantities which can be computed directly, we can obtain very accurate results with the aid of tables of but few significant figures.

This method has moreover the advantage that it separates in the final result the differences due to each of the successive surfaces.

It only remains to transform the expression

$$D = -\sin i + \sin i' - \sin u + \sin u',$$

so that it may become available for logarithmic calculation.

Now, in the first place, the differences of the sines can be readily transformed into products, thus :

$$D = -2 \cos \frac{i + i'}{2} \sin \frac{i - i'}{2} + 2 \cos \frac{u + u'}{2} \sin \frac{u' - u}{2}.$$

Again from the relation $i - i' = u' - u$ we find

$$D = -2 \sin \frac{i - i'}{2} \left(\cos \frac{i + i'}{2} - \cos \frac{u + u'}{2} \right).$$

Repeating the process with respect to the expression within brackets and noting that

$$i + i' + u + u' = 2(u + i) = 2\phi\,;\ i + i' - u - u' = 2(i' - u),$$

the expression for D assumes the simple form

$$D = 4 \sin \frac{i - i'}{2} \sin \frac{i' - u}{2} \sin \frac{\phi}{2}. \qquad \ldots \quad \text{(vii)}$$

This provides a convenient expression for the logarithmic calculation of the value of D.

Kerber, nevertheless, has proposed another expression, the derivation of which we also give here.

$$\sin\frac{\phi}{2} = \sin\frac{i+u}{2} = \sin\left(\frac{i-u}{2}+u\right) = \sin\frac{i-u}{2}\cos u + \cos\frac{i-u}{2}\sin u$$

$$= \frac{\frac{1}{2}\sin(i-u)\cos u + \cos^2\frac{i-u}{2}\sin u}{\cos\frac{i-u}{2}}$$

$$= \frac{\sin(i-u)\cos u - \sin\frac{i-u}{2}\cos\frac{i-u}{2}\cos u + \cos(i-u)\sin u + \sin^2\frac{i-u}{2}\sin u}{\cos\frac{i-u}{2}}$$

$$= \frac{\sin i - \sin\frac{i-u}{2}\left(\cos\frac{i-u}{2}\cos u - \sin\frac{i-u}{2}\sin u\right)}{\cos\frac{i-u}{2}}$$

$$= \frac{\sin i + \sin u}{2\cos\frac{i-u}{2}} .$$

Further noting that

$$\frac{e}{r} = \frac{s-r}{r}\sin u + \sin u = \sin i + \sin u ,$$

and introducing in the above expression for D that just found for $\sin\frac{\phi}{2}$, Kerber's formula becomes

$$D = 2\,\frac{e}{r}\,\frac{\sin\frac{i-i'}{2}\sin\frac{i'-u}{2}}{\cos\frac{i-u}{2}} . \qquad \ldots \text{(viii)}$$

It will thus be seen that the expression for D can be given a convenient form for logarithmic calculation.

Our next step would now appear to be to give the corresponding formulæ for skew rays. Unfortunately, we are not able to do so.

Seidel (*6.*) seems to have been in possession of these formulæ, though he did not publish them. This would appear from the following reference :—

" In conclusion, however, let me remark that, in my opinion, the proper method to be adopted—that is, the method best suited to

the nature of the conditions in the computation of rays through optical combinations within and without the axial plane—differs radically from that which seeks to determine the integral quantities by which the position and inclination of a ray is defined after any number of refractions. In fact, the object of the practical method is to ascertain their departures from the standard values which result from the application of the approximate formulæ (of the first order). In this practical method the procedure is confined to small quantities which can be expressed with sufficient exactness by a small number of decimal places, since they provide a direct measure of what appears as a defect in the optical image. By this treatment the problem is likewise susceptible of an elegant solution by convenient expressions, which are related to the defects of the third order, as formulated by me in a previous investigation (in connection with the general treatment of rays in space), in the same manner as the equations of finite differences are distinguished from the differential formulæ. On the other hand, the procedure to which I am here referring departs somewhat radically from the accustomed practice of opticians, whose requirements I have sought to serve in this investigation. I shall therefore reserve further communications on the subject here indicated for another occasion."

References to the formulæ investigated in the preceding articles have already been given in the text. It will therefore be sufficient to indicate publications of numerical examples of the use of these formulæ.

Detailed applications of Seidel's method have been published by Steinheil and Voit (*3.*). Examples respecting rays proceeding from a point on the axis are given by Lummer (*2.526*). Similar examples have also been supplied by Gleichen (*4.430*); also, by the same author, calculations respecting astigmatic image-points on principal rays of finite inclination (*443*). Attention should also be drawn to the hints contributed by Wanach (*1.*) on the numerical computation of objectives.

The difference formula of Kerber (*7.7*) is likewise illustrated by a numerical example.

CHAPTER III.

ABBE'S GEOMETRICAL THEORY OF THE FORMATION OF OPTICAL IMAGES.

(E. WANDERSLEB.*)

1. THE PROBLEM AND ITS VARIOUS ASPECTS.

47. In the first chapter we have dealt fairly exhaustively with all those problems relating to reflection and refraction which are of material interest from a physical point of view. In the following discussions we may therefore confine our attention to the application of these investigations. These comprise on the one hand the explanation of certain natural phenomena, which mostly fall under the heading of meteorological optics, and provide on the other hand a basis for the construction of optical instruments.

In the succeeding chapters we shall occupy ourselves with the latter.

The ultimate object of an optical instrument is invariably to form images of objects, or images of such primary images, through the agency of reflections or refractions, or combinations of both, by an appropriate arrangement of suitably formed and combined optical media. In every case the image is called into existence by a process whereby some of the rays issuing from every individual point A of an object are so modified by reflection and refraction as to reconverge to another point, the so-called **image-point** A'.

Now, the laws which establish the relations between an object and its image have until quite recently been studied exclusively by

* This chapter is based upon the 2nd chapter of Czapski's "Theorie der Optischen Instrumente." Extensive departures will be met with in the reduction of the general image formation formulæ and in the section dealing with the classification of images, the latter being essentially that proposed by Eppenstein.

investigating in detail all special cases in which homocentrically diverging pencils are made to reconverge to a point, and then generalising the results of these special investigations by a process of induction. Even Gauss, to whom is due the greatest and most significant progress in the generalisation of the special theorems, proceeded in his famous "Dioptrische Untersuchungen" (*3.*) from special assumptions, confining his investigations to spherical surfaces ranged along a common axis and to paraxial pencils of rays of narrow section, both in the object-space and the image-space, besides which he accepted the truth of the law of refraction itself as the basis of his deductions, abandoning only the restriction of the investigation to infinitely thin lenses in contact, or spherical surfaces coinciding at their vertices. His avowed endeavour was to reduce the laws of optical image formation in a system of lenses of any composition to such simple expressions as those obtaining in the case of a single refracting surface or of a single lens of negligible thickness. Though he showed that the initial particulars of the system, viz., the radii, thicknesses and refractive indices, play a very subsidiary part in these laws, and that the fact of the formation of images depends upon constants of a much more general nature, he does not appear to have realised that the formation of optical images is fundamentally independent of the special means by which it is brought about.

Moebius (*4.*) seems to have been the first to suggest that the formation of axial images through refraction at a single spherical surface embodies the principle of collinear relationship, whence he concluded that all theorems on the optical effect of any system of reflecting and refracting spherical surfaces, arranged in any way whatever, express nothing more or less than the direct consequences of this relation between an object and its image due to a single refraction. His conclusions were taken up and his theory further developed and confirmed in the light of modern geometrical analysis by Lippich (*1.*), Beck (*1.*) and Hankel (*1.*), but, like Moebius, these authors retained the assumption that some kind of dioptric process was a necessary condition for establishing collinear relations in two spaces.

The first to abandon explicitly all reference to physical conditions in his deduction of the laws of the formation of images and to enunciate a purely geometrical theory was Clerk Maxwell (*1.2.*). He applied his investigation to a hypothetical "perfect" instrument which he defined without regard to the possibility of the means of its realisation by the following three requirements :—

I. *Every ray of a pencil proceeding from a single point of the object must, after passing through the instrument, converge to, or diverge from, a single point of the image.*

II. *If the object is a plane surface normal to the axis of the instrument, the image of any point of it must lie in a plane at right angles to the axis* (This definition expressly postulates the presence in the instrument of an axis, and, as will be seen by the next requirement, the axis is necessarily an axis of symmetry);

III. *The image of an object in this plane must be similar to the object.*

The relations connecting the magnitudes and positions of object and image which Clerk Maxwell (*I.*) deduces from these three purely geometrical premises agree with those derived by Gauss from the special physical conditions from which he proceeded.

The last remaining step was taken by Abbe, who about the year 1872, without any knowledge of the previous work of Moebius and Clerk Maxwell, developed in the course of his university lectures the geometrical theory of the formation of optical images by assuming nothing more than that optical transformation from one space into another *does* occur; in other words, that the quadruply infinite rays of one space are determinately conjugate to the rays in the other space in such a manner that to each pencil of rays passing through a single point in the first space there is a corresponding pencil of rays which passes through a single point in the other space.

This condition is sufficient to determine the essence and genesis of an optical image and is alone significant for the process by which an image comes into existence by the intervention of reflecting or refracting surfaces. Any geometrical and physical conditions which enter into the process are incidental only and merely concern, in the first place, the numerical values of the constants appearing in the general equations and left there necessarily undetermined; secondly, the geometrical disposition of the mutually related spaces and their position with respect to the boundaries between the physical media employed; and thirdly, the particular circumstances and restrictions governing the application, to any given case, of the general principle.

It appears, however, by no means superfluous to embody this aspect in the theory of optical instruments, and accordingly, to distinguish with the utmost precision between relations which can be deduced from the fundamental principles of optical transformation and those which are the consequences of special dioptric conditions. For, obviously, no investigation can be regarded as capable of intelligent application which omits to define clearly and to circumscribe the necessary assumptions and conditions upon which it rests.

2. THE GENERAL CHARACTERISTICS OF OPTICAL IMAGE-FORMATION.

A. The General Form of the Equations of Image-Formation.

48. We shall now investigate by purely mathematical means the optical transformation of one space into another, by virtue of which the quadruply infinite rays in one space are determinately conjugate to those of the other space in such a manner that each homocentric pencil of rays in the one space has corresponding to it one, *and only one*, homocentric pencil in the other space. The relation between the two spaces is completely equivalent and reversible. If, nevertheless, we retain the terms *object-space* and *image-space*, *object-ray* and *image-ray*, *object-point* and *image-point* we do so solely to facilitate expression.

Proceeding, accordingly, from the single assumption that corresponding to every homocentric pencil of rays in one space there is a pencil in the other, we are at once able to formulate the following important theorems :

1. Points on a straight line situated in one space have corresponding to them in the other space image-points which lie likewise on a straight line.

For, according to our hypothesis, the ray R passing through points P_1, P_2, P_3 on a straight line has corresponding to it a ray R' in the image-space which passes through points P_1', P_2', P_3', said to be conjugate to P_1, P_2, P_3, and since a ray is geometrically a straight line it follows that P_1', P_2', P_3' lie also on a straight line.

2. A plane S in the object-space has corresponding to it a plane S′ in the image-space. Since a plane is defined in position by two intersecting straight lines it follows from (1) that two straight lines a, b in the one space have corresponding to them two straight lines a', b' in the other space. Any other straight line c lying in the first plane, and therefore intersecting the other two straight lines, has corresponding to it in the image-space a straight line c', which, in accordance with the original hypothesis, will necessarily cut the other two straight lines in the image-space and hence must lie in the same plane.

3. Object planes which all contain the same straight line have corresponding to them image-planes all of which have likewise a straight line in common. This is evident since the trace of the pencil in the object-plane, *i.e.*, the ray which is contained in each of the object-planes, must have corresponding to it an image-ray contained in each of the image-planes jointly and hence constituting the trace of a pencil in the image-plane.

In geometry two spaces which are related as here defined are said to be **homographic** or **collinear.** The analytical condition for this correlation of the two spaces can be expressed by the following three equations, as given by Salmon (*1*.), viz. :

$$\left.\begin{aligned} x' &= \frac{a_1 x + b_1 y + c_1 z + d_1}{ax + by + cz + d} \\ y' &= \frac{a_2 x + b_2 y + c_2 z + d_2}{ax + by + cz + d} \\ z' &= \frac{a_3 x + b_3 y + c_3 z + d_3}{ax + by + cz + d} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (1)$$

In these equations x, y, z and x', y', z' are rectangular co-ordinates in the object-space and image-space respectively.

The solution of the equations (1) in terms of x, y, z yields three analogous equations, in which the expressions on the right sides are the quotients of two linear expressions in terms of x' y' z', the denominator being the same in all three quotients. The equations are :

$$x = \frac{a_1'x' + b_1'y' + c_1'z' + d_1'}{a'x' + b'y' + c'z' + d'}\text{, etc.} \quad \ldots \quad (2)$$

where the accented coefficients are functions of the corresponding unaccented coefficients in the first set of equations.

From these equations and also as a necessary condition that the object-plane and the image-plane may be definitely conjugate, it follows that finite planes are in general conjugate to finite planes. In the case only of the finite object-plane

$$F = ax + by + cz + d = 0 \quad \ldots \quad \ldots \quad (3)$$

we find that it has corresponding to it in the image-space a plane at an infinite distance, whilst the finite image-plane

$$F' = a'x' + b'y' + c'z' + d' = 0 \quad \ldots \quad (4)$$

corresponds to an infinitely distant plane in the object-space.

Owing to this discrepant property, the planes $F = 0$ and $F' = 0$, will therefore be referred to as the **planes of discontinuity** of the object-space and the image-space respectively.

When $a = b = c = 0$, every finite plane has a finite plane corresponding to it, and the planes of discontinuity recede to infinity, so that the two planes at infinity, which are generally symbolised by the equations $d = 0$ and $d' = 0$, are mutually conjugate. This singular case embodies a **telescopic system,** so-called because it is realised in the class of optical instruments known as telescopes.

B. The Reduction of the Equations of Image Formation to their Simplest Fundamental Forms.

49. The analytical discussion of the equations of image-formation at once suggests an appropriate choice of co-ordinates within the two spaces which will enable us to define the relations connecting the object and image by means of very simple equations. The same object may be attained more expeditiously and clearly by purely geometrical means. We shall therefore choose this method.

By our definition of the planes of discontinuity, object-rays comprised within the infinitely distant object-plane are conjugate to image-rays contained within the plane of discontinuity $F' = 0$ of the image-space. Hence, the infinitely distant straight line of the image-plane $F' = 0$ is the only infinitely distant straight image-line which has corresponding to it an infinitely distant straight object-line. Conversely it will be seen that this straight object-line must be identical with the infinitely distant straight line of the plane of discontinuity $F = 0$ of the object-space. The infinitely distant straight lines of the two planes of discontinuity constitute therefore the only pair of infinitely distant conjugate straight lines. (It is only in telescopic image-formation that each infinitely distant straight line of the object-space has corresponding to it an infinitely distant straight line of the image-space.) Now since infinitely distant straight lines are the traces of the pencils of parallel planes, and since in the image-formation which we are here considering conjugate rays are the traces of conjugate plane pencils, we can at once formulate the following theorem:

The two pencils of parallel planes having corresponding to them the planes of discontinuity $F = 0$ and $F' = 0$, respectively, constitute a pair of conjugate pencils and are, moreover, the only pair of conjugate parallel plane pencils. Every other parallel plane pencil in the object-space has conjugate to it a pencil of non-parallel image-planes, whose finite line of intersection is situated in the plane of discontinuity of the image-space, and conversely. (The telescopic system again presents the only instance in which every parallel plane pencil has corresponding to it another parallel plane pencil.)

Disregarding for the present the case of telescopic image-formation, we shall choose the direction of the pencils of normals to the two planes of discontinuity as the axes of x and x'. Since by the theorem which we have just enunciated every plane normal to the direction of x (for which $x = constant$) has corresponding to it a plane normal to the direction of x' (for which $x' = constant$) it follows that x' no longer depends upon y and z, but solely upon x, so that in equation (1) we have $b_1 = c_1 = b = c = 0$.

The pencil of parallel rays which is formed by the normals to the plane of discontinuity in the object-space and whose trace is an infinitely distant point, must have corresponding to it a homocentric pencil of image-rays having a finite trace situated in the plane of discontinuity $F' = 0$ in the image-space. Only one single ray of this pencil is normal to the plane $F' = 0$; in other words, among the infinitely numerous object-rays which are normal to the plane of discontinuity $F = 0$, there is only one single ray having corresponding to it an image-ray which is normal to the plane of discontinuity $F' = 0$ of the image-space. We shall choose these definitely defined rays for the axes of x and x'. Then for $y = 0$, $z = 0$ we shall also have $y' = 0$, $z' = 0$, that is to say, in the equations (1) we shall have $a_2 = a_3 = d_2 = d_3 = 0$.

In our choice of the directions of y and y' and those of z and z', we are guided by the following consideration:

The pencil of the object-plane whose trace is the axis of x, has corresponding to it the pencil of the image-plane whose trace is the axis of x', since the axes of x and x' are conjugate to one another. As we are solely concerned with homographic relations it follows from a well-known theorem of geometry that in the pencil of the object-plane which we are considering there is always a pair, and in general only one pair, of planes normal to one another, which has conjugate to it a pair of mutually perpendicular planes in the corresponding pencil of the image-plane. We shall choose the four planes thus defined as the xy, xz, $x'y'$, $x'z'$ planes. Then, when $y = 0$ it follows that $y' = 0$, and $z' = 0$ when $z = 0$; that is to say, in equation (1) the constants c_2, b_3 become zero.

The equations of image-formation (1) will now assume the forms

$$x' = \frac{a_1 x + d_1}{ax + d}; \quad y' = \frac{b_2 y}{ax + d}; \quad z' = \frac{c_3 z}{ax + d}. \quad \ldots \quad (5)$$

We have not as yet reached any conclusion as regards the planes of reference $x = 0$ and $x' = 0$. For this purpose we have at our disposal two particularly simple and characteristic conventions, both of which are of great importance in the application of the theory.

1. For our planes of reference we shall choose the planes of discontinuity $F = 0$ and $F' = 0$. Then $x' = \infty$ when $x = 0$, and $x = \infty$ when $x' = 0$. This choice of the planes of reference will cause the co-efficients d and a_1 to vanish. Putting $\frac{d_1}{a} = \mathsf{A}$, $\frac{b_2}{a} = \mathsf{B}$, $\frac{c_3}{a} = \mathsf{C}$, we shall obtain the equations of image-formation in the following form:

$$x' = \frac{\mathsf{A}}{x}; \quad y' = \frac{\mathsf{B}\, y}{x}; \quad z' = \frac{\mathsf{C}\, z}{x}. \quad \ldots \quad \ldots \quad (6)$$

2. For zero planes we shall choose conjugate planes. Then x and x' must simultaneously reduce to zero, so that $d_1 = 0$, and the equations assume the form:

$$x' = \frac{a_1 x}{ax + d}\,;\; y' = \frac{b_2 y}{ax + d}\,;\; z' = \frac{c_3 z}{ax + d}, \quad \ldots \qquad (7)$$

in which there are still four arbitrary coefficients, one of which will be employed in the special choice of the conjugate planes of reference.

We have expressly excluded from our consideration the case of the *telescopic image-formation*, where $a = b = c = 0$, in which the infinitely distant planes of the two spaces are conjugate, since in the case of the telescopic system the reasoning here given cannot be applied. In our choice of a definitely suitable system of co-ordinates we shall make use of a readily demonstrable theorem in co-ordinate geometry which states that when two spaces are collinearily related there is in every homocentric pencil of object-rays a single set of three rays forming a rectangular corner which has conjugate to it a similarly arranged set of three rays in the corresponding homocentric pencil of the image-space. Now, the three directions thus defined in the two spaces are always the same in a telescopic system, from whatever pair of conjugate homocentric pencils we may proceed. This is not difficult to see if we consider that in a telescopic system every infinitely distant point has corresponding to it an infinitely distant point, so that for every pencil of parallel rays at infinity in one space there is a similar corresponding pencil at infinity in the other space.

This suggests an appropriate choice of a system of co-ordinates:

In the first place we shall choose two conjugate points as our points of origin, such that when $x = y = z = 0$ then $x' = y' = z' = 0$. This will cause the coefficients d_1, d_2, d_3 in the equations (1) to vanish, so that $d_1 = d_2 = d_3 = 0$. Next, we will adopt the definitely defined conjugate edges of conjugate rectangular corners as the respective directions of the co-ordinate axes, which will cause the coefficients $b_1 c_1$; $c_2 a_2$; $a_3 b_3$ to vanish in the equations (1). The coefficients a, b, c are zero, since we are considering a telescopic system. For the telescopic image-formation we shall accordingly have the following simple equations:

$$x' = \frac{a_1}{d}\,x\,;\; y' = \frac{b_2}{d}\,y\,;\; z' = \frac{c_3}{d}\,z\,, \quad \ldots \qquad (8)$$

for which we may write in abbreviated form

$$x' = px\,;\; y' = qy\,;\; z' = rz\,. \quad \ldots \qquad (9)$$

The equations may thus be regarded as a special case of equations (7) in which $a = 0$.

In its most general aspect the formation of images is accordingly characterised by three coefficients. The axis of x of our system of co-ordinates is the principal axis whilst the axes of y and z are the corresponding secondary axes. In the special case of the telescopic system the three axes are equivalent.

C. Equations of Image-formation referred to the Planes of Discontinuity.

50. In our further investigations we shall proceed from the simplified equations (6), where the origins of the systems are located in the planes of discontinuity. The theorems which can be deduced from these equations must likewise be contained in the second set of equations (7), since, as geometrical relations, they cannot be independent of the choice of the system of co-ordinates which determines their analytical form.

51. The Different Species of Optical Image-formation. —When considering the choice of appropriate axes (§ 49) it was remarked that in every homocentric pencil of object-rays there are three rays at right angles to each other corresponding with which there are three rays of equal properties in the conjugate image-pencil. The two rectangular corners which are formed by these mutually perpendicular rays are related to one another by a point-to-point correspondence. We will consider two conjugate octants. These are related in two essentially different ways.

If we cause two pairs of conjugate edges of the octants to coincide, which is always possible, the third pair of conjugate edges will either likewise coincide, in which case the entire octants will coincide; or they will merely meet at the vertex of the octants, so that one will be the continuation of the other. The octants will not then coincide but will merely be in contact over one of their lateral planes.

In the former case the conjugate octants, and hence also the conjugate rectangular corners, are *congruent*, that is, correspond in every respect, whilst in the latter case they are *symmetrical* with respect to one another. Both expressions are here used in a sense which differs from the usual meaning, in that the absolute amounts of the linear relationship of the conjugate edges are not essential for the classification of different species of image-formation, and are left out of account.

From the continuity of all related quantities it follows immediately that in homographically related spaces there can exist either only congruence, or only symmetry of all pairs of conjugate rectangular corners. An exception to this relation is conceivable

only at the planes of discontinuity; but, as we shall now show, such an exception does not arise.

The relations of (1) congruence or correspondence in all respects, and (2) symmetry, are indicated in Fig. 25A.

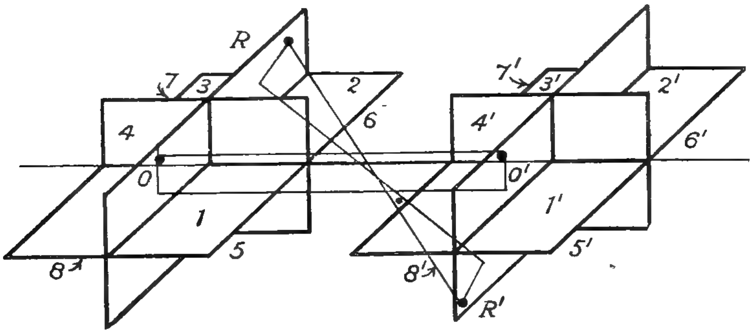

FIG. 25A.

Diagram showing the division of the object and image spaces into octants, and their mutual relations in "obverse" and "reverse" image-formations.

To show this, we will consider the two families of conjugate rectangular corners having their vertices on the axes of x and x' respectively. In accordance with our special choice of co-ordinate systems, the edges are necessarily always parallel to the three co-ordinate axes, and hence we may write the equations of image-formation (6) in the form:

$$dx' = -\frac{\mathsf{A}}{x^2}dx; \quad dy' = \frac{\mathsf{B}}{x}dy; \quad dz' = \frac{\mathsf{C}}{x}dz, \ldots \qquad (10)$$

where dx and dx', dy and dy', dz and dz' represent pairs of conjugate elements of the rectangular edges. If for all positive and negative values of x we retain the algebraical signs of dx, dy, dz, it follows from the above equations (10) that when $x > 0$ the linear elements dx', dy', dz' retain their algebraical signs, that is to say, for all positive values of x the conjugate rectangular corners here considered are either **all congruent** in pairs or **all symmetrically similar** in pairs. This conclusion has already been more generally derived from the principle of continuity. In the transition from positive to negative values of x, *i.e.*, at the plane of discontinuity, dy' and dz' change their signs likewise, whilst dx' retains its sign. From this it follows that at the plane of discontinuity ($x = 0$) there is no exception to the rule, *i.e.*, in the relation of the conjugate rectangular corners no transition occurs from congruence in every respect to symmetrical similarity.

This proves that within the limits of a complete image-formation the conjugate rectangular corners are either all congruent or all symmetrical, the terms *congruent* and *symmetrical* being taken in the above expressly extended sense. In the former case a right-handed screw will be reproduced in the image-space as a right-handed screw, and hence we shall refer to this case as **obverse** or **identically similar image-formation.*** In the latter case a right-handed screw is reproduced in the image space as a left-handed screw, and we shall refer to this species of image-formation as **reverse** or **symmetrically similar image-formation.***

So long as we consider the formation of images in a purely mathematical manner, dispensing with any assumptions respecting the relative positions of the two spaces, these are the only two essentially different aspects which present themselves for the classification of the image-formation.

To ascertain whether or how these distinctive properties manifest themselves in the algebraical signs of the constants of the image-formation, it will be necessary to decide upon the sense of the direction conforming to positive or negative values of linear intercepts on the co-ordinate axes. We must for this purpose establish relations between the purely geometrical and the physical process of image-formation. In terms of the latter the sense of direction along any straight line, regarded as a ray, is defined by the direction in which the light is propagated. This applies in particular to the two principal axes. We shall accordingly adopt the following conventions of sign :

About both principal axes the positive sense of direction agrees with the direction of the propagation of the light. There will then be two possible ways in which the light may proceed through a series of conjugate points on the two axes, in that it may do so in the same sense or in the opposite sense. In the former case, where $\frac{dx'}{dx} > 0$, we shall describe the image-formation of the axes with respect to each other as **progressive,**† whilst in the second case, where $\frac{dx'}{dx} < 0$, we shall refer to it as **recessive.**†

We shall now consider an object-ray whose intercept is positive on the axis of x, and with respect to which the components of the light measured along the secondary axes of the object-space

---

* Eppenstein introduced the terms "rechtwendig" and "rückwendig," and these terms were incorporated in the second edition of Czapski's work. Here an endeavour has been made to convey their meaning by the terms "obverse or identically similar," and "reverse or symmetrically similar." Trans.

† The words "progressive" and "recessive" are used in the endeavour to convey the meaning of the specially constructed words "rechtläufig" and "rückläufig." Trans.

are positive, the positive directions on these latter axes having been chosen quite arbitrarily in the first instance. We shall then define the system of co-ordinates in the image-space by the condition that the luminous motion along the image-ray which is conjugate to the object-ray shall have positive components on the secondary axes.

Also, the two rays which are inclined to the axis conform to either progressive or recessive image-formations.

In view of the continuity of the relations, all pairs of conjugate rays involved in one image-formation give rise to either an exclusively progressive series or to an exclusively recessive range of images, so that we may, without further qualification, speak of progressive and recessive image-formations.

Our conventions respecting the sense of direction along the axes may be defined by the following inequalities:

1. In the case of **progressive** image-formation:

$$\frac{dx'}{dx} > 0\,;\ \frac{dy'}{dy} > 0\,;\ \frac{dz'}{dz} > 0\,;$$

i.e.,

$$\mathsf{A} < 0\,;\ \mathsf{B} > 0\,;\ \mathsf{C} > 0\,.$$

2. In the case of **recessive** image-formation:

$$\frac{dx'}{dx} < 0\,;\ \frac{dy'}{dy} < 0\,;\ \frac{dz'}{dz} < 0\,;$$

i.e.,

$$\mathsf{A} > 0\,;\ \mathsf{B} < 0\,;\ \mathsf{C} < 0\,.$$

The employment of the ordinary dioptric and catoptric means, namely, refracting and reflecting surfaces separating homogeneous media, results invariably in progressive image-formation. We shall therefore disregard the second case and confine ourselves to the discussion of progressive image-formation, where

$$\mathsf{A} < 0\,;\ \mathsf{B} > 0\,;\ \mathsf{C} > 0 \quad \ldots \quad \ldots \quad (11)$$

The difference between obverse and reverse image-formation does not become manifest in the coefficients of image-formation. By our assumptions we have embodied this difference in the system of co-ordinates itself, in that in the case of an obverse image-formation a *canonic* object-system of co-ordinates has conjugate to it a canonic image-system, whilst in a reverse image-formation

a canonic system has an *acanonic* image-system of co-ordinates conjugate to it.*

If the distinction between the two species of image-formation is to appear in the signs of the coefficients, we may fix the positive direction with respect to two pairs of axes only, say, the axis of x and x' and the axis of y and y', by reference to the direction of light along two suitably chosen conjugate rays, whilst the positive direction of z and z' may be defined by the requirement, say, that the systems of co-ordinates shall be canonic in both spaces. Assuming these conditions to be given, we shall have, in addition to the conditions $\mathsf{A} < 0$ and $\mathsf{B} > 0$,

for obverse image-formations $\mathsf{C} > 0$,

for reverse image-formations $\mathsf{C} < 0$.

In what follows we shall suppose the image-formation to be of the former kind, where B and C have the same algebraical sign, so as to obviate frequent reference to the algebraical signs in the case to which we shall subsequently confine ourselves, where the image-formation is symmetrical with respect to the axis of x and where $\mathsf{B} = \mathsf{C}$.

We shall now briefly summarise the results of our discussion of the different species of image-formation.

From a purely geometrical aspect there are only two fundamentally different types of image-formation, viz. :

1. The **Obverse image-formation.**
2. The **Reverse image-formation.**

From the physical aspect reference to the direction of propagation of the light along a ray furnishes another classification. We have accordingly made a distinction between

1. **Progressive image-formation** and
2. **Recessive image-formation.**

Recessive image-formations are not realisable with the ordinary dioptric means. We shall therefore disregard them.

We shall choose the algebraical signs of the co-ordinate axes in such a manner that all progressive image-formations will be subject to the inequalities :

$$\mathsf{A} < 0, \mathsf{B} > 0, \mathsf{C} > 0,$$

so that, assuming the co-ordinate system of the object-space to be

* Supposing the positive directions of the axes of x and y to point to the South and East respectively ; then the positive direction of the axis of z will point towards the zenith in a canonic system and towards the nadir in an acanonic system

canonic, the image-space will have a canonic system of co-ordinates in the case of obverse image-formation and an acanonic system in the case of reverse image-formation.

In a chapter on the characteristics of the different species of image-formations and image-forming systems, Czapski (*3*, *35*) has reproduced two errors.

In one place it is there stated that the recessive image-formation is realisable by catoptric means, viz. by refractions in combination with an odd number of reflections.

In another place he states that, without assuming a definite linear relationship of the two spaces, the image-forming systems may be divided into "converging" and "diverging" systems.

The former error was pointed out by Runge immediately after the publication of the first part of the work, thereby affording an opportunity of rectifying the error before its completion.

The latter error was corrected by Eppenstein in the revised edition of Czapski's book.

52. Magnifications and their Restriction to Image-formations which are Symmetrical with respect to the Principal Axis.—The ratio of two conjugate lengths in the two spaces is termed the **magnification.** In particular, the ratio of conjugate lengths situated on the principal axis, is called the **longitudinal** or **axial magnification** and represents the magnification in depth.

Proceeding from equations (6) in § 49, which are the equations of image-formation referred to the planes of discontinuity, the ratio of infinitely small conjugate lengths will be

$$\frac{dx'}{dx} = a = -\frac{\mathsf{A}}{x^2} = -\frac{x'^2}{\mathsf{A}} \quad \ldots \quad \ldots \quad (12)$$

It will be seen that this ratio varies as x and x' vary.

For finite axial intercepts it follows from equations (6) that

$$\frac{x_2' - x_1'}{x_2 - x_1} = -\frac{\mathsf{A}}{x_1\, x_2} = -\frac{x_1'\, x_2'}{\mathsf{A}}.$$

The ratio of intercepts normal to the axis of x is called the **lateral magnification** or, simply, the **magnification.** In the general case of tri-axial image-formation it varies from azimuth to azimuth. In practice, we are almost exclusively concerned with the formation of images which are symmetrical about the axis of x. We shall therefore in what follows deal with this case only, and, to indicate this restriction, we shall replace the letters x, y, z; x', y', z' by the symbols x_s, y_s, z_s; x_s', y_s', z_s'.

We must now put $\mathsf{B} = \mathsf{C}$, and we need make no further distinction between y_s and z_s. Every pair of mutually perpendicular

meridian planes is transformed into a pair of meridian planes which are likewise mutually perpendicular, and the choice of secondary axes becomes indeterminate. Nevertheless, our formulæ are not thereby rendered ambiguous, since we retain the condition that the plane of $x_s\,y_s$ shall correspond to the plane of $x_s'\,y_s'$, and the plane of $x_s\,z_s$ to the plane of $x_s'\,z_s'$.

The lateral magnification, which we shall denote by the letter β, will now be constant for all planes at right angles to the axis of x_s, so that

$$\beta = \frac{dy_s'}{dy_s} = \frac{y_s'}{y_s} = \frac{\mathsf{B}}{x_s} = \frac{\mathsf{B}}{\mathsf{A}}\,x_s' \qquad \ldots \qquad (13)$$

and, therefore, is independent of y_s and y_s'.

From this it follows that plane figures at right angles to the axis of x_s are transformed in the image space into similar figures.

A comparison of equations (12) and (13) shows that

$$\alpha = -\frac{\mathsf{A}}{\mathsf{B}^2}\,\beta^2 . \qquad \ldots \qquad \ldots \qquad (14)$$

Thus the longitudinal magnification is everywhere proportional to the square of the lateral magnification, as was already noted by Toepler (*1*) with respect to dioptric image-formations, and hence in any given image-formation the ratio

$$\frac{\beta^2}{\alpha} = -\frac{\mathsf{B}^2}{\mathsf{A}} \qquad \ldots \qquad \ldots \qquad (15)$$

is constant for all positions in the space.

By (12) the longitudinal magnification is inversely proportional to the square of the distance between the object and the plane of F, and directly proportional to the square of the distance between the image and the plane of F'. It can here only assume all values from zero to infinity, since we are confining ourselves to progressive image-formations.

The lateral magnification is inversely proportional to the distance of the object from the plane of F and directly proportional to the distance of the image from the plane of F'. It can assume all values from $-\infty$ to $+\infty$.

The connection between the lateral magnification and the longitudinal magnification and their relation to the values of x_s and x_s' are best demonstrated graphically as shown in Figs. 26 to 28.

In the object-space (Fig. 26), let a series of co-ordinates y_s and z_s of equal length be described at equidistant points on the axes of x_s, so that their terminal points lie on straight lines, one being accordingly in the plane of $x_s\,y_s$, the other in the plane of $x_s'\,z_s'$, and both parallel to the axis of x_s. The terminal points of the corresponding co-ordinates in the image-space as represented in

Figs. 27 and 28, are situated on straight lines contained in the planes of $x_s'\ y'_s$ and $x_s'\ z_s'$ and which intersect the axis of x_s' at a point $x_s' = 0$ in the plane of discontinuity. The points of intersection of

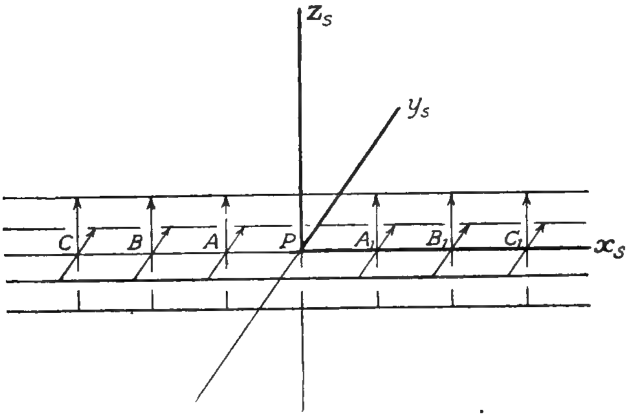

FIG. 26.

Graphic representation of magnifications and the significance of focal planes and of the axes of symmetry.
(Object-space).

the ordinates lie more closely together the nearer they are to the plane of discontinuity.

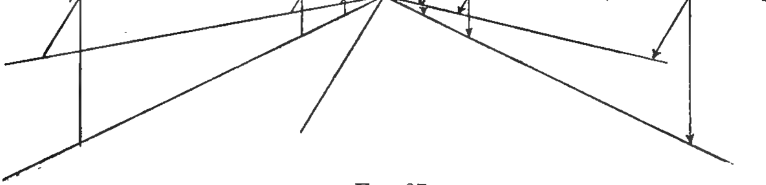

FIG. 27.

Graphic representation of magnifications and the significance of focal planes and of the axes of symmetry.
(Image-space.)

The points A', B', C' ; A_1', B_1', C_1' in Figs. 27 and 28 are the images of the points A, B, C ; A_1, B_1, C_1 in Fig. 26. The arrows in the figures indicate the relations of the algebraical signs between the conjugate linear elements. Since in all cases $\mathsf{A} < 0$ and $\mathsf{B} = \mathsf{C} > 0$, it follows from the equations (6) in § 49 that

Positive values of y_s and z_s with a positive value of x_s, are always associated with positive values of y_s' and z_s' and a negative value of x_s' :

and

Positive values of y_s and z_s with a negative value of x_s, are always associated with negative values of y_s' and z_s' and a positive value of x_s'.

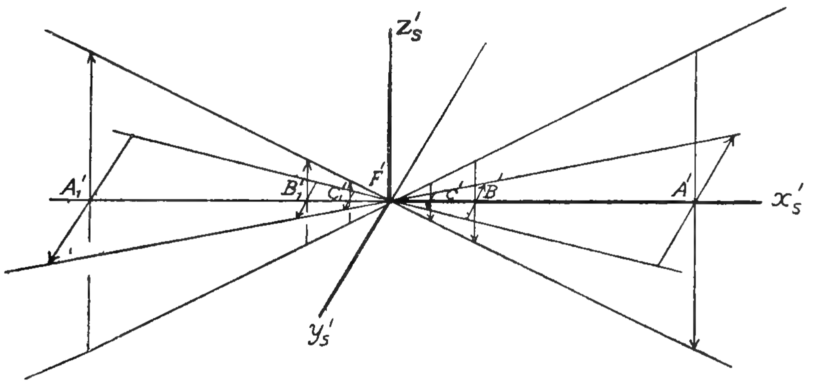

Fig. 28.

Graphic representation of magnifications and the significance of focal planes and of the axes of symmetry. (Image-space.)

Figs. 26 and 27 represent an obverse image-formation, the co-ordinate systems of both spaces being canonic, whilst Figs. 26 and 28 represent a reverse image-formation, the co-ordinate system in Fig. 28 being acanonic.

A family of equidistant y_s' and z_s' co-ordinates, having their terminal points on four straight lines inclined at a certain angle to the axis of x_s' and passing through F', will have corresponding to them in the object space y_s and z_s co-ordinates of similar magnitude which lie closer together the nearer they are to the plane of F.

The most striking and important feature of these graphic representations is that they assist us in visualising the significance of the focal planes. It will be seen that they divide the two spaces into symmetrical halves which are transformed in an identical manner.

53. Mutual Correspondence of Straight Lines and Pencils.—The points of the plane of discontinuity in the object-space have corresponding to them the points of the infinitely distant image plane ; in other words, a homocentric pencil of object-rays, whose centre is situated in the focal plane, say at a distance

$h = \sqrt{h^2_1 + h^2_2}$ from the axis, has corresponding to it in the image-space a pencil of parallel rays which we will suppose to have an inclination u' with respect to the axis of x_s'. Since the image-formation is supposed to be symmetrical with respect to the principal axis the quantity h will have the same value for all azimuths if the corresponding pencil of parallel rays in the image-space is inclined at the same angle u' to the axis. Supposing, therefore, there is given any image-ray inclined at an angle u' to the axis, it follows that the ordinate h where the corresponding object-ray cuts the focal plane is completely determined by u'. Obviously, the analogous quantities h' and u are mutually determined in a similar manner.

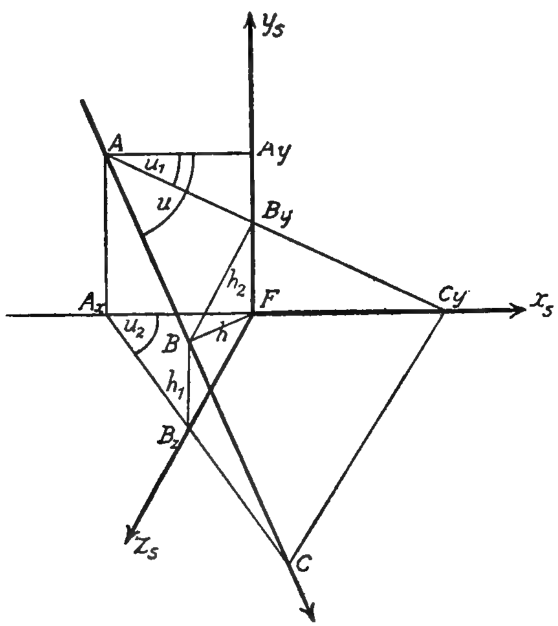

FIG. 29.

$FB = h$; $B_z B = h_1$; $B_y B = h_2$

The constituents of a ray inclined at a skew angle to the axis.

We will now prove that the quotients $\dfrac{h}{\tan u'}$ and $\dfrac{h'}{\tan u}$ are independent of the quantities h, h', u, u' and are constant for any given image-formation.

Let any ray AC inclined at a skew angle to the axis of x_s be given in terms of its projections on the planes $x_s\, y_s$ and $x_s\, z_s$. Let $y_s = h_1 - x_s \tan u_1$ be the equation of the projection AC_x, and

$z_s = h_2 - x_s \tan u_2$ that of the projection $A_x C$, where u_1 and u_2 are the angles comprised between the projections and the axis of x_s; h_1 and h_2 being the ordinates of the points where these projections intersect the plane of discontinuity F (Fig. 29).

The ray itself then cuts the plane of discontinuity at the distance $h = \sqrt{h_1^2 + h_2^2}$ from the axis of x_s and is inclined to the axis of x_s at an angle u, which is determined by the equation $\tan^2 u = \tan^2 u_1 + \tan^2 u_2$.

The angle u_1 may be completely defined in the following manner. Its absolute magnitude must be less than π, and the angle must be reckoned positive if the direction of rotation, which brings, by the shortest route, the direction of motion of the luminous ray into coincidence with the positive direction of x_s, has the same sense as the rotation which brings the positive x_s direction by the shortest route into the positive y_s direction. The angle u_2 in the plane of $x_s\, z_s$ may be determined in a similar way.

The choice of signs for the angles conforming to that advocated in § 28, differs from that usually adopted in co-ordinate geometry. We might, of course, have followed the usual convention. It will, however, be noted that our present object is solely to prove that our theory furnishes completely defined quantities.

For the angle u, the plane of which is not parallel to any of the co-ordinate planes, no definite algebraical sign can be given. It is, however, completely determined by the projections u_1 and u_2 (in Fig. 29 $u_1 > 0$, $u_2 < 0$). Angles in the image-space are naturally determined by an analogous rule.

The image of the object-ray AC which we are considering, *i.e.*, the ray which is conjugate to the incident ray, is supposed to be defined by the equations of its projections on the planes of $x_s' y_s'$ and $x_s' z_s'$ respectively. These traces of the ray in the image-space are obviously the images of the traces of the ray in the object-space. Their equations are:

$$y_s' = h_1' - x_s' \tan u_1',$$
$$z_s' = h_2' - x_s' \tan u_2',$$

where h_1', h_2' and u_1', u_2' denote quantities which are analogous to h_1, h_2, u_1, u_2. They assume, in accordance with the equations (6) of § 49, viz., $x_s' = \frac{\mathsf{A}}{x_s}$, $y_s' = \frac{\mathsf{B} y_s}{x_s}$ and $z_s' = \frac{\mathsf{B} z_s}{x_s}$, the following forms:

$$y_s' = \frac{\mathsf{B}}{x_s}\,(h_1 - x_s \tan u_1) = -\,\mathsf{B} \tan u_1 + \frac{\mathsf{B}}{\mathsf{A}}\, h_1\, x_s',$$

and

$$z_s' = \frac{\mathsf{B}}{x}\,(h_2 - x_s \tan u_2) = -\,\mathsf{B} \tan u_2 + \frac{\mathsf{B}}{\mathsf{A}}\, h_2\, x_s'.$$

Comparing these with the preceding equations, we now see that

$$\left.\begin{aligned} h_1' &= -\mathsf{B}\tan u_1; \quad h_2' = -\mathsf{B}\tan u_2 \\ \tan u_1' &= -\frac{\mathsf{B}}{\mathsf{A}} h_1; \quad \tan u_2' = -\frac{\mathsf{B}}{\mathsf{A}} h_2 \end{aligned}\right\} \tag{16}$$

The angle of inclination u' comprised between the image-ray and the axis of x_s', and the distance from the axis of x_s' of the point where the image-ray meets the plane of F' $(x_s' = 0)$ are defined by the equations

$$\tan^2 u' = \tan^2 u_1' + \tan^2 u_2'$$

and

$$h'^2 = h_1'^2 + h_2'^2.$$

Substituting the resulting values of h_1', h_2', $\tan u_1'$, $\tan u_2'$, we have

$$\tan^2 u' = \left(\frac{\mathsf{B}}{\mathsf{A}}\right)^2 (h^2_1 + h^2_2) = \left(\frac{\mathsf{B}}{\mathsf{A}} h\right)^2$$

$$h'^2 = \mathsf{B}^2 (\tan^2 u_1 + \tan^2 u_2) = (\mathsf{B}\tan u)^2,$$

for which we may write

$$\pm \frac{h}{\tan u'} = \frac{\mathsf{A}}{\mathsf{B}}; \quad \pm \frac{h'}{\tan u} = \mathsf{B}. \quad \ldots \tag{17}$$

The double sign arises from the square root which occurs in the calculation. If we repeat the calculation with respect to conjugate rays contained in the axial planes, say with respect to the conjugate projections of the rays already considered, no radical terms will make their appearance, and hence we derive from equations (16) the following expressions :

$$-\frac{h_1}{\tan u_1'} = \frac{\mathsf{A}}{\mathsf{B}}; \quad -\frac{h_2}{\tan u_2'} = \frac{\mathsf{A}}{\mathsf{B}},$$

$$-\frac{h_1'}{\tan u_1} = \mathsf{B}; \quad -\frac{h_2'}{\tan u_2} = \mathsf{B}.$$

In the general formulæ we need therefore only consider the negative sign.

54. Focal Lengths.—The relations embodied in equations (17) give the coefficients of image-formation a further significance. The relation $-\mathsf{B} = h'/\tan u$ indicates the ratio of the height at which a ray in the image-space cuts its plane of discontinuity and the tangent of the angle which its conjugate ray in the object-space makes with its principal axis. The relation

$$-\mathsf{A}/\mathsf{B} = h/\tan u'$$

denotes the ratio of the height at which a ray in the object-space

cuts the plane of discontinuity and the tangent of the angle which its conjugate ray in the image-space makes with its principal axis.

From this, as indeed from the equations of image-formation (6), it follows that the coefficients A and B are not similar and that, whereas B is a linear quantity, A is the square of a linear quantity. It is therefore advantageous to replace these unlike constants by the like quantities B and $\frac{A}{B}$. We shall put $B = f$ and $\frac{A}{B} = f'$. The quantities f and f', which in the theory of optical instruments are accepted as characteristic of the process of image-formation, are called the *focal lengths*, whilst the planes of discontinuity are called the *focal planes* of the object and image-spaces respectively. Both names have their origin in the historical development of geometrical optics, but there is no intimate connection between them and the essence of the subject. The definition of the focal lengths can only be obtained appropriately from the equations:

$$f = -\frac{h'}{\tan u}; \quad f' = -\frac{h}{\tan u'}, \quad \dots \qquad \dots \quad (18)$$

whilst their relation to the properties of the image-formation which they characterise follows from the equations which now take the place of the general equations (6), viz.:

$$\left.\begin{array}{l} x_s' = \dfrac{ff'}{x_s} \\ \text{or} \\ x_s x_s' = ff' \end{array}\right\} \quad \dots \quad \dots \quad \dots \quad (19)$$

$$\left.\begin{array}{l} y_s' = \dfrac{f}{x_s} y_s = \dfrac{x_s'}{f'} y_s \\ \text{or} \\ \beta = \dfrac{y_s'}{y_s} = \dfrac{f}{x_s} = \dfrac{x_s'}{f'} \end{array}\right\} \quad \dots \quad \dots \quad \dots \quad (20)$$

The nearest approach to a concrete conception of the essence of focal lengths is afforded by a definition which conforms to the above equations (18) (if we disregard the algebraical sign), and which was already referred to as the only appropriate definition by Gauss (3) in connection with his analysis of lens systems. This definition states:—

The first focal length, that of the object-space, is the ratio of the linear magnitude of an image in the focal plane of the image-space to the apparent angular magnitude of the corresponding infinitely distant object. The second focal length,

that of the image-space, is equal to the ratio of the linear magnitude of an object in the focal plane of the object-space to the apparent magnitude of its infinitely distant image.

Since we are concerned exclusively with progressive image-formations, which conform to the inequalities (11) given in § 51, we conclude therefrom respecting the signs of f and f' that

$$f > 0 \text{ and } f' < 0 .$$

55. Angular Magnification.—We now return to the subject of the transformation of a straight line into another straight line and, in particular, to the consideration of the case in which the straight line is situated in a meridian plane, so that it intersects the axis. Its image is situated in a meridian plane of the image-space. Again let u, u' be the angles contained between the axis and the intersecting rays in the object and image-spaces respectively, and let h, h' be the incidence heights at which these rays intersect the respective focal planes. Then, as will be seen from Fig. 29,

$$h = x_s \tan u \quad \text{and} \quad h' = x_s' \tan u',$$

and from equations (18) we know that

$$h = -f' \tan u' \quad \text{and} \quad h' = -f \tan u ,$$

whence we deduce the ratio

$$\gamma = \frac{\tan u'}{\tan u} = -\frac{x_s}{f'} = -\frac{f}{x_s'} . \qquad \ldots \qquad (21)$$

This ratio is independent of the angles u and u', at which the two conjugate rays intersect the axes, and is therefore constant with respect to any pair of conjugate axial intercepts x_s and x_s'.

In the theory of optical systems this very important ratio is known as the **angular magnification** or the **ratio of convergence.**

56. The relations connecting the Three Magnifications.—A comparison of the formulæ which we have derived for β and γ discloses at once the simple connection between these two quantities, namely

$$\beta . \gamma = -\frac{f}{f'} . \quad \ldots \qquad \ldots \qquad \ldots \qquad (22)$$

This signifies that the product of the ratio of magnifications in two conjugate planes and the angular magnification of the rays at the points where the axis intersects these planes is constant for any given image-formation.

We now briefly recapitulate the formulæ deduced hitherto with respect to the planes of discontinuity.

For the **Focal Abscissæ** of conjugate points on the axis :

$$x_s x_s' = ff'. \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

For the **Longitudinal Magnification**

$$\alpha = \frac{dx_s'}{dx_s} = -\frac{ff'}{x_s^2} = -\frac{x_s'}{x_s}. \quad \ldots \quad \text{(ii)}$$

For the **Lateral Magnification**

$$\beta = \frac{y_s'}{y_s} = \frac{f}{x_s} = \frac{x_s'}{f'}. \quad \ldots \quad \ldots \quad \text{(iii)}$$

For the **Angular Magnification** (or **Ratio of Convergence**)

$$\gamma = \frac{\tan u'}{\tan u} = -\frac{x_s}{f'} = -\frac{f}{x_s'}. \quad \ldots \quad \text{(iv)}$$

By appropriate combinations of these relations we derive therefrom

$$\left.\begin{aligned} \frac{\alpha}{\beta^2} = -\frac{f'}{f}; \quad \frac{\alpha}{\beta} = -\frac{f'}{f}\beta = \frac{1}{\gamma} \\ \beta\gamma = -\frac{f}{f'}; \quad \frac{\beta}{\gamma} = \alpha = -\frac{f}{f'}\cdot\frac{1}{\gamma^2} \end{aligned}\right\} \quad \ldots \quad (23)$$

and finally,

$$\frac{\alpha\gamma}{\beta} = 1. \quad \ldots \quad \ldots \quad \ldots \quad \text{(v)}$$

These relations apply to all correlated spaces in which rectilinear rays in one space are transformed into rectilinear rays in the other, where every homocentric pencil of rays in the object-space has corresponding to it a homocentric pencil in the image-space, and where the image-forming system has an axis of symmetry.

57. The Cardinal Points of an Optical System.—The formulæ (19), (20), (21) indicate that the quantities β and γ may assume all values ranging from $+\infty$ to $-\infty$, if x_s and x_s' can be varied without restriction. The quantity α, on the other hand, can only assume values ranging from 0 to $+\infty$, since we are restricting ourselves to progressive image-formations.

Some of these values are of importance partly as a means of simplifying formulæ to be derived later, and partly in their practical application. Particular interest attaches to the following points on the axis.

1. The points where the *longitudinal magnification* $\alpha = +1$, *i.e.*, points such that an infinitely small displacement of the object on the axis involves an equal displacement of the image. In every image-formation there are two pairs of these points, since the expression for the longitudinal magnification $\alpha = -\frac{ff'}{x_s^2}$ is quadratic with respect to x_s.

2. The points where the *lateral magnification* $\beta = +1$ or $\beta = -1$, *i.e.*, points at which the object and image at right angles to the axis are of similar magnitude and occupy similar or inverse positions with respect to the positive lateral axes.

3. The points where the *angular magnification* $\gamma = +1$ or $\gamma = -1$, *i.e.*, points such that a ray which in the object-space proceeds from one of them has corresponding to it in the image-space a ray which proceeds from the conjugate point at the same or the opposite inclination to the axis.

The following table shows the corresponding values of a, β, γ, x_s and x_s'. It shows, amongst other things, that $\gamma = 1$ conforms to those points on the axis where $a = \beta$, *i.e.*, where a thin laminar object and image at right angles to the axis will be similar in all three dimensions; and that $\beta = \gamma$ at those points where $a = +1$.

a	β	γ	x_s	x_s'
$+1$	$\pm\sqrt{-\frac{f}{f'}}$	$\pm\sqrt{-\frac{f}{f'}}$	$\pm\sqrt{-ff'}$	$\mp\sqrt{-ff'}$
$-\frac{f'}{f}$	$+1$	$-\frac{f}{f'}$	$+f$	$+f'$
$-\frac{f'}{f}$	-1	$+\frac{f}{f'}$	$-f$	$-f'$
$-\frac{f}{f'}$	$-\frac{f}{f'}$	$+1$	$-f'$	$-f$
$-\frac{f}{f'}$	$+\frac{f}{f'}$	-1	$+f'$	$+f$

... (24)

With reference to lenses and systems of lenses, the points for which $\beta = +1$ were called by Gauss (*3*) **Principal Points**, and the planes at right angles to the axis at these points **Principal Planes**, whilst the points corresponding to $\gamma = +1$ were called by Listing (*1*) **Nodal Points**. By analogy, Toepler (*1.*) called the points corresponding to $\beta = -1$ *Negative Principal Points*, the planes at right angles to the axis at these points *Negative Principal Planes*, and, similarly, the points corresponding to $\gamma = -1$ *Negative Nodal Points*. We shall use these terms in referring to the general case of optical image formation.

Apart from these names, there are a few others which apply to the same points, and additional pairs of conjugate or merely analogously situated points have been introduced, which may be of value

in special cases, but, if introduced into the general theory of image-formation, would only interfere with the succinct form of the investigation. We shall therefore confine our references to the works bearing on the study of the cardinal points of optical systems. Among these we may mention, apart from the works of Gauss (*3.*), Listing (*1.*) and Toepler (*1.*) already mentioned, two other papers by Listing (*3 . 5.*), also papers on the same subject by Biot (*1 . 3.*), L. Moser (*1.*), Bravais (*2.*), Neumann (*1.*), Grunert (*2.*), Lippich (*1.*), Casorati (*1.*), Matthiessen (*2 . 3 . 9 . 11.*), Hoppe (*1.*), Guébhard (*1.*), Hällstén (*1.*), Monoyer (*1.*), Kessler (*3.*), Govi (*1 . 2.*), Drews (*1.*), E. Schmidt (*1.*), Hederich (*1.*), Henke (*1.*), Lefebvre (*2 . 3.*).

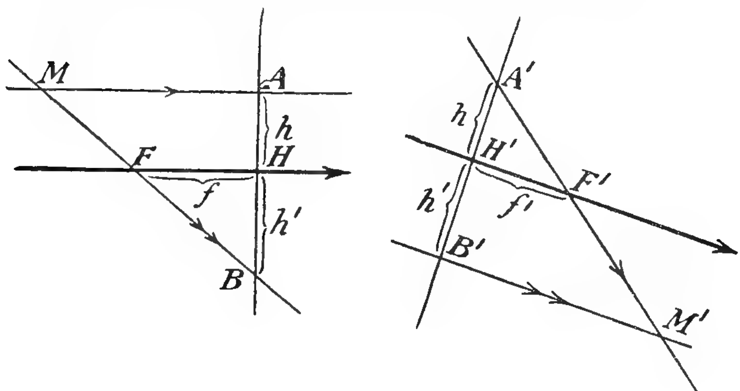

Fig. 30.

$FH = f$; $F'H' = f'$; $HA = H'A' = h$; $HB = H'B' = h'$.

Graphic determination of the image-point M' corresponding to an object-point M with the aid of the focal points and principal planes.

58. Graphic Methods.—We have already seen that a system which forms images symmetrical about the axis is completely determined by four elements, viz., the positions of the focal planes and the magnitudes of the focal lengths. It is likewise defined by two of the pairs of cardinal points indicated above, or by one of them in conjunction with a pair of the elements just referred to, as we shall see presently. The so-called cardinal points enable us to determine graphically in a very simple manner the point or ray conjugate to any given point or ray.

We shall here deal with only one problem, viz.:—Having given an object-point M, the focal points F and F', and the focal lengths f, f'; to find graphically the corresponding image-point M'.

From the table (24) § 57, it will be seen that a knowledge of the focal points and focal lengths at once gives us the positions of the principal points H and H' (Fig. 30). They are found by marking on the axis the distance f from F in the direction

of increasing values of x_s, *i.e.*, towards the right, and the distance $-f'$ from F' in the direction of diminishing values of x_s', *i.e.*, towards the left. At H and H' describe planes at right angles to the axes. These are the principal planes. From M let a ray proceed in a direction parallel to the axis of x_s and let it meet the principal plane H at a distance $h = HA$ from the axis. The conjugate ray will then meet the principal plane H' at the same distance $h = H'A'$ from the axis and will pass through the second focus F', whereby it is completely determined. Now, let another ray proceed from M through F and let it meet H at a distance $h' = HB$ from the axis. The conjugate ray will cut H' at a similar distance $h' = H'B'$ and will be parallel to the axis of x_s', so that it will likewise be completely determined in position and direction. M' is the point of intersection of these two rays.

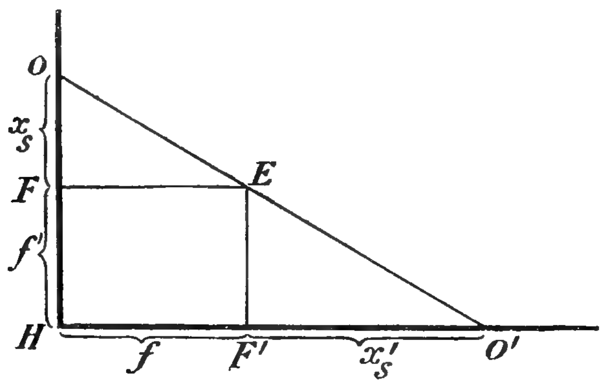

FIG. 31.

$HF' = f$; $HF = f'$; $FO = x_s$; $F'O' = x_s'$

Graphic determination of the distance between the image-point and the image focal-plane, the distance between the object-point and the object focal-plane being given in addition to the focal lengths.

This construction fails when the object-point is situated on the object-axis and consequently also when the image-point is on the image-axis. In this case the distance x_s' of the image-point from the image focal-plane may be found for example by the auxiliary method indicated in Fig. 31.

Let $HF'EF$ be a rectangle having sides $HF' = f$ and $HF = f'$. FO is equal to the distance x_s of the object-point from the object focal-plane. $F'O'$ is then equal to the distance x_s' of the image-point from the image focal-plane.

Thus, $$F'O' \,/\, FE = F'E \,/\, FO\,,$$
and $$x_s' \,/\, f = f' \,/\, x_s\,.$$

Other analogous problems may be solved in a similar manner. Some are very interesting from a geometrical point of view, but as they are of little significance for the purpose in hand, we must refer the reader to the appropriate literature.

Apart from the papers, already mentioned, which deal with Gauss's theory, especially with the cardinal points, the following writers have contributed notable papers on the graphic treatment of image-forming systems :—Gavarret (*1*), Reusch (*5*), Lissajous (*1*), Martin (*1*), Bender (*1*), Lebourg (*1*), Ferraris (*1*), Gariel (*1*), Kobald (*1*), D'Ocagne (*1*), Lefebvre (*1*), Schiller (*1*), Matthiessen (*12*), Cole (*1*), Barton (*1*).

D.—The Equations of Image-formation referred to Conjugate Planes.

59. Hitherto we have expressly confined ourselves to the discussion of equations (6), in which the abscissæ in both spaces are referred to the planes of discontinuity. To be able to investigate the particular case of telescopic image-formation, and for other reasons of a practical nature, we now proceed to express the results hitherto obtained in the form of equations (7), in which the abscissæ are measured from a pair of conjugate points.

60. General Non-Telescopic Image-formation. — To dispense with the necessity of having to repeat the investigations contained in the preceding articles with such slight modifications as their application to the equations (7) demands, we shall simply proceed from the relations and equations already established, merely displacing the system of co-ordinates by appropriate amounts in the direction of the axes. Let the abscissæ of a pair of conjugate points—which will now become the new origins—referred to the focal planes, be $x_{s,0}$ and $x'_{s,0}$, and let x_s, x_s' be those of any other pair likewise referred to the same planes. The abscissæ of the latter pair of points referred to the first pair as origins will then be $x_s - x_{s,0} = \mathsf{A}$ and $x_s' - x'_{s,0} = \mathsf{A}'$, and by (19) we now have the following relations between these quantities :

$$x_{s,0} \cdot x'_{s,0} = f \cdot f',$$

also

$$x_s \cdot x_s' = f \cdot f',$$

or

$$(x_{s,0} + \mathsf{A})\,(x'_s + \mathsf{A}') = f \cdot f',$$

hence by their combination

$$x'_{s,0}\,\mathsf{A} + x_{s,0}\,\mathsf{A}' + \mathsf{A}\mathsf{A}' = 0$$

or

$$\frac{x'_{s,0}}{\mathsf{A}'} + \frac{x_{s\,0}}{\mathsf{A}} + 1 = 0 \quad . \quad \ldots \qquad \ldots \qquad (25)$$

This equation gives the abscissæ of conjugate points with respect to a pair of conjugate points of reference in terms of the distances of the points of reference from the focal planes.

Helmholtz (*1.* 49; *2.* 69; *5.* ii. 94) and others adopted the converse mode of reference, expressing the abscissæ in terms of the distance of the focal planes from the conjugate points of reference; hence the difference of the signs in the constant term of our equations from those of Helmholtz and others.

With respect to the magnification, it follows immediately from euatqion (20) that

$$\beta = \frac{y'_s}{y_s} = \frac{x'_{s,o} + \mathsf{A}'}{f'} = \frac{f}{x_{s,o} + \mathsf{A}}, \qquad \dots \ (26)$$

and the angular magnification becomes

$$\gamma = \frac{\tan u'}{\tan u} = -\frac{x_{s,o} + \mathsf{A}}{f'} = -\frac{f}{x'_{s,o} + \mathsf{A}'}. \qquad \dots \ (27)$$

To obtain expressions which do not involve the abscissæ $-x_{s,o}$ and $-x'_{s,o}$ of the focal planes referred to the conjugate points of reference of the new system of co-ordinates, we may substitute the focal lengths f and f' and the magnifications at the points of reference for which $\mathsf{A} = 0$ and $\mathsf{A}' = 0$, viz.,

$$\beta_0 = \frac{x'_{s,o}}{f'} = \frac{f}{x_{s,o}}.$$

By (25), (26), (27) the equation in terms of the intercepts then becomes

$$\frac{f'}{\mathsf{A}'}\beta_0 + \frac{f}{\mathsf{A}} \cdot \frac{1}{\beta_0} + 1 = 0;$$

while that for the ordinates is

$$\beta = \frac{y'_s}{y_s} = \frac{f'\beta_0 + \mathsf{A}'}{f'} = \frac{f\beta_0}{f + \mathsf{A}\beta_0} \qquad (28)$$

and for the angular magnification

$$\gamma = \frac{\tan u'}{\tan u} = -\frac{f + \mathsf{A}\beta_0}{f'\beta_0} = -\frac{f}{f'\beta_0 + \mathsf{A}'}.$$

These equations assume particularly simple forms when the points of reference are so chosen that β_0 has a suitable value.

The simplest and most important of these equations is that referred to the principal points, where $\beta_0 = +1$. Our equations will now assume the forms :

$$\left.\begin{aligned} &\frac{f'}{\mathsf{A}'} + \frac{f}{\mathsf{A}} + 1 = 0, \\ \beta = \frac{y_s'}{y_s} &= \frac{f}{f + \mathsf{A}} = \frac{f' + \mathsf{A}'}{f'} = -\frac{f}{f'}\frac{\mathsf{A}'}{\mathsf{A}}, \\ \gamma = \frac{\tan u'}{\tan u} &= -\frac{f}{f' + \mathsf{A}'} = -\frac{f + \mathsf{A}}{f'} = \frac{\mathsf{A}}{\mathsf{A}'}. \end{aligned}\right\} \quad (29)$$

To obviate confusion with equations formulated by other writers, as, for example, by Helmholtz, the reader is again reminded that in both spaces the abscissæ referred to the respective points are reckoned positive in the direction of the motion of the light.

Referring the abscissæ to the negative principal planes, for which $\beta = -1$, we have similarly

$$\frac{f'}{\mathsf{A}'} + \frac{f}{\mathsf{A}} - 1 = 0$$

and

$$\beta = \frac{y_s'}{y_s} = \frac{f}{f - \mathsf{A}} = \frac{\mathsf{A}' - f'}{f'}, \text{ \&c.}$$

It will be seen that these expressions are similar to those which Helmholtz obtained for the positive principal planes, in that he reckoned A' in the opposite sense to A and made the signs of the focal lengths opposite to those used by us.

Similarly, we might introduce γ_0 in place of β_0 and in this way derive a set of simple equations referred to the nodal points, where $\gamma_0 = \pm 1$. These equations, however, are of value in special cases only.

61. Telescopic Image-formation.—The equations derived in the preceding article hold good likewise in the case of telescopic image-formation, for we have shown that these equations are special cases of the general equations referred to the conjugate points when $a = 0$. The characteristic feature of telescopic image-formation, expressed in terms of the new constants f and f', as shown by the fundamental equations (18), consists in the fact that the focal lengths are both infinitely great, but nevertheless retain, as we shall see presently, their constant finite ratio m.

By division with f or f' respectively, let the equations (28) be reduced to a form in which $\frac{f'}{f}$, or f and f' alone occur as factors

or divisors and make f and f' infinite. The equation in terms of the abscissæ then becomes

$$\frac{\mathsf{A}'}{\mathsf{A}} = -\beta_o^2 \frac{f'}{f} = p \quad \dots \quad \dots \quad (30)$$

and, similarly

$$\beta = \frac{y_s'}{y_s} = \beta_o = q \quad \dots \quad \dots \quad (31)$$

in conformity with the equations (9) which we established in § 49 for this case. The angular magnification becomes

$$\gamma = -\frac{f}{f'}\frac{1}{\beta_o} \quad \dots \quad \dots \quad \dots \quad (32)$$

and is therefore likewise constant. This ratio is of special practical significance in telescopic systems, for when speaking of magnification in reference to the transformation of an infinitely distant object into an infinitely distant image the term can only apply to the *angular magnification* γ.

Expressed in terms of their angular magnification, the **ratio of conjugate abscissæ** referred to a pair of conjugate points takes the form

$$\frac{d\mathsf{A}'}{d\mathsf{A}} = a_o = \frac{\mathsf{A}'}{\mathsf{A}} = -\frac{f}{f'}\frac{1}{\gamma_o^2} \quad \dots \quad \dots \quad (33)$$

and the **ratio of the conjugate ordinates** is

$$\beta_o = \frac{y_s'}{y_s} = -\frac{f}{f'}\frac{1}{\gamma_o} \quad \dots \quad \dots \quad (34)$$

By combining the three quantities a, β, γ there is obtained the simple equation

$$\frac{a\gamma}{\beta} = 1,$$

which agrees with that already established for non-telescopic systems.

3.—THE LAWS OF IMAGE-FORMATION IN COMPOSITE SYSTEMS.

62. The image-space in any given first system may in its turn become the object-space of the succeeding system, and so forth. The resulting effect of these two, or more, successive systems can be regarded as a single system whose constituents can be defined in position, direction and magnitude in terms of the constituents

of the component systems and their relative positions. This is of great practical importance, since the formation of images, as brought about by physical means, is almost invariably accomplished by the formation of a series of successive component systems.

Accordingly, when we shall have studied the process by which we may compute the formation of images in a composite system from the particulars of the component systems and their relative positions, it will suffice in further operations to investigate special image-formations in the component systems. This will enable us to determine completely in all cases the resultant effect of a system of any degree of complexity.

A.—Composition of Two Image-forming Systems.

63. Composition of Two Finite Systems into a Single Finite System.—We shall at the outset introduce a restriction which exists nearly always in practice, by assuming that the axis of the image-space of the first system coincides with the axis of the object-space in the second system. For the investigation of cases which are not subject to this restriction we refer the reader to the works of Neesen (*I.*), Casorati (*I.*), Beck (*I.*) and Schwarz (*I.*).

Let F_1, F_1' be the focal planes and f_1, f_1' the focal lengths of the first system. Similarly, let F_2, F_2', f_2, f_2' be the focal planes and the focal lengths of the second system, the plane of F_2 being parallel to that of F_1'; and, finally, let the positions of the two systems with respect to one another be given by the distance of the object focal-plane of the second system from the image focal-plane of the first system, reckoned as positive in the direction of the motion of the light, *i.e.*, by the distance $F_1' F_2 = \triangle$.

Then the image-formation of the whole system composed of the Systems I and II is completely determined when the positions of the focal planes F, F' and the magnitudes of the focal lengths f, f' are known.

In the first place, it follows from our assumptions that the focal plane F is parallel to F_1, and that F' is parallel to F_2', since the planes which are parallel to the plane F_1 have, corresponding to them in the image-space of System I, planes which are parallel to F_1' and hence also parallel to F_2 by our initial assumption. Now, the latter have corresponding to them planes which are parallel to F_2', whence it follows finally that the planes parallel to F_1 have corresponding to them planes which are parallel to F_2'. In § 49 it was shown, however, that in every optical image-formation there is in general only a single family of parallel planes in one space which has corresponding to it a similar family in the other space, and that these families correspond to the planes of discontinuity of the spaces appropriate to them.

If the object-axis of the first system, and the image-axis of the second system are respectively the object-axis and the image-axis of the whole system, then these axes are images of one another, and according to the general investigation there is only one pair of straight lines at right angles to the focal planes which are mutually conjugate, these being the straight lines which we have chosen as the principal axes of the image-forming system.

The positions of the focal planes F and F' follow from the consideration that F', the plane which is conjugate to the infinitely distant plane of the object-space with respect to the whole system, must be conjugate to the plane F_1' with respect to System II. Let σ' be the interval between F' and F_2' reckoned as positive when measured in the direction of the incident light. Then σ' is the focal distance conjugate to $F_2F_1' = -\triangle$ with respect to the System II, so that by (19)

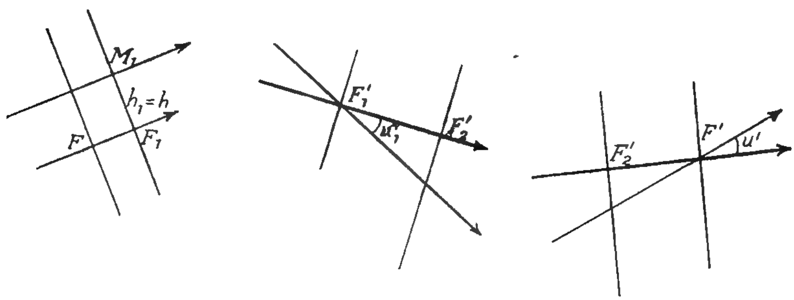

FIG. 32.

$F_1'F_2 = \Delta$; $F_1F = \sigma$; $F_2'F' = \sigma'$; $F_1M_1 = h_1 = h$

Composition of two finite systems to form a single finite system.

$$\sigma' = -\frac{f_2 f_2'}{\triangle} \quad \dots \quad \dots \quad \dots \quad (35)$$

Similarly

$$\sigma = +\frac{f_1 f_1'}{\triangle}, \quad \dots \quad \dots \quad \dots \quad (36)$$

this being the distance of the plane F from F_1, reckoned in the same sense as when F is regarded as the plane conjugate to F_2 with respect to System I.

To determine the values of the focal lengths we revert to their fundamental equations (18)

$$f = -\frac{h'}{\tan u}; \quad f' = -\frac{h}{\tan u'}.$$

A ray parallel to the object-axis of the first component system, and hence also of the whole system, and incident at a height $h = h_1$

from the axis, after traversing System I, cuts its image-axis, *i.e.*, the object-axis of the second system, at a point F_1' and at an angle u_1', the value of which is obtained from the fundamental equation $f_1' = -\frac{h_1}{\tan u_1'}$. Now, when this ray has traversed the second system we know already that it intersects its image-axis at F' at a distance σ' from F_2' and at an angle u', whose value is determined from the equation for the angular magnification at conjugate points on the axis, viz. by (21)

$$\frac{\tan u'}{\tan u_1'} = -\frac{x_{s2}}{f_2'} = +\frac{\triangle}{f_2'}.$$

This equation, when combined with that for u_1', gives the following expression

$$f' = -\frac{h}{\tan u'} = +\frac{f_1' f_2'}{\triangle}. \quad \dots \quad (37)$$

Similarly, by tracing backwards a ray which emerges in a direction parallel to the image-axis of the second component system, and hence also of the whole system, we find that

$$f = -\frac{h'}{\tan u} = -\frac{f_1 f_2}{\triangle}. \quad \dots \quad \dots \quad (38)$$

These four quantities, σ, σ', f, f', in conjunction with the determined relative positions, suffice to define completely the optical properties of the whole system.

The formulæ here given show in a very simple manner the way in which the positions of the resultant focal planes and the magnitudes and signs of the resultant focal lengths depend upon the focal lengths of the component systems and the quantity $\triangle$, the **optical interval** of the component systems. While reserving this discussion for a later occasion, when we shall be able to deal with concrete cases, we shall here only draw attention to the great variability of the final values arising from the variation of $\triangle$ for given values of f_1, f_2, f_1' and f_2'.

64. Combination of two Finite Systems to form a Telescopic System. As a particular case, it may arise that $\triangle = 0$, *i.e.*, the front focus of the second system coincides with the back focus of the first system. In this case $f = \infty$, and also $f' = \infty$, so that we have the condition of a telescopic system. The ratio of f to f' remains, however, finite, since from the above equations (37) and (38)

$$\frac{f'}{f} = m = -\frac{f_1' f_2'}{f_1 f_2}.$$

To determine in this case the constants of the telescopic system from those of the component systems, we have only to consider that a ray which is incident in a direction parallel to the axis will pass through the common focus of both component systems and will again emerge from the second component system in a direction parallel to the axis. The magnification $\frac{y_s'}{y_s} = \frac{h_2'}{h_1} = \beta$, which is constant for all points of the axis, now becomes $\frac{h_2'}{\tan u_2} / \frac{h_1}{\tan u_1'}$, where $u_2 = u_1'$ denotes the angle at which a ray

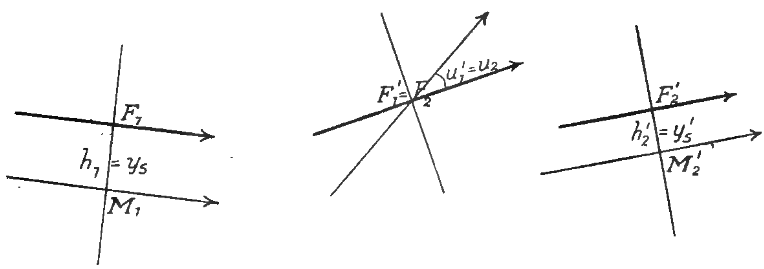

FIG. 33.

$F_1 M_1 = h_1 = y_s$; $F_2' M_2' = h_2' = y_s'$.

Composition of two finite systems into a telescopic system.

entering the system in a direction parallel to the axis at a height h_1 intersects the axis at a point between the two component systems. Accordingly,

$$\frac{y_s'}{y_s} = \beta_0 = \frac{f_2}{f_1'} \quad \dots \quad \dots \quad (39)$$

Hence the **angular magnification** is

$$\gamma = \frac{\tan u'}{\tan u} = \gamma_0 = -\frac{1}{m\,\beta_0} = \frac{f_1}{f_2'} \quad \dots \quad (40)$$

and the **ratio of conjugate abscissæ**, in conformity with our previous notation, becomes

$$\frac{\mathsf{A}'}{\mathsf{A}} = -\,m\beta_0^{\,2} = \frac{f_2\,f_2'}{f_1\,f_1'} \;. \quad \dots \quad \dots \quad (41)$$

The position of a pair of conjugate points requires to be determined specially. For such a pair we shall choose in all cases the front focus of the first system and the back focus of the second system.

65. Combination of a Telescopic and a Finite System. Let the first system be telescopic and let it be defined by the value of β_1 or γ_1 as well as by the positions of two conjugate points O_1, O_1' and the ratio $f_1' / f_1 = m_1$. Let the second system be defined by F_2, F_2', f_2, f_2' and the relative position of the two systems by the interval between F_2 and the point O_1', viz., $O_1'F_2 = \delta$. The image-axis of the front system will again be supposed to coincide with the object-axis of the back system. Then the back focus of the second system is likewise that of the whole system, since rays which at incidence are parallel to the axis remain so between the two systems, and hence are also parallel to the axis when incident upon the second system. The front focus of the whole system can be readily calculated, being conjugate to the front focus of the second system with respect to the telescopic front system. Its distance a from O_1 may be determined from equations (33) and (34) in § 61 :

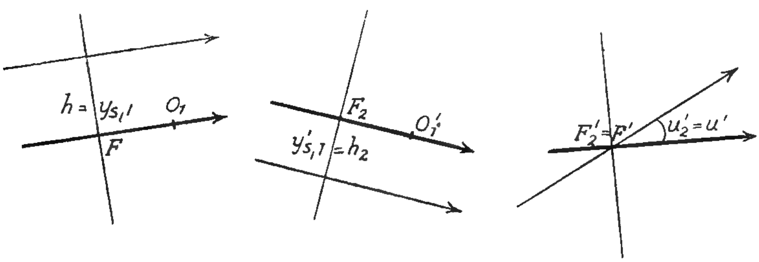

FIG. 34.

$O_1F = a$; $O_1'F_2 = \delta$.

Combination of a telescopic and a finite system.

$$a = -\frac{f_1'}{f_1}\gamma_1^2\delta = -\frac{f_1'}{f_1}\cdot\frac{f_1^2}{f_1'^2}\cdot\frac{1}{\beta_1^2}\cdot\delta = -\frac{f_1}{f_1'}\cdot\frac{1}{\beta_1^2}\delta = -\frac{\delta}{m_1\beta_1^2}$$

or

$$a = -m_1\delta\gamma_1^2 \dots \qquad \dots \qquad \dots \quad (42)$$

The **focal length** of the image-space is

$$f' = -\frac{h}{\tan u'} = -\frac{y_{s1}}{y'_{s1}}\cdot\frac{h_2}{\tan u'} = \frac{1}{\beta_1}\cdot f_2' = -\frac{f_1'}{f_1}\gamma_1\cdot f_2'$$

or

$$f' = -m_1\gamma_1 f_2', \quad \dots \qquad \dots \quad (43)$$

that of the object-space by (18), (21) and 34) being

$$f = -\frac{h'}{\tan u} = -\frac{h'}{\tan u_2}\cdot\frac{\tan u_1'}{\tan u} = f_2\cdot\gamma_1 = -\frac{1}{m_1\beta_1}\cdot f_2. \quad (44)$$

The significance of the intermediate quantities y_{s1}, $y'^{s}_1 = h_2$ and $u_2 = u_1'$ will suggest itself.

The same reasoning applies to the reverse case when the front system is finite and the back system telescopic.

66. Combination of Two Telescopic Systems. Let each system be defined in terms of β or γ, the ratios $m_1 = \frac{f_1'}{f_1}$, $m_2 = \frac{f_2'}{f_2}$ and the positions of a pair of conjugate points O_1; O_1'; O_2, O_2', and let the relative position of the component systems be denoted by the interval $O_1'O_2 = \delta$.

The resulting image, as may be seen at once, is likewise telescopic. The magnification ratio β as well as the angular magnification γ is equal to the product of the corresponding ratios of the component systems.

Thus

$$\beta = \frac{y_s'}{y_s} = \frac{y_s'}{y_{s1}} \cdot \frac{y_{s1}}{y_s} = \beta_2 \cdot \beta_1 \quad \ldots \quad \ldots \quad (45)$$

and, similarly,

$$\gamma = \frac{\tan u'}{\tan u} = \frac{\tan u' \tan u_1'}{\tan u_2 \tan u} = \gamma_2 \cdot \gamma_1 . \quad \ldots \quad (46)$$

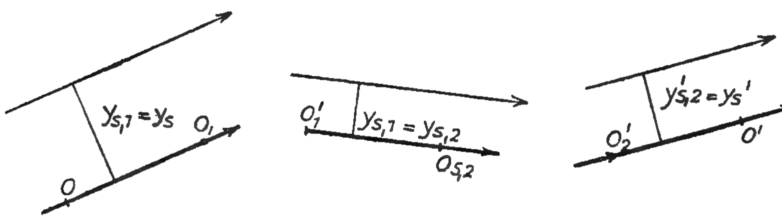

FIG. 35.

$O_1'O_2 = \delta$.

Combination of two telescopic systems.

The position of one, and even two, pairs of conjugate points is likewise easily determined; for, by (30), the point O', which is conjugate to O_1' with respect to System II, is separated from O_2' by a distance

$$a' = \delta \, \beta_2^2 \frac{f_2'}{f_2} = m_2 \, \delta \, \beta_2^2 \quad \ldots \quad \ldots \quad (47)$$

and is conjugate to O_1 with respect to the whole system. Similarly, the point O, which is conjugate to O_2 with respect to System I, is

separated from O_1 by a distance reckoned positive in the direction of the incident light, whose value is

$$a = \frac{\delta}{\beta_1^2} \cdot \frac{f_2}{f_1'} = - \frac{\delta}{m_1 \beta_1^2} . \quad \ldots \quad \ldots \quad (48)$$

The general result may be stated as follows :—The combination of two finite systems, in general, gives rise to a finite system, and in one case only to a telescopic system. The combination of two telescopic systems results invariably in a telescopic system ; whilst the combination of a finite and a telescopic system produces invariably a finite system.

Conversely, from similar considerations and by the use of the same formulæ, any given finite system can be resolved into two finite systems or into a finite and a telescopic system. Similarly, any given telescopic system may be resolved either into two finite systems having their adjacent focal planes coincident or into two telescopic systems.

The formulæ admit of their immediate extension to any number of systems. Of these we shall, however, confine ourselves to combinations of finite systems only.

B. Combination of any number of Finite Systems.

67. Let σ be again the distance of the front focal plane of the whole system from the front focal plane of the first system, and let σ' be the distance of the back focal plane of the whole system from that of the last system ; let also f_1, f_1' ; f_2, f_2' . . . . . f_k, f_k' be the focal lengths of the component systems, and f and f' those of the whole system.

The positions of the focal planes of the component systems and the directions of the principal axes of the whole system follow from similar considerations to those used in the case of two component systems. In fact, they coincide with the object focal-plane and the object-axis of the first, and the image focal-plane and the image-axis of the last component system.

In order to determine the positions of the focal planes and the magnitudes of the focal lengths of the whole system from those of the component systems, let the intervals between the adjacent foci of two consecutive systems be denoted by

$$F_1'F_2 = \triangle_1 ; \ F_2'F_3 = \triangle_2 ; \ F'_{k-1}F_k = \triangle_{k-1}.$$

The back focal-plane of the whole system is the image of the back focal-plane of the first system formed successively by the succeeding systems. Denoting by σ_h' the interval between the back focal-plane

of the system resulting from the combination of the first p systems and the back focal-plane of the pth system, there is obtained from (35) the following set of equations for the determination of $\sigma' = \sigma_k'$:

namely, $\sigma_2' = -\dfrac{f_2 \cdot f_2'}{\triangle_1}$; $\sigma_3' = -\dfrac{f_3 \cdot f_3'}{\triangle_2 - \sigma_2'}$, etc.;

and, generally,

$$\sigma_p' = -\frac{f_p \cdot f_p'}{\triangle_{p-1} - \sigma'_{p-1}},$$

whence σ' is obtained in the form of a continued fraction, viz.:

$$\sigma' = -\cfrac{f_k \cdot f_k'}{\triangle_{k-1} + \cfrac{f_{k-1} \cdot f'_{k-1}}{\triangle_{k-2} + \cdots \qquad \cdots \qquad \cdots \atop \vdots \atop \triangle_2 + \cfrac{f_2 \cdot f_2'}{\triangle_1}}}, \qquad (49)$$

Similarly, we find by (36)

$$\sigma = +\cfrac{f_1 \cdot f_1'}{\triangle_1 + \cfrac{f_2 \cdot f_2'}{\triangle_2 + \cdots \qquad \cdots \qquad \cdots \atop \vdots \atop \triangle_{k-2} + \cfrac{f_{k-1} \cdot f'_{k-1}}{\triangle_{k-1}}}}. \qquad (50)$$

To find the focal length of the whole system we may proceed, as in the combination of two systems, to determine the successive angular magnifications at the points where an image of F_1' is formed in rotation, and finally to multiply all these angular values into one another and by f_1'. We shall thus obtain by (37) the following expression for f' viz.:

$$-\frac{h}{\tan u_1'} \cdot \frac{\tan u_1'}{\tan u_2'} \cdot \frac{\tan u_2'}{\tan u_3'} \cdots\cdots \frac{\tan u'_{k-1}}{\tan u'_k} = -\frac{h}{\tan u'}, = f'. \quad (51)$$

Denoting by $f_1'{}_p$ the focal length of the system composed of the first p systems, we obtain by (37) the following successive expressions:

$$f'_{1\,2} = \frac{f_1' \cdot f_2'}{\triangle_1}; \quad f'_{1\,3} = \frac{f'_{1\,2} \cdot f_3'}{\triangle_2 - \sigma_2'}; \quad \cdots\cdots f'_{1\,k} = f' = \frac{f'_{1\,k-1} \cdot f_k'}{\triangle_{k-1} - \sigma'_{k-1}};$$

hence

$$f' = \frac{f_1' \cdot f_2' \cdot f_3' \cdots\cdots f_k'}{\triangle_1 (\triangle_2 - \sigma_2') \cdot (\triangle_3 - \sigma_3') \cdots\cdots (\triangle_{k-1} - \sigma'_{k-1})}. \quad (52)$$

The denominator of this expression can be calculated directly with the aid of the formulæ established above for σ_p'. We shall denote it by N_k. It will then be seen from the above deduction that N_k is connected with the preceding values of N_{k-1}, N_{k-2}, etc., by the relations :

$$f' = \frac{f'_{1\,k-1} \cdot f_k'}{\triangle_{k-1} - \sigma'_{k-1}}\ ; \quad N_k = \triangle_{k-1} - \sigma'_{k-1}\ ; \quad N_{k-1} = \triangle_{k-2} - \sigma'_{k-2}\ ;$$

whence

$$N_k = \frac{\triangle_{k-1} N_{k-1} + f_{k-1} \cdot f'_{k\,1}}{N_{k-1}} \ldots \qquad \ldots \quad (53)$$

and so forth. With the aid of this relation it becomes still easier to calculate N_k. We may also express the result of this calculation directly in the form of a continued fraction, or we may have recourse to other devices. We shall not here go further into these details of procedure, which have been fully discussed in the works of Mœbius (*2.*), Bessel (*1.*), Matthiessen (*2. 8.*), Casorati (*1.*), Günther (*1.* § 27), Ferraris (*2.*), Monoyer (*1.*).

Similarly, we find by (38)

$$f = (-1)^{k-1} \frac{f_1 \cdot f_2 \cdot f_3 \cdot \cdots \cdot f_k}{N_k}, \qquad \ldots \quad (54)$$

where N has the same value as in the expression for f'.

If other quantities are given instead of the principal foci and focal lengths, and also in place of the distance $\triangle$ between the focal points for the determination of the component images and the relative positions of the component systems, the corresponding elements of the composite system may be obtained by methods analogous to those applied above. Instead of discussing the procedure at length we must content ourselves with a reference to the literature quoted above.

For our present purpose it is sufficient to know that the elements of a composite image-forming system may be determined from the corresponding elements of the component systems, and to indicate the method of investigation.

4.—IMAGE-FORMATION IN AXIALLY SYMMETRICAL SPACES.

68. Hitherto we have not made any assumptions respecting the relative positions of the object-space and the image-space. We shall do so now, and we shall consider a case which occurs in nearly all optical systems in which the resulting images are symmetrical

relatively to the axis. These systems themselves are symmetrical with respect to the axis, in that they can be generated by the revolution about the axis of any of their meridian sections, *i.e.*, sections containing the axis. In view of this property of symmetry, it follows in the first place, that the principal axes of the object-space and image-space are in a straight line, which is the axis of the system; and, moreover, every ray which, in the object-space, lies within a meridian plane must remain within this meridian plane whilst traversing the system and after emerging from it. Hence, every meridian plane of the object-space must coincide with the meridian plane in the conjugate image-space, and therefore the plane of $x_s y_s$ must coincide with that of $x_s' y_s'$ and the plane of $x_s z_s$ with that of $x_s' z_s'$. We shall place the plane of $y_s z_s$ and that of $y_s' z_s'$ in the positions of the corresponding focal planes, in which case the equations of the image-formation assume the form of equations (6) in § 49.

The axis of y_s will now be parallel to the axis of y_s' and the axis of z_s to that of z_s'. As regards the respective algebraical signs of these conjugate secondary axes we have only to distinguish two possible cases:—

(1) The positive direction of y_s coincides with the positive direction of y_s', and at the same time the positive direction of z coincides with that of z_s' both in position and sense.

(2) The positive direction of y_s' coincides with the negative direction of y_s, and at the same time the positive direction of z_s' coincides with the negative direction of z_s.

In view of the axial symmetry of the image-forming system, no case can arise in which for one pair of conjugate secondary axes the sense of direction is the same in the object and image and opposite for the other pair.

In the first case, that half of the object-space in which $x_s > 0$ appears erect in the image-space, whilst the other half in which $x_s < 0$ appears inverted. Such an image-forming system is called a **converging system.**

In the second case, that half of the object-space for which $x_s > 0$ appears inverted in the image-space, whilst that half for which $x_s < 0$ appears erect. Such an image-forming system is called a **diverging system.**

In either case the image-formation itself may be *obverse* or *reverse.* It was shown in § 51 that in obverse image-formations the co-ordinate systems of the object-space and image-space must be of the same kind, *i.e.*, either both *canonic* or both *acanonic*, whilst in the case of reverse image-formations they must be dissimilar. Hence, under the conditions which we have assumed respecting the

relative positions of the object and image-spaces it is necessary to distinguish the following two cases ;—

(*a*) If the image-formation is to be *obverse*, then, both in converging and diverging systems, the positive direction of x_s must coincide both in position and in sense with the positive direction of x_s'.

(*b*) If the image-formation is to be *reverse*, then the positive direction of x_s must be opposite to that of x_s'.

It will be well to recall here that the coefficients of the equations (6) must in all cases satisfy the conditions $a < 0$, $b = c > 0$, since we have defined the algebraical signs of the conjugate co-ordinate axes in terms of the direction of the incident light. If the image-formation is obverse and is produced by a converging system, then, and only then, do the co-ordinate directions of the object-space coincide with the conjugate directions of the image-space, both in direction and sense, and the transition from the co-ordinate system of the object to that of the image consists solely in a displacement of the origin of the abscissæ.

This simple transition readily suggests its application to the other three cases, and accordingly, instead of defining the algebraical signs of the axes in terms of the direction of the light, we might do so by assuming every co-ordinate axis of the object-space to coincide with its conjugate axis in the image-space, both in direction and in sense. Proceeding from this convention it will be seen that the equations of image-formation are subject to the following conditions :—

$$\left.\begin{array}{ll} \text{1. For } \textit{converging} \text{ systems :} & \mathsf{B} = \mathsf{C} > 0, \\ \text{2. For } \textit{diverging} \text{ systems :} & \mathsf{B} = \mathsf{C} < 0, \\ \text{and in addition} & \\ \quad (a) \text{ in } \textit{obverse} \text{ image-formations :} & \mathsf{A} < 0, \\ \quad (b) \text{ in } \textit{reverse} \text{ image-formations :} & \mathsf{A} > 0 \end{array}\right\} \quad \ldots \quad (55)$$

Since, in accordance with the definition of the focal length given in § 54, $\mathsf{A} = f.f'$; $\mathsf{B} = \mathsf{C} = f$, we derive the following rules for the algebraical signs of the focal lengths :—

$$\left.\begin{array}{l} \text{1. In } \textit{converging} \text{ systems—} \\ \quad (a) \text{ in } \textit{obverse} \text{ image-formations: } f > 0\, ;\, f' < 0, \\ \quad (b) \text{ in } \textit{reverse} \text{ image-formations : } f > 0\, ;\, f' > 0, \\ \text{2. In } \textit{diverging} \text{ systems—} \\ \quad (a) \text{ in } \textit{obverse} \text{ image-formations: } f < 0\, ;\, f' > 0, \\ \quad (b) \text{ in } \textit{reverse} \text{ image-formations : } f < 0\, ;\, f' < 0. \end{array}\right\} \ldots \quad (56)$$

In systems of this kind, if, in addition to any number of refracting surfaces, there is an even number of reflecting surfaces, the resulting image-formation will be obverse. If, on the other hand, the number of reflecting surfaces is odd the resulting image-formation will be reverse. These facts may be noted provisionally at this stage.

Bibliography.

69. Until quite recently the theory of the formation of optical images, as stated above, was always dealt with under special assumptions. Apart from the numerous papers already referred to, the following are of interest as regards the development of the theory :—

From the period preceding 1841, which witnessed the publication of the *Dioptrische Untersuchungen* of Gauss, we may note the papers of Kepler (*1.*), Cotes (*1.*), Euler (*2.*), Harris (*1.*), Lagrange (*1.*), Piola (*1.*) Mœbius (*1.*).

From the period subsequent to 1841 we may note the papers and treatises of Biot (*2.*), Encke (*1.*), Seidel (*2.*), Mœbius (*3.*), Mossotti (*1.*), Albrich (*1.*), von Lang (*1.*), Zech (*1. 2.*), Hansen (*1.*), Hirschberg (*1.*), Parow (*1.*), Giraud-Teulon (*1.*), Rœthig (*1. 2.*), Battelli (*1.*), Mesquita (*1.*), Pendlebury (*1.*), Brockmann (*1.*), Gariel (*1.*), van den Berg (*1.*), Isely (*1.*), Sampson (*1.*), Pfaundler (*1.*), Sissingh (*1.*).

Chapter IV.

THE FORMATION OF OPTICAL IMAGES.

(P. Culmann.*)

1. THIN PENCILS NEAR THE AXES OF SPHERICAL SURFACES.

Fundamental Properties of Lenses and Systems of Lenses.

70. Hitherto we have only assumed that an optical image is formed, without concerning ourselves, or making any detailed statements regarding the manner in which it is produced. We accepted the occurrence of optical images as an established fact, and, proceeding therefrom, we investigated the general laws which govern the formation of images as the result of the intersection of rectilinear rays at points, irrespective of the means by which they are formed.

In this chapter we propose to study in detail some of the most important ways in which optical images are formed. We shall discuss the physical conditions under which an optical image can be produced; also the restrictions imposed by the behaviour of pencils of finite angular magnitude, which in practice always take the place of image-formations in infinite space; and, finally, we shall study the means by which the focal lengths and the positions of the focal planes may be computed from the data that are usually available. As already explained, this is all that remains to complete the application of the general theory.

Since from the last section it follows that the resultant effect of any optical system, whatever its composition, can always be resolved into the effects of successive component systems, we shall be justified in confining ourselves to the simplest of the possible cases of the formation of images; and we shall accordingly deal only with the formation of images by the reflection or refraction of rays at a single surface separating two different media. We shall first investigate the formation of images by the intervention of a spherical surface.

* This chapter is essentially a revised edition of the corresponding chapter of Czapski's book on the *Theory of Optical Instruments*. Entire pages have been retained without change, whilst others have been modified more or less extensively. The following paragraphs only have been added; viz., the articles on the zero invariant; two formulæ due to Seidel; the theorem of Cotes; the special cases of astigmatic refraction; sections dealing with double-curved surfaces; an article on anamorphotic image-formation, and the historical notes.

A.—Refraction at a Spherical Surface.

71. Graphic Method.—The following method, which is originally due to Young (*3. 73*), furnishes an exceedingly simple means of determining the position of the refracted ray.

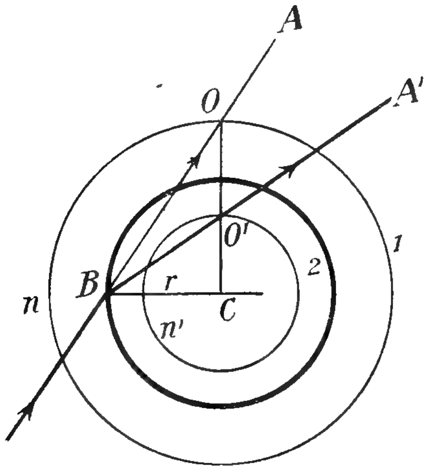

FIG. 36.

Graphic method of determining the refracted ray.

About C as centre describe circles (or spheres) with radii $r_1 = \frac{n'}{n}r$ and $r_2 = \frac{n}{n'}r$ respectively. Join the point O, where the ray produced meets the first circle, with C; also join O', where OC meets the second circle, with B. Then BO' represents the refracted ray. Since from the construction $CO'/CB = CB/CO$, it follows that the triangles CBO' and COB are similar, so that the angle $BOC = O'BC$. Now, in the triangle COB, $\sin CBO/\sin BOC = CO/CB = n'/n$, whence it follows that CBO' is the angle of refraction i' corresponding to CBO.

72. Trigonometrical Tracing of a Refracted Ray.*—Let a pencil of light proceeding from, or converging towards,

* It will be noticed that the first part of this article contains a repetition of the investigation of a ray intersecting the axis, as indicated in § 28. This part had already been written when it was decided to let the chapter on the computation of optical systems precede that on the general formation of images. Since these repetitions are not very extensive they have been allowed to remain, as it was thought that to many readers it might be a welcome assistance to have before them all that is necessary for the comprehension of the subject.

O, meet a spherical surface having its centre at C and separating two media whose refractive indices are n and n'. We shall in the first instance confine our attention to the tracing of a single ray of this pencil, which we shall assume to meet the spherical surface at B. Let the plane of the paper be the plane which contains the ray BO and the centre of the sphere C (Fig. 37). Since BC

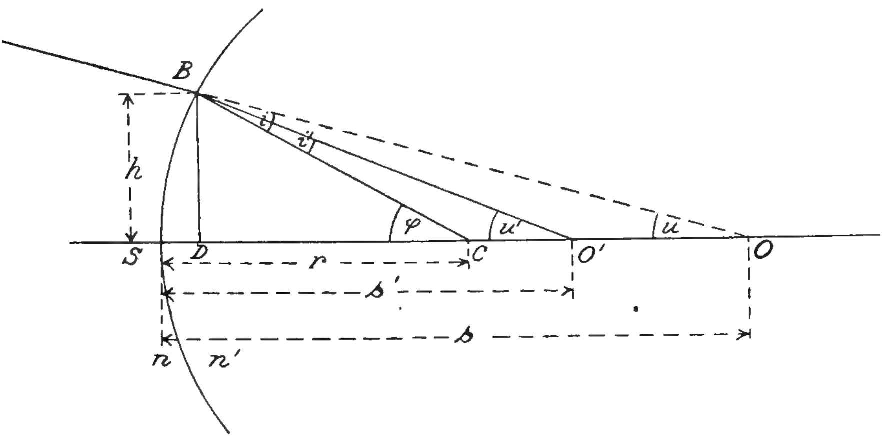

FIG. 37.

Refraction at a spherical surface of a ray of finite aperture which intersects the axis at O.

coincides with the normal of incidence, the plane BOC is the plane of incidence of the ray BO, which remains in the plane after refraction. The line joining C and O is the centre-line or **axis** of the refracting surface, and the point S where it meets the surface, is its **vertex**.

Our object is to calculate the data of the refracted ray from those of the incident ray.

Now a ray is completely determined by the position of its point of intersection with the axis and its inclination to the axis at this point. We shall adopt the following notation:

Let s, c and p be respectively the distances of the point O from the vertex S, from the centre C, and from the point of incidence B. These distances will be regarded as abscissæ and may accordingly have positive or negative values. We shall regard the direction of the incident light as positive. In our diagrams we shall always assume that this direction is from left to right, so that s, c, p are positive when O is on the right of S, C or B, as in Fig. 37.

We shall regard the radius r of the sphere as the abscissa of the centre C with respect to the vertex. Accordingly r is positive when the spherical surface presents its convex side to the incident light.

The vertical distance of the point of incidence B from the axis, *i.e.*, the *incidence height*, will be denoted by the letter h and will be reckoned as positive when above the axis. The angle BOS comprised between the incident ray and the axis is the inclination of the ray with respect to the axis and will be denoted by u. The angle u has the same algebraical sign as the quotient $\frac{h}{s}$.*

The angle $BCS = \phi$ subtended at the centre of the sphere by the ordinate h, has the same sign as $\frac{h}{r}$.

The corresponding data of the refracted ray are distinguished from those of the incident ray by accented letters. O' is accordingly the point where the refracted ray intersects the axis. s', c', p' are the abscissæ of O' with reference to S, C and B, whilst u' is the angle comprised between the axis and the refracted ray. The angle of incidence is denoted by i, the angle of refraction by i' and the deviation of the ray by δ. The sign of these three angles is determined by the formulæ which serve for their trigonometrical calculation.

When the refractive indices n and n' of the media in front of and behind the refracting surface, the radius r of the sphere, the axial intercept s, and the inclination u of the incident ray are given, it remains to determine the corresponding values of s' and u'. This is accomplished by the following formulæ, which follow from the diagram, with the exception of the third equation, which expresses the law of refraction :—

$$c = s - r \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

$$\frac{c}{r} = \frac{\sin i}{\sin u} \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

$$n \sin i = n' \sin i' \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

$$\delta = i - i' \quad \ldots \quad \ldots \quad \ldots \quad \text{(iv)}$$

$$u' = u + \delta \quad \ldots \quad \ldots \quad \ldots \quad \text{(v)}$$

$$\frac{c'}{r} = \frac{\sin i'}{\sin u'} \quad \ldots \quad \ldots \quad \ldots \quad \text{(vi)}$$

$$' = c' + r \quad \ldots \quad \ldots \quad \ldots \quad \text{(vii)}$$

* This sign of u, as defined by the writers of other chapters of this book, differs from that adopted by Czapski, but agrees with the convention of modern exponents of optical computation (§ 28).

The first equation gives the distance c of the object-point from the centre, the second the angle of incidence i, the third the angle of refraction i', the fourth the deviation δ, the fifth the angle of inclination u' of the refracted ray, the sixth the distance c' of the image-point from the centre; and, finally, the seventh the distance of the same point from the vertex.

In addition, the figure furnishes the following equations :

$$\phi = i + u = i' + u' \quad \ldots \quad \ldots \quad \text{(viii)}$$

$$h = r \sin \phi \quad \ldots \quad \ldots \quad \ldots \quad \text{(ix)}$$

and from the triangle BOO',

$$\frac{p'}{p} = \frac{\sin u}{\sin u'}, \quad \ldots \quad \ldots \quad \ldots \quad \text{(x)}$$

from which we derive with the aid of equations (ii), (iii) and (vi)

$$\frac{p'}{p} = \frac{n'c'}{n\,c} \quad \ldots \quad \ldots \quad \ldots \quad \text{(xi)}$$

or

$$n' \frac{s' - r}{p'} = n \frac{s - r}{p}. \quad \ldots \quad \ldots \quad \text{(xii)}$$

The corresponding equations for the case of reflection at a spherical surface may be obtained directly from the above equations if we put $\frac{n'}{n} = -1$.

Thus

$$\left.\begin{aligned} c &= s - r \\ \frac{c}{r} &= \frac{\sin i}{\sin u} \\ i &= -i' \\ \delta &= 2\,i \\ u' &= u + 2\,i \\ \frac{c'}{r} &= -\frac{\sin i}{\sin (u + 2\,i)} \\ s' &= c' + r \end{aligned}\right\} \ldots \quad \ldots \quad \text{(xiii)}$$

In these equations we must always consider the direction of the incident light as positive. The inclinations with respect to the axis give us only the position of the refracted ray, they do not indicate its direction.

The formulæ for the computation of the refracted ray do not suffice for the determination of any number of successive

refractions unless all the planes of incidence are coincident, *i.e.*, unless the centres of all refracting spheres lie upon a straight line contained within the first plane of incidence and hence meeting the incident ray when produced, if necessary. In this case the spheres are said to be **centred.** We then have always $s_k = s'_{k-1} - d_{k-1}$, and $u_k = u'_{k-1}$, where d_{k-1} is the interval between the vertex of the k^{th} sphere and that of the $(k-1)^{th}$ surface, whilst s_k, s'_{k-1}, u_k, u'_{k-1} have the same meaning with respect to the $(k-1)^{th}$ and k^{th} surface respectively that the corresponding unaccented letters had with respect to the single surface considered above.

If refractions and reflections occur in the same system the direction of the incident light, which determines all signs, will be from *right to left* for a refraction that occurs after a first reflection. It is then best to make a new diagram in which the light proceeds again from *left to right*, and to give all the distances and angles occurring in the calculation new signs in conformity with the reversed direction of the light. In this chapter we shall, however, always assume that the light meets the first surface of the system in a positive direction, and we shall accordingly leave the positive direction of the co-ordinate axes unchanged.

The formulæ here given do not suffice when the surfaces are not centred, or when their common axis is at a skew angle with the incident ray. In these cases we must proceed as in § 36 to § 43.

B.—Normally Incident Finite Pencils. Spherical Aberration.

73. Let a pencil of rays converging to any point O or proceeding from it, be incident on a spherical surface whose centre is at C. If we join C and O by a straight line, it will be sufficient for the tracing of all rays of a pencil having its apex at O, to consider all the rays contained in a plane P passing through CO. For the whole pencil may readily be conceived as being generated by the rotation about the axis CO of the bundle of rays contained within the plane P. The complete system of rays is, in fact, symmetrical with respect to the axis CO.

Let g be any ray in the pencil O of rays situated in the plane P. Then the conjugate refracted ray g' is likewise contained in the plane P. If now both rays are made to revolve about the axis CO together with the plane P containing them, each will generate a cone. O is the apex of the cone of incident rays, whilst O' is the apex of the cone of refracted rays, being the point at which g' cuts the axis CO. The position of this point O' is determined by the formulæ in the preceding article, and depends therefore in general

upon the inclination u of the incident ray g. If we conceive the whole of the rays proceeding from O as constituting a series of cones having a common apex O, these will likewise have corresponding to them, after refraction, a series of cones having the same axis but having, in general, different apices. Rays inclined to the axis at the same angle u converge accurately to a single point after refraction, whereas rays which are originally inclined to the axis at different angles do not, in general, all meet at one point.

To demonstrate more strikingly this fact, as well as the relation of u and s', it should be noted that in formula § 72 (xii), viz.

$$\frac{n'(s'-r)}{p'} = \frac{n(s-r)}{p}$$

$$p^2 = (s-r)^2 + r^2 + 2r(s-r)\cos\phi$$

and, similarly,

$$p'^2 = (s'-r)^2 + r^2 + 2r(s'-r)\cos\phi\,.$$

From this it will be seen that similar values of s have corresponding to them different values of s' according to the magnitude of ϕ, i.e., the angular aperture of the rays g. This diversity in the position of the apex of the cones of refracted rays is referred to as *spherical aberration* (see the next chapter). We have already seen that it is possible to devise other surfaces, as for example, the Cartesian surfaces discussed in § 23, the property of which is such that a homocentric pencil having a given position and any angular aperture may be so refracted as to re-converge to a single point. The sphere belongs to surfaces of this type, inasmuch as it is likewise capable of transforming pencils occupying a definite position into homocentric pencils.

C.—Normally Incident Elementary Pencils. Points on the Axis.

74. By expanding $\cos\phi$ in terms of powers of ϕ it will be seen that for values of ϕ whose squares and higher powers are negligible in comparison with unity we can put $p = s$ and $p' = s'$. Using the symbols s, s' to indicate that we are dealing with paraxial rays, in accordance with the notation introduced in § 30, formula § 72 (xii) now becomes

$$\frac{n(s-r)}{s} = \frac{n'(s'-r)}{s'}. \quad \ldots \quad \ldots \qquad \text{(i)}$$

We see therefore that within a cone of rays of very small angular magnitude there is a definite relation between s and s', that is to say, the refracted rays form a homocentric pencil, and hence give rise to an *optical image*, in so far, at least, as points on the axis are concerned.

The union of normally incident pencils refracted at a spherical surface is of the *second order*; that is, to maintain an identical relation between the axial intercepts, it is necessary to neglect the second powers of ϕ (or u). Under these conditions the sphere coincides with any surface of revolution tangential to it at the vertex, so that the relation established by the last formula holds within the same degree of approximation for these surfaces as well, and in our formulæ r will then denote the radius of curvature of the surface tangential at the vertex.

Incidentally it may be pointed out that the equation § 72 (xi), viz., $\frac{p'}{p}=\frac{n'c'}{nc}$ which holds for any pair of conjugate points on the axis, may be expressed in the following forms with respect to two such pairs of points, $O_1O'_1$ and O_2O_2' ·

$$\frac{c_1'}{c_1} \bigg/ \frac{s_1'}{s_1}=\frac{n}{n'},$$
$$\frac{c_2'}{c_2} \bigg/ \frac{s_2'}{s_2}=\frac{n}{n'},$$

since, in the case of rays which are very near to the axis, $p=s$ and $p'=s'$. These equations signify, however, that the points $O_1\,S\,C\,O_1'$ and $O_2\,S\,C\,O_2'$ have the same cross-ratio and accordingly have homographic properties, and similarly $O_1\,S\,C\,O_1'$ is homographic with respect to $O_3\,S\,C\,O_3'$, etc. With the aid of the elements of projective geometry this homographic relation furnishes elegant expressions for the laws governing the formation of images by the refraction of thin pencils at centred spherical surfaces. This, in fact, is the method which was adopted by Mœbius (*4.*), Lippich (*1.*), Beck (*1.*) and Hankel (*1. 146*).

The equation § 72 (xii), $n'\frac{s'-r}{p'}=n\frac{s-r}{p}$, which expresses the relation between conjugate points on the axis, can readily be stated in terms of the **optical invariant** I. By the optical invariant is understood the value

$$I=n\sin i=n'\sin i', \qquad \ldots \qquad \ldots \qquad \text{(ii)}$$

which remains unaltered by the refraction. For infinitely small angles this expression assumes the form:

$$ni=n'i'.$$

Now, $i = \phi - u$, $i' = \phi - u'$ and, since the angles are infinitely small,

$$\phi = \frac{h}{r}; \quad u = \frac{h}{s}; \quad u' = \frac{h}{s'}.$$

Substituting these values in $ni = n'i'$ and dividing by h throughout, we shall have, as in § 31,

$$n\left(\frac{1}{r} - \frac{1}{s}\right) = n'\left(\frac{1}{r} - \frac{1}{s'}\right). \quad \ldots \quad \ldots \quad (1)$$

Since for the particular rays under consideration, $p = s$ and $p' = s'$, this equation differs only from equation § 72, (xii) as regards the factor $\frac{1}{r}$. It may be written in the form

$$\frac{n'}{s'} - \frac{n}{s} = \frac{n' - n}{r} \ldots \quad \ldots \quad \ldots \quad (1a)$$

and, if we denote the difference of the values of an expression before and after refraction by prefixing the symbol Δ, we may write briefly :

$$\Delta\left(\frac{n}{s}\right) = \frac{1}{r}\Delta n. \quad \ldots \quad \ldots \quad \text{(iii)}$$

D.—The Zero Invariant.

75. We shall call the expression $n\left(\frac{1}{r} - \frac{1}{s}\right) = n'\left(\frac{1}{r} - \frac{1}{s'}\right)$ the **zero invariant**, and denote it briefly by the symbol Q_s. It will be extensively used in the theory of aberrations. Some of the formulæ* which will then be required may now be deduced here with advantage.

Since

$$Q_s = n\left(\frac{1}{r} - \frac{1}{s}\right) = n'\left(\frac{1}{r} - \frac{1}{s'}\right),$$

$$\frac{1}{s'} = \frac{1}{r} - \frac{Q_s}{n'},$$

$$\frac{1}{s} = \frac{1}{r} - \frac{Q_s}{n}.$$

---

* These formulæ were communicated to the author by Drs. A. Koenig and M. v. Rohr, and the proofs are inserted here in accordance with their wishes.

Subtracting the second equation from the first and introducing the symbol Δ, we obtain

$$\Delta \frac{1}{s} = - Q_s \Delta \frac{1}{n} \quad \ldots \quad \ldots \quad \text{(i)}$$

or

$$Q_s = - \frac{\Delta \left(\frac{1}{s}\right)}{\Delta \left(\frac{1}{n}\right)}. \quad \ldots \quad \ldots \quad \text{(ii)}$$

Dividing the first equation by n', and the second by n and subtracting the second equation from the first, we obtain

$$\Delta \frac{1}{ns} = \frac{1}{r} \Delta \left(\frac{1}{n}\right) - Q_s \Delta \left(\frac{1}{n^2}\right). \quad \ldots \quad \text{(iii)}$$

Finally by squaring both equations and again subtracting the second from the first it follows that

$$\Delta \frac{1}{s^2} = - \frac{2\,Q_s}{r} \Delta \frac{1}{n} + Q_s^2 \Delta \frac{1}{n^2}, \quad \ldots \quad \text{(iv)}$$

whence, by substituting $- \Delta \frac{1}{s}$ for $Q_s \Delta \frac{1}{n}$, we obtain

$$Q_s^2 \Delta \frac{1}{n^2} = - \frac{2}{r} \Delta \frac{1}{s} + \Delta \frac{1}{s^2}, \quad \ldots \quad \text{(v)}$$

or, finally, with the aid of the equations $\Delta \frac{1}{ns} = \frac{1}{r} \Delta \frac{1}{n} - Q_s \Delta \frac{1}{n^2}$ and $Q_s \Delta \frac{1}{n} = - \Delta \frac{1}{s}$,

$$Q_s \Delta \frac{1}{ns} = \frac{1}{r} \Delta \frac{1}{s} - \Delta \frac{1}{s^2}. \quad \ldots \quad \ldots \quad \text{(vi)}$$

E.—The Formation of Images of Extra-axial Points and of Surfaces by Normal Incident Thin Pencils.

76. The image-formations discussed hitherto relate solely to points on a specified axis of a sphere. The relations so established obviously apply equally to any axis of the sphere. If, as an actual fact or in imagination, we consider none but exceedingly thin pencils of rays and cause all their axes to pass through C, we shall be able, within the limits of approximation here assumed, to transform by refraction at a single spherical surface, the entire

space whose index is n in front of the sphere, into the space whose index is n', part of which lies behind the refracting surface and is real, whilst another part is situated in front of the sphere and is virtual.

From the zero invariant it will be readily seen that spheres described about C in the object-space have likewise corresponding spheres described about C in the image-space. Corresponding figures on these spheres are similar and in mutual perspective

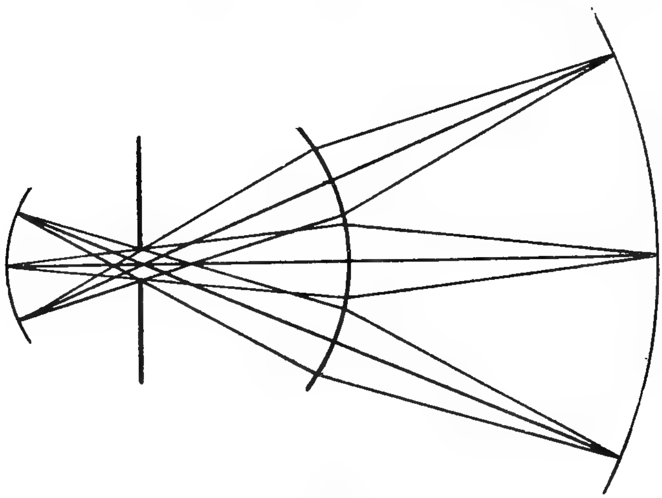

FIG. 38.
Formation of images by normally incident thin pencils.

with C as their centre of perspective, since the corresponding points all lie on rays passing through C. Infinitely distant elements have corresponding to them the elements of a sphere described about C as centre. It can easily be demonstrated that a plane has corresponding to it a surface of revolution of the second order. There is no particular optical axis since all the centre-lines are equally significant.

These conclusions show that images formed in this way do not belong to the type of homographic systems considered in the preceding chapter. Moreover, the assumptions from which we proceeded in the investigation of homographic systems are not all embodied in the present case. There is, indeed, a single point O' corresponding to a point O, but rays inclined at a finite angle to the axis passing through O do not, in general, pass through the point O' which corresponds to O, owing to the effect of spherical aberration. The principle of homographic image-ormation is therefore not applicable to these rays.

In fact, as we shall presently show, the image in this case is the aggregate of an infinite number of component bundles of rays (each pencil having a centre-line as its principal axis), each of which satisfies homographic conditions within the paraxial space.

F.—The Special Case of Paraxial Points. Collinear (Homographic) Image-formation.

77. If we consider one of these centre-lines OC, then, as was indicated above, all the points O situated on this centre-line correspond with points O' on this same centre-line, so long as we confine ourselves to rays that are inclined at infinitely small angles to the axis. If now we consider a point O_1 (Fig. 39), which, though not on the axis OC, is still infinitely near to it, we know that a corresponding single point O'_1 will be formed by rays which are infinitely near to the axis, since all these rays are infinitely near to its particular centre-line O_1C.

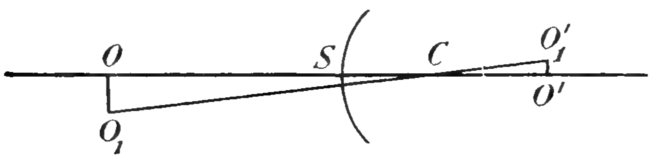

Fig. 39.

Formation of paraxial image-points.

We may imagine the axis to be surrounded by an infinitely narrow space in which all the homographic or collinear conditions of image-formation are fulfilled, so that every point in one space has corresponding to it one, and only one, point in the other space, and a ray passing through three points in the object-space has corresponding to it a ray which passes through the corresponding points in the image space. This, however, is true so long only as this ray is inclined at an infinitely small angle to the original optical axis, so that it always remains infinitely near to the centre-line which proceeds from each of these three points.

So long as we confine ourselves to points and rays of this kind, which will be called **paraxial** points and rays, we shall be free to apply all the relations deduced in the preceding chapter. It only remains to apply them to the special case now under consideration.

We have advisedly introduced this case of paraxial homographic image-formation by a non-collinear, though less restricted, process, in order to illustrate the applicability of the general laws

established in the preceding chapter and to indicate the reasoning which governs the range of conditions under which they can be applied.

G.—The Fundamentals of Image Formation by a Refracting Surface.

78. In an image-forming system of paraxial rays the axes of the object and image-spaces coincide with the centre-line of the sphere to which all points and rays with which we are concerned are supposed to be infinitely near. In the first place, the centre-line fulfils the condition of being conjugate to itself, in that a ray which is incident normally to the sphere proceeds without deviation; in the second place, planes normal to the centre line are transformed into similar planes. In fact, we have seen that in the case of non-collinear image-formation by any "central" pencils (§ 76), concentric spherical surfaces about the centre of the sphere are transformed into similar concentric surfaces. For spaces infinitely near to the axis, the two systems of image-formation which we have considered are identical in nature and accordingly in their resultant effect. Within this limited region a spherical surface normal to the axis may be regarded as identical with a normal plane element.

Every point of the conjugate spaces, *i.e.*, the paraxial spaces, may be regarded as a point of the object-space as well as of the image-space. This is an ideal condition which does not by any means arise in every actual optical image-forming system. According to its position with regard to the refracting surface, every point is the real or virtual focus of a pencil of rays. In the case of refraction the space in front of the refracting surface contains the real object-points and the space behind it the real image-points; whereas in the case of reflection for both object and image spaces, the real points are in front and the virtual points are behind the reflecting surface.

Proceeding from these considerations and the relationships of the conjugate axes it is possible, by a process of induction, to derive, for the type of system under consideration, the same general laws of image-formation which we have established deductively in the preceding section. This is, indeed, the procedure which was adopted by Gauss and Helmholtz, and after them, by the great majority of investigators. As we are already in possession of the general results we shall now adapt them particularly to the present case.

The focal planes have already been defined as the images of the infinitely distant planes. Their positions relatively to the vertex of the refracting surface may be found by equating to infinity s in the one case and s' in the other in the formula (1a) of § 74.

We then have for the distances of the foci F, F' from the vertex of a spherical surface of radius r:

$$SF = -\frac{nr}{n'-n}; \qquad SF' = \frac{n'r}{n'-n}. \quad \ldots \quad \text{(i)}$$

Also, by equations (18) in § 54,

$$f = -\frac{h'}{\tan u}; \; f' = -\frac{h}{\tan u'}.$$

If now B is the point of incidence of a ray parallel to the axis at a height h, and if from B a perpendicular is drawn to the axis, then, since the inclination of the ray to the axis is assumed to be infinitely small, it follows that the foot of this perpendicular coincides with the vertex S, and hence $\frac{SB}{SF'} = \tan u'$, and therefore $f' = -SF'$. Similarly $f = -SF$. Accordingly, **The focal-length of the object-space is equal to the distance, reckoned as negative, of the vertex from the object-focus, that of the image-space is equal to the distance, similarly reckoned as negative, of the vertex from the image-focus.** (This circumstance accounts for the introduction of the term *focal length* as the equivalent of a constant which we have already defined as a *ratio* of two quantities.)

We have accordingly

$$f = \frac{nr}{n'-n}; \; f' = -\frac{n'r}{n'-n}; \quad \ldots \quad \text{(ii)}$$

hence

$$f/f' = -n/n'. \quad \ldots \quad \ldots \quad \text{(iii)}$$

This equation shows that in a dioptric system the focal lengths have always opposite signs and that their ratio, which has already been shown to be constant, is equal to the ratio of the indices of refraction of the two media. The first conclusion places the dioptric system here considered in the main category of *obverse* systems, as defined in § 51 and § 68.

Moreover, from the expressions for f we are able to decide under what conditions a surface has a "converging" or a "diverging" effect, in the sense in which these terms have been employed previously. Thus, for example, the image-formation due to a single refraction is *converging* when

$$\left.\begin{array}{l} n' > n \text{ and } r > 0 \\ n' < n \text{ and } r < 0 \end{array}\right\};$$

and, conversely, it is *diverging* when

$$\left.\begin{array}{l} n' > n \text{ and } r < 0 \\ n' < n \text{ and } r > 0 \end{array}\right\}.$$

The appropriateness of the terms "converging" and "diverging" is most clearly indicated by the modification which every pencil undergoes in one or the other of these cases. For it will be readily seen that a pencil of rays is invariably rendered more convergent by refraction at a converging surface, whilst it always becomes more divergent when it is refracted at a diverging surface.

After the manner of Helmholtz, let $n'u'$ and nu denote the *optical convergence* before and after refraction. Then, since $i = \phi - u$, $i' = \phi - u'$, and for infinitely small angles $n'i' = ni$,

$$n'u' - nu = (n' - n)\,\phi \quad \ldots \quad \ldots \quad \text{(iv)}$$

is the change of the optical convergence, which, independently of the distance of the luminous point, is therefore proportional to the semi-aperture angle ϕ subtended at the centre of the sphere by the ray. The angle $\phi = \frac{h}{r}$ has the same sign as r (assuming that h is always positive), hence the optical convergence increases when $n' > n$ and $r > 0$, or when $n' < n$ and $r < 0$, that is when the optically denser medium is on the concave side of the sphere. (During this change the inclination of the ray, which can likewise be taken as a measure of the convergence, may increase or decrease.)

All the other quantities which serve to define the formation of images by a single refracting surface may readily enough be deduced from our assumptions with special reference to the above case; but, as we have already pointed out, they follow directly from the general equations of the theory of collinear image-formation, which are here capable of immediate application. It is only necessary to introduce the values of f and f' above determined and to take into consideration the positions of F and F'.

In particular, it should be observed that the two principal points coincide at the vertex of the refracting surface, and the nodal points at the centre of the sphere.

Since the principal points coincide with the vertex, and hence $\mathsf{A} = s$ and $\mathsf{A}' = s'$, the equations (29) in § 60 in conjunction with equation (iii), $f\,/\,f' = -\,n\,/\,n'$, assume the following forms:

$$\frac{f'}{s'} + \frac{f}{s} + 1 = 0, \quad \beta = \frac{ns'}{n's}, \quad \gamma = \frac{s}{s'}. \quad \ldots \quad \text{(v)}$$

The discussion of the formation of images by a single refracting surface is particularly interesting, inasmuch as the principal aim of the Gaussian theory was to express the effect of any number of refracting and reflecting surfaces in such terms as would render them analogous and comparable with the case of the formation of images by a single refracting or reflecting surface.

H.—Refraction at a Plane.

79. The formulæ which hold in the case of the plane can easily be established independently, but we shall derive them here from the expressions obtained for a spherical surface by putting $r = \infty$. The principal foci will then be situated at infinity, and the focal lengths become infinitely great. The system is therefore telescopic. The ratio of the two focal lengths, on the other hand, remains finite, being as before $f / f' = - n / n'$.

By equation (1a) in § 74 or by (v) in § 78

$$\frac{n'}{s'} = \frac{n}{s} \text{ or } \frac{s'}{s} = \frac{n'}{n}.$$

The abscissæ of the object and image-points measured with reference to the refracting plane are proportional for all values of these abscissæ. The image and the object are on the same side of the plane, since n'/n is always positive in the case of refraction. According to the above equations (v) the lateral magnification $\beta = 1$ and the angular magnification $\gamma = \frac{n}{n'}$.

I.—Reflection at a Spherical Surface.

80. The formulæ for reflection at a spherical surface are derived from those established for refraction by putting $\frac{n'}{n} = -1$. We then have by § 78 (i)

$$SF = SF' = \frac{r}{2},$$

and therefore

$$f = f' = -\frac{r}{2}.$$

Both principal foci coincide at the middle point of the radius vector between the centre and vertex. The two focal lengths are equal in magnitude, being $-\frac{r}{2}$. This is subject to the convention that the positive direction along the axis is always reckoned from left to right, so that in the image-space it does not conform with the direction of the incident light. The systems of co-ordinates of the object-space and image-space are completely coincident, and thus there arises the case discussed at the end of the preceding chapter in § 68, where both axes are parallel, and the system is of the *reverse* kind, since the focal lengths have the same sign.

Since f and r have opposite signs it follows that the surface has a converging effect when r is negative, *i.e.*, in the case of mirrors which present their concave side to the incident light, whereas the effect is diverging when the convex side faces the incident light.

The equation for the optical convergence is by § 78 (iv)

$$u' + u = 2\,\phi. \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

When interpreting this equation, it should, however, be noted that the angle u' determines merely the position of the reflected ray and not its direction. If we consider that reflection implies a reversal of the direction of the light it will be seen that for positive values of u' the rays diverge from the mirror, whereas positive values of u imply convergence and negative values of u' divergence. If, accordingly, we write the above equations in the form $u' - (-u) = 2\,\phi$ it signifies that reflection causes an increase equal to $2\,\phi$ in the optical divergence of the rays. In convex mirrors ϕ is positive and the divergence increases, whilst in concave mirrors ϕ is negative and the divergence diminishes.

Since the principal foci of the image and object spaces coincide, whilst their focal lengths are equal, it follows that a given point O has always the same point corresponding to it, whether we regard it as a point in the object-space or one in the image-space. The object and image spaces have a property of involution in the language of geometry of position.

The equation for the abscissa with reference to the vertex becomes in the case of reflection, by § 78 (v),

$$\frac{1}{s'} + \frac{1}{s} = -\frac{1}{f} = \frac{2}{r}. \quad \ldots \quad \ldots \quad \text{(ii)}$$

The lateral magnification is :

$$\beta = -\frac{s'}{s}; \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

and the angular magnification

$$\gamma = \frac{s}{s'}. \quad \ldots \quad \ldots \quad \ldots \quad \text{(iv)}$$

J.—Reflection at a Plane Surface.

81. When the reflecting surface is a plane we must again make r infinite, in which case

$$s = -s', \qquad \beta = +1, \qquad \gamma = -1.$$

The object and the image are then situated symmetrically with respect to the reflecting plane and are of equal magnitude.

It is easy to see that in the case of reflection at a plane surface the formation of the image is not restricted in extent or effect to a narrow space about a straight line normal to the surface. On the contrary, a strictly geometrical image is formed without superficial limitations by pencils of any width. This is, indeed, the only case which exists of such extended images being formed. Since, however, it provides no scope for modifying the conditions of position and magnitude in the two spaces it is obviously of very restricted importance as regards the essential requirements of optical instruments. Reflection at a plane or several planes provides a means of transposing an object or an image formed by other optical systems in an unchanged form to another position in space. The image formed by an odd number of plane mirrors is symmetrical with respect to the object, and that due to an even number of reflections is therefore equal in every respect to the object. This is self-evident from the preceding discussion.*

K.—Refraction at Several Surfaces (Centred Optical System).

82. The formation of images by the refraction or reflection of elementary paraxial pencils at any number of spherical surfaces separating media of different refractive properties, can be investigated in terms of that due to a single surface, and the resulting relations can be deduced from our general laws of collinear image-formation and the combination of these systems. Supposing the first surface, which we have already investigated, to be succeeded by a second surface having its centre within the elementary space, in which alone it can form collinear images, the image-space of the first surface may be regarded as the object-space of the second surface ; for, in the second case also, the axes and rays of all incident pencils are inclined at very small angles to the centre-line of this second surface. The same argument may be applied to any number of successive surfaces. If the centres of these surfaces lie, not only within the elementary space associated with the first surface, but also on the same straight line, the surfaces are said to be centred. It follows from the previous discussion that this straight line itself is the principal axis of the image-formation or simply the axis of the system.

The positions of the foci of such a system are not difficult to calculate. To accomplish this we may either apply formulæ (1) or (1*a*) of § 74 in rotation, or we may combine the successive formulæ into a continued fraction ; or, finally, we may use the method of

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* Images formed by multiple reflection have been fully described in text-books on geometrical optics by Lloyd (*1.*), Parkinson (*1.*), Meisel (*1.*), Heath (*1. 2.*) ; while Maurer (*1.*) gives a list of special references in a paper on several interesting questions relating to the optical square.

determinants. The same applies to the calculation of the focal lengths f and f' of the compound system. We shall not go further into this subject, as we have already discussed the principles of the problem in § 67. We must accordingly content ourselves with drawing attention to the references there given, noting that the subject is treated by the writers quoted rather from the special aspect in which we regard it here than as a general investigation.

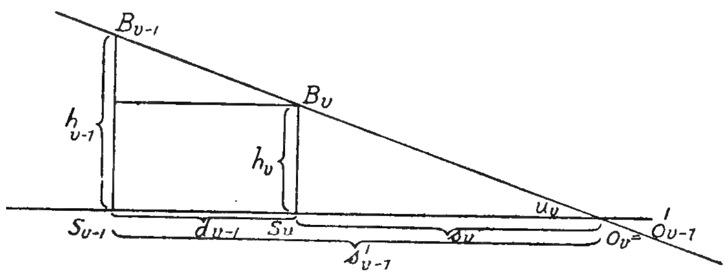

FIG. 40.

$S_{v-1}B_{v-1} = h_{v-1}$; $S_v\ B_v = h_v$; $S_{v-1}O_v = s'_{v-1}$; $S_v\ O_v = s_v$

Computation of the focal length.

Incidentally it may be shown how the focal lengths can be obtained from the abscissæ referred to the vertex, as given in the preceding chapter. Let s_v and s_v' as before, be the axial intercepts before and after refraction at the v^{th} surface, and let h_v be the ordinate of the point where the ray meets the v^{th} surface. If now we ascribe to the last surface of the system the suffix k, it follows from the definition of the focal length in equation (18) in § 54 that

$$f' = -\frac{h_1}{\tan u_k'} = -\frac{h_1}{h_2} \cdot \frac{h_2}{h_3} \cdot \frac{h_3}{h_4} \cdot \cdots\cdots \frac{h_k}{\tan u_k'} \qquad \text{(i)}$$

but from Fig. 40

$$\frac{h_{v-1}}{h_v} = \frac{s_v'{}_{-1}}{s_v} \qquad \ldots \qquad \ldots \qquad \ldots \qquad \text{(ii)}$$

and for the last surface

$$h_k = s_k' \tan u_k', \qquad \ldots \qquad \ldots \qquad \text{(iii)}$$

hence

$$f' = -\frac{s_1' \cdot s_2' \cdot s_3' \ldots s_k'}{s_2 \cdot s_3 \ldots s_k} \cdot \ldots \qquad \ldots \qquad \text{(iv)}$$

To determine the focal length in the object-space it is only necessary to trace in the reverse direction a ray which enters the system in a

direction parallel to the axis. The signs should obviously be so chosen as to conform to this inverse direction of the light, and in the final result the signs must again be changed conformably to the original direction of incidence. The equation $f/f' = -n/n'$ (§ 78, iii), which will presently be shown to hold for a composite system as well as for a single surface, may serve as a check.

The lateral and the angular magnifications can likewise be expressed in terms of the axial intercepts of the paraxial rays. With the aid of the equations given in § 78 (v), viz.

$$\beta = \frac{ns'}{n's} \text{ and } \gamma = \frac{s}{s'},$$

and noting that $n'_{v-1} = n_v$ and $y'_{s,v-1} = y_{s,v}$, we find at once that

$$\beta = \frac{y'_{s,k}}{y_{s,1}} = \frac{y'_{s,1}}{y_{s,1}} \cdot \frac{y'_{s,2}}{y_{s,2}} \cdots \frac{y'_{s,k}}{y_{s,k}} = \beta_1 \beta_2 \ldots \beta_k = \frac{n_1 s_1'}{n'_1 s_1} \cdot \frac{n_2 s_2'}{n_2' s_2} \cdots \frac{n_k s_k'}{n_k' s_k}$$

$$= \frac{n_1}{n_k'} \cdot \frac{s_1' \, s_2' \ldots s_k'}{s_1 \, s_2 \ldots s_k} \qquad \ldots \qquad \ldots \qquad \ldots \text{ (v)}$$

and, similarly,

$$\gamma = \gamma_1 \gamma_2 \ldots \ldots \gamma_k = \frac{s_1 \, s_2 \ldots s_k}{s_1' s_2' \ldots s_k'} \, . \qquad \ldots \qquad \ldots \text{ (vi)}$$

L.—Two Formulæ due to Seidel.

83. At this point it will be convenient to introduce two formulæ evolved by Seidel (*2.*), which establish the relation between two paraxial rays traversing the same optical system. These two expressions, interesting enough in themselves, will be employed in the theory of spherical aberration.

We shall trace the paths of two paraxial rays proceeding respectively from two points O_1, P_1 on the axis, through the whole optical system. (Generally, O_1 is the point where the axis meets the object-plane and P_1 the point where it meets the entrance-pupil.) Before and after refraction at the v^{th} surface let the ray proceeding from O_1 intersect the axis at O_v and O_v' respectively, and let the ray proceeding from P_1 meet it at P_v and P_v' respectively. Let s_v and s_v' be the abscissæ of O_v and O_v' with respect to the vertex S_v of the v^{th} surface, and, similarly, let x_v and x_v' be those of P_v and P_v'. Let the ray proceeding from O_1 intersect the v^{th} surface at a height h_v from the axis, and let that proceeding from P_1 meet the v^{th} surface at a height y_v from the axis. Let d_v be the distance $S_v S_{v+1}$, as before. Then, according to § 31 and § 75, if we distinguish the zero invariants of the v^{th}

surface relating to O and P respectively, by the suffixes s and x, the expressions for the zero invariants will assume the following forms:

$$Q_{vx} = n_v'\left(\frac{1}{r_v} - \frac{1}{x_v'}\right) = n_v\left(\frac{1}{r_v} - \frac{1}{x_v}\right),$$

$$Q_{vs} = n_v'\left(\frac{1}{r_v} - \frac{1}{s_v'}\right) = n_v\left(\frac{1}{r_v} - \frac{1}{s_v}\right);$$

hence

$$Q_{vx} - Q_{vs} = n_v'\left(\frac{1}{s_v'} - \frac{1}{x_v'}\right) = n_v\left(\frac{1}{s_v} - \frac{1}{x_v}\right),$$

or, since

$$s_v = s'_{v-1} - d_{v-1}, \quad x_v = x'_{v-1} - d_{v-1} \text{ and } n_v = n'_{v-1}.$$

$$Q_{vx} - Q_{vs} = n'_{v-1}\left(\frac{1}{s'_{v-1} - d_{v-1}} - \frac{1}{x'_{v-1} - d_{v-1}}\right)$$

$$= \frac{n'_{v-1}\, s'_{v-1}\, x'_{v-1}\,(x'_{v-1} - s'_{v-1})}{s'_{v-1}\, x'_{v-1}\, s_v\, x_v}$$

$$= n'_{v-1}\left(\frac{1}{s'_{v-1}} - \frac{1}{x'_{v-1}}\right)\frac{s'_{v-1}\, x'_{v-1}}{s_v\, x_v} = \frac{s'_{v-1}\, x'_{v-1}}{s_v\, x_v}\left(Q_{v-1,x} - Q_{v-1,s}\right).$$

In place of the ratios of s and x we may introduce those of h and y, in which case

$$\frac{s'_{v-1}}{s_v} = \frac{h_{v-1}}{h_v} \text{ and } \frac{x'_{v-1}}{x_v} = \frac{y_{v-1}}{y_v}$$

hence

$$Q_{v,x} - Q_{v,s} = \frac{h_{v-1}\, y_{v-1}}{h_v\, y_v}\left(Q_{v-1,x} - Q_{v-1,s}\right). \quad \dots \quad (1)$$

Applying this formula in the reverse order to successive transitions from the last, or k^{th}, to the k-1^{th} surface, thence to the k-2^{th} surface, and so forth, and multiplying together all the successive equations, we arrive finally at the following equation:

$$Q_{k,x} - Q_{k,s} = \frac{h_1 y_1}{h_k y_k}\left(Q_{1,x} - Q_{1,s}\right) \quad \dots \quad (2)$$

Also, from Fig. 40, it follows that

$$\frac{h_{v-1} - h_v}{h_{v-1}} = \frac{d_{v-1}}{s'_{v-1}}$$

and, similarly,

$$\frac{y_{v-1} - y_v}{y_{v-1}} = \frac{d_{v-1}}{x'_{v-1}}.$$

Subtracting the first equation from the second,

$$\frac{h_v}{h_{v-1}} - \frac{y_v}{y_{v-1}} = d_{v-1}\left(\frac{1}{x'_{v-1}} - \frac{1}{s'_{v-1}}\right),$$

or, multiplying by $\frac{y_{v-1}}{h_v}$,

$$\frac{y_{v-1}}{h_{v-1}} - \frac{y_v}{h_v} = d_{v-1}\frac{y_{v-1}}{h_v}\left(\frac{1}{x'_{v-1}} - \frac{1}{s'_{v-1}}\right)$$

$$= - d_{v-1}\frac{y_{v-1}}{h_v}(Q_{v-1,x} - Q_{v-1,s})\frac{1}{n'_{v-1}},$$

or, in conjunction with equation (2),

$$\frac{y_v}{h_v} = \frac{y_{v-1}}{h_{v-1}} + h_1 y_1 (Q_{1,x} - Q_{1,s})\frac{d_{v-1}}{h_{v-1}\, h_v\, n_v} \quad . \quad \text{(ii)}$$

By summing all the equations which result if we substitute for v its successive values $v = k$, $v = k - 1$, $v = k - 2$ . . . . $v = 2$, and multiplying both sides by $\frac{h_k}{y_1}$, we obtain the equation :

$$\frac{y_k}{y_1} = \frac{h_k}{h_1} + h_1 h_k (Q_{1,x} - Q_{1,s})\sum_{v=2}^{k}\frac{d_{v-1}}{n_v\, h_{v-1}\, h_v} \quad . \quad \ldots \text{(3)}$$

If we assume the refractive indices, the radii and the thicknesses to be given, the position of a ray in front of and behind the vth surface is determined by the vth zero invariant and the incident height of the vth point, since from the zero invariants, the intercepts of the rays on the axis may be derived directly, and conversely from the latter the zero invariants may be computed. Knowing the path of a ray, as for example the ray proceeding from O determined by the values of $Q_{v,s}$ and h_v, we can use this ray of reference commencing with the values y_1 and $Q_{1,x}$, to compute for any other ray the value of y_k with the aid of formula (3), and then $Q_{k,x}$ with the aid of formula (2). It will thus be seen that we are able to compute the terminal position of the second ray from its initial position.

Seidel (*2.*) employed these formulæ for the production of the fundamental dioptric formula, which we can only mention here.

M.—The Smith-Helmholtz Equation.

84. We now proceed to prove the following theorem, which has already been mentioned above :

In every optical system the absolute values of the focal lengths f and f' of the object and image-spaces are in the ratio

of the refractive indices n and n' of these two spaces. The absolute values of the focal lengths are therefore equal when the first and last media have the same refractive index.

The two focal lengths have opposite signs if the system consists solely of refractions or if the number of reflections is even ; and they have the same sign if there is an odd number of reflections.

To prove this theorem we shall employ the formulæ (54) and (53) established in § 67 for the combination of optical systems. These formulæ were :

$$f = (-1)^{k-1} \frac{f_1 f_2 \ldots \ldots f_k}{N_k}, \quad \ldots \qquad \text{(i)}$$

$$f' = \frac{f_1' f_2' \ldots \ldots f_k'}{N_k},$$

hence

$$\frac{f}{f'} = (-1)^{k-1} \frac{f_1 f_2 \ldots \ldots f_k}{f_1' f_2' \ldots \ldots f_k'}.$$

Now, by § 78 (iii) $\dfrac{f_v}{f_v'} = -\dfrac{n_v}{n_v'}$, hence

$$\frac{f}{f'} = (-1)^{2k-1} \frac{n_1 n_2 \ldots \ldots n_k}{n_1' n_2' \ldots \ldots n_k'} = -\frac{n_1 n_2 \ldots \ldots n_k}{n_1' n_2' \ldots \ldots n_k'}. \qquad \text{(ii)}$$

If there are reflections only, $n_v' = n_{v+1}$, so that

$$\frac{f}{f'} = -\frac{n_1}{n'} = -\frac{n}{n'}, \quad \ldots \quad \ldots \qquad \text{(iii)}$$

where $n = n_1$ and $n' = n_k'$ denote the refractive indices in front of and behind the entire system.

If the vth surface is a reflecting surface, $\dfrac{n_v}{n_v'} = -1$, $n'_{v-1} = n_{v+1}$, so that

$$\frac{f}{f'} = \frac{n}{n'}. \quad \ldots \quad \ldots \quad \ldots \qquad \text{(iv)}$$

Every reflection implies a change of sign. The arrangement of the system must be supposed to be such that the initial direction of light remains the positive direction, even when the direction has been reversed by the occurrence of a reflection. We shall suppose the focal lengths of all the individual

surfaces to be fixed with reference to the original direction of the light, say the direction from left to right, and we shall define the terms **front** and **back focal length** as the focal lengths in the object and image spaces respectively, in conformity with the direction of the light as here defined. If now, in consequence of the occurrence of a reflection, we have to traverse a surface from right to left, it follows that the back focal length becomes that of the object-space and the front focal length that of the image-space.

In § 56, equation (22) it was shown that the product of the lateral and angular magnifications is always $\beta \,.\, \gamma = -\frac{f}{f'}$. If in this equation we introduce the value just found for the ratio of the focal lengths, viz., $\frac{f}{f'} = \mp \frac{n}{n'}$, then

$$\beta \,.\, \gamma = \pm \frac{n}{n'} \quad \cdots \quad \cdots \quad \cdots \quad \text{(v)}$$

or

$$n' y_s' u' = \pm n y_s u \,. \quad \cdots \quad \cdots \quad (4)$$

The positive sign applies when there are refractions only or an even number of reflections, and the negative sign when there is an odd number of reflections.

Equation (4) is usually referred to as the *Helmholtz-Lagrange formula.* It would be more correct to call it the *Smith-Helmholtz equation.*

When there are refractions only and $n = n'$, so that $f' = -f$, the principal points coincide with the nodal points. If in this case reference be made to the sign of the focal length, without specifying one or other focal length, the statement is understood to refer to the value of f.

The simple geometrical meaning which attaches to the term focal length in the case of refraction at a single surface, and also at several surfaces separated by infinitely small intervals, ceases to be applicable to a system of surfaces when these are separated by finite intervals: for the "focal length" cannot in this case be defined as the negative distance of the principal focus from the refracting surface. The principal foci and all other cardinal points previously considered may then assume any positions relatively to the refracting surfaces and relatively to each other, without prejudice to the general laws which we have already established.*

* The reader is referred to the discussions of Matthiessen (*9.*), Brockmann (*1.*), and others.

N.—Lenses of Finite Thickness.

85. Special practical importance attaches to the case of a system consisting of two refracting surfaces bounded on either side by a medium of the same refractive index which we may make equal to unity. The importance of this case arises from the fact, that nearly all technically produced optical systems consist almost exclusively of binary elements of this kind, whilst all others may be resolved into such binary components or *lenses.*

According to the curvature of the surfaces bounding the lenses externally, we describe the lenses as *bi-convex*, *bi-concave*, *plano-convex*, *plano-concave*, *concavo-convex.* Frequently these terms are employed in such a way as to indicate the position of the lens relatively to the incident rays, which may be done by connecting the terms referring to the respective surfaces in the order in which they are traversed by the incident light, making thus a distinction between *plano-convex* and *convexo-plane* lenses and between *concavo-convex* and *convexo-concave* lenses.

If the first and second surfaces are distinguished by suffixes 1 and 2 respectively, and if the refractive index of the substance of the lens relatively to the external medium is denoted by n, and the distance between the vertices of the two lens surfaces that is the thickness of the lens, by d, then

$$\left.\begin{aligned} f_1 &= \frac{r_1}{n-1}\,; \quad f_1' = -\frac{nr_1}{n-1} \\ f_2 &= -\frac{nr_2}{n-1}\,; \quad f_2' = \frac{r_2}{n-1} \end{aligned}\right\} \quad \ldots \qquad \text{(i)}$$

The distances from the vertices to the principal foci of the individual surfaces are equal to the respective focal lengths reckoned negatively. The quantity $\triangle = F_1'F_2$, which occurs in the formulæ for the position of the two centred systems, *i.e.*, the distance of the front principal focus of the second system from the back principal focus of the first system, becomes accordingly in this case

$$\triangle = F_1'\,S_1 + S_1\,S_2 + S_2\,F_2 = f_1' + d - f_2,$$

or, substituting the above values for f_1' and f_2,

$$\triangle = \frac{R}{n-1}, \quad \ldots \quad \ldots \quad \ldots \qquad \text{(ii)}$$

where, for brevity, we put

$$R = n\,(r_2 - r_1) + d\,(n-1). \quad \ldots \qquad \text{(iii)}$$

Substituting this value of $\triangle$ and those of f_1, f_1', f_2 and f_2' in the equations (37) and (38), § 63, we obtain the following formulæ for the focal length of the lens, namely :

$$f = \frac{nr_1r_2}{(n-1)R}; \quad f' = -\frac{nr_1r_2}{(n-1)R} = -f; \tag{5}$$

and for the distances of the principal foci of the lens from the front principal focus of the first surface and from the back principal focus of the second surface, by equations (36) and (35) § 63.

$$\sigma = F_1F = -\frac{nr_1^2}{(n-1)R}; \quad \sigma' = F_2'F' = -\frac{nr_2^2}{(n-1)R}.$$

From σ and σ' we can compute the distances of the principal foci of the lens from the vertices of the respective surfaces, viz :

$$\left.\begin{aligned} s_F &= S_1F = S_1F_1 + F_1F = -f_1 + \sigma = \frac{-r_1}{n-1} - \frac{nr_1^2}{(n-1)R} \\ &= -\frac{r_1(nr_1+R)}{(n-1)R} \\ s_{F'} &= S_2F' = S_2F_2' + F_2'F' = f_2 + \sigma' = \frac{-r_2}{n-1} + \frac{nr_2^2}{(n-1)R} \\ &= \frac{r_2(nr_2-R)}{(n-1)R}. \end{aligned}\right\} \tag{6}$$

The principal and the nodal points coincide since $n = n'$. Their co-ordinates referred to the principal foci F and F' are f and f' and their distances from the vertex of the lens are :

$$\left.\begin{aligned} s_H &= S_1H = S_1F + f = s_F + f = -\frac{r_1d}{R} \\ s_{H'} &= S_2H' = S_2F' + f' = s_{F'} + f' = -\frac{r_2d}{R} \end{aligned}\right\} \tag{7}$$

As we shall see later, it is frequently necessary to operate with the reciprocals of the focal lengths in the object-space, the so-called **powers** of the lenses, viz., $\frac{1}{f} = \phi$. From the above it will be seen that the power may be expressed in terms of the reciprocals of the radii, *i.e.*, the curvatures $\frac{1}{r_1} = \rho_1$ and $\frac{1}{r_2} = \rho_2$ as follows :

$$\phi = (n-1)(\rho_1 - \rho_2) + \frac{(n-1)^2}{n}\rho_1\rho_2 d. \tag{8}$$

86. Rayleigh (*2.* art. 5), proceeding from the principle of stationary paths, has derived an expression for the focal length of a convex lens in terms of its effective aperture, its thickness and the index of refraction. In order that the plane wave RPQ may reach F' with unchanged phase, it follows, on reference to Fig. 41, that the reduced path $[PDF']$ must be equal to $[RAF']$. Now the reduced path becomes

$$[PDF'] = DF' + n\,.\,PD = DF' + nd = PC + CF' + (n-1)\,d\,.$$

In a thin lens CF' does not sensibly differ from f, so that the reduced path is

$$[PDF'] = PC + f + (n-1)\,d\,.$$

Denoting the semi-aperture of the lens by y, the path of the ray which touches the sharp edge of the lens will be given by

$$[RAF'] = RA + AF' = PC + \sqrt{f^2 + y^2}$$
$$= PC + f\left(1 + \frac{1}{2}\frac{y^2}{f^2} + \dots\dots\right).$$

Hence, if we neglect all powers of $\frac{y}{f}$ above the second,

$$[RAF'] = PC + f + \frac{1}{2}\frac{y^2}{f}\,.$$

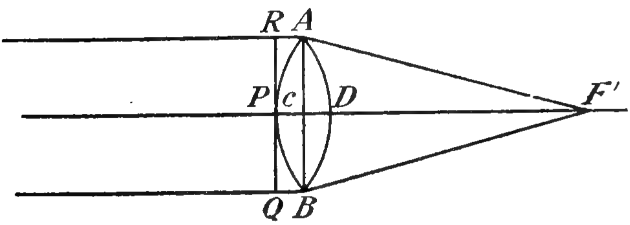

FIG. 41.
$PD = d$; $CA = y$,
Rayleigh's determination of the focal length.

In order that $[RAF']$ may be equal to $[PDF']$, we must have

$$(n-1)\,d = \frac{1}{2}\frac{f}{y^2}$$

and hence

$$f = \frac{1}{2}\frac{y^2}{(n-1)\,d}\,.$$

In the case of a lens made of glass having a refractive index $n = 1.5$, the semi-aperture of the lens is then the mean proportional between the thickness and the focal length, which is a well known rule.

In the case of lenses which are not bounded by a sharp-edged periphery, d should be reckoned as the difference of the thicknesses at the edge and at the middle of the lens.

87. The sign of f or ϕ, as the case may be, depends not only on the radii but also on the thickness of the lens. In the case of a bi-convex lens, for instance, $r_1 > 0$, $r_2 < 0$, so that f_1 and f_2 are both positive, hence $f = -\dfrac{f_1 f_2}{\triangle}$ is positive when $\triangle$ is negative, and, since $\triangle = d + \dfrac{n}{n-1}(r_2 - r_1)$, this is the case so long as

$$d < \frac{n\,(r_1 - r_2)}{n-1}.$$

When d exceeds this limiting value, f becomes negative and the lens becomes diverging despite the fact that it is composed of two converging elements. When $d = \dfrac{n(r_1 - r_2)}{n-1}$, so that $\triangle = 0$ and $f = \infty$, the lens forms a telescopic system.

The magnitude and sign of the focal length of a simple lens of any other form may be investigated in a similar manner.

O.—Thin Lenses.

88. These formulæ become especially simple when the thickness of the lens is small in comparison with its focal length so that it can be equated to zero. In this case, by equations (5) and (8)

$$f = \frac{r_1\, r_2}{(n-1)\,(r_2 - r_1)}; \quad \phi = (n-1)\,(\rho_1 - \rho_2). \quad \dots \quad (9)$$

The power of the lens is then proportional, on the one hand, to the refractive index diminished by unity, and, on the other, to the difference of the two curvatures taken in the order of the direction of incidence. The latter alone determines the signs of f and ϕ. The lens has a converging or diverging effect according as the outwardly directed convex or concave surface respectively of the lens has the greater curvature.

In many cases, to arrive at a preliminary rough estimate of the optical effect of the system, by way of a first approximation, the lenses may be regarded as infinitely thin. It will therefore be useful here to refer briefly to the theory of thin lenses.

A comparison of the formulæ (7) in § 85 shows that if we put $d = 0$, the focal lengths of such a lens are equal to the distances of the foci from the lens, reckoned negatively. The theory of infinitely thin lenses thus becomes very similar to that

of a single refracting surface, except that the focal lengths of the lenses assume equal absolute values in this theory. The foci are accordingly situated symmetrically with respect to the lens, and the principal points and the nodal points coincide at the vertex of the lens.

The distances s and s' of conjugate points from the vertex of the lens are obtained from the formulæ (29) of § 60.

Since $f' = -f$, $\mathsf{A} = s$ and $\mathsf{A}' = s'$, it follows from this equation that

$$\frac{1}{s'} - \frac{1}{s} = \frac{1}{f} \quad \text{or} \quad f = \frac{ss'}{s - s'}. \qquad \ldots \qquad (10)$$

As before, s and s' are reckoned positive in the direction of the incident light.

For the lateral and angular magnifications we obtain similar expressions as before, in accordance with § 82, viz.:

$$\beta = \frac{y'_s}{y_s} = \frac{s'}{s}; \quad \gamma = \frac{u'}{u} = \frac{s}{s'}. \qquad \ldots \qquad (11)$$

In the case of two infinitely thin lenses separated by a finite distance d, their optical interval $\triangle$ can be calculated by means of an expression similar to that obtained for two surfaces in § 85 and § 63 (37, 38).

$$\triangle = -f_1 - f_2 + d,$$

where f_1 and f_2 denote the focal lengths in the object-space of the two lenses respectively. The focal length f of a system composed of these two lenses becomes accordingly by § 63 (37, 38).

$$f = \frac{f_1 f_2}{f_1 + f_2 - d} \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

and the power ϕ of the system is

$$\phi = \frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2} = \phi_1 + \phi_2 - \phi_1 \phi_2 d. \qquad (12a)$$

In discussing this equation we have to distinguish three cases viz.:

(1) *f_1 and f_2 both positive*, so that both lenses are converging. In this case the focal length of the compound system has its smallest positive value, and the system accordingly attains its highest power when $d = 0$, *i.e.*, when the lenses are in contact. As d increases f does so likewise and ultimately becomes infinite when $d = f_1 + f_2$. In this case the two lenses form a telescopic system. As d increases still further f assumes a large negative value, which diminishes without limit with the further increase of d.

(2) f_1 *and* f_2 *both negative*, so that both lenses are diverging. In this case $d = 0$ causes f to assume its greatest value, its sign being negative. As d increases f diminishes, but always remains negative.

(3) *One lens converging, the other diverging*, say f_1 positive, f_2 negative. In this case, if $d = 0$, the focal length of the compound system is positive or negative according as the absolute value of f_1 is less or greater than f_2.

(a) When $f_1 > f_2$, and therefore, $f_1 + f_2$ is positive, f assumes increasing negative values as d increases, and when $d = f_1 + f_2$ it becomes infinite, *i.e.*, the system becomes telescopic. A further increase of d causes f to assume decreasing positive values.

(b) When $f_1 < f_2$, and therefore, $f_1 + f_2$ is negative, f has its greatest positive value when the lenses are in contact. As d increases f diminishes, but always remains positive.

In these cases, whenever $d = f_1 + f_2$ the front focus of the second lens coincides with the back focus of the first lens; f becomes infinite and the system is telescopic. The magnifications of such a telescopic system are expressed by the equations given in § 64, viz.:

$$\beta = \frac{y_s'}{y_s} = -\frac{f_2}{f_1}\,; \quad \gamma = \frac{u'}{u} = -\frac{f_1}{f_2}.$$

When the distance d between the lenses is zero, the power of the system is equal to the sum of the powers of the two lenses, thus:

$$\phi = \phi_1 + \phi_2 \quad \text{or} \quad \frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}.$$

This result is applicable to any number of infinitely thin lenses in contact if the total thickness of the system in comparison with its focal length may be neglected. Supposing there are k such surfaces, whose focal lengths are $f_1, f_2 \ldots\ldots f_k$, then the power of the whole system is equal to the algebraical sum of the powers of the component lenses.

Thus
$$\phi = \phi_1 + \phi_2 + \ldots\ldots \phi_k \quad \ldots \quad \ldots \quad (13)$$

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} + \ldots\ldots \frac{1}{f_k}, \quad \ldots \quad \ldots \quad (13a)$$

or, when by (8) expressed in terms of the radii and curvatures,

$$\phi = (n_1 - 1)\left(\frac{1}{r_1} - \frac{1}{r_1'}\right) + (n_2 - 1)\left(\frac{1}{r_2} - \frac{1}{r_2'}\right) + \ldots\ldots$$

$$\phi = \sum_{v-1}^{k} (n_v - 1)(\rho_v - \rho_v'), \quad \ldots \quad \ldots \quad (13b)$$

the first and second surfaces of each lens being distinguished by the upper indices, whilst the lenses themselves correspond to the lower indices.

P.—Cotes' Theorem.

89. In view of its historical interest and its elegant form we shall give here an account of Cotes' theorem relating to infinitely thin lenses. This theorem gives the apparent distance of an object as seen through several infinitely thin lenses. By the *apparent distance* of an object Cotes understands the distance at which it should be placed to be seen by the unaided eye at the same angle of view which it subtends at the eye when seen through the optical system. Let OM (Fig. 42) be the object and let it be viewed through three infinitely thin lenses L_1, L_2 and L_3, having their vertices at S_1, S_2, S_3 by an eye placed at Q_4. Let the ray which reaches the eye from the extreme point M of the object meet the lenses at the points P_1, P_2 and P_3. Through M, P_1 and P_2 draw lines parallel to the axis and produce them to meet the ray P_3Q_4 at the points R, R_1 and R_2. From these points let fall the perpendiculars RG, R_1G_1 and R_2G_2 upon the axis. Then G_2Q_4 is the apparent distance of the object S_2P_2 as seen through the lens L_3; G_1Q_4 is the apparent distance of the object S_1P_1 as seen through the lenses L_2 and L_3, and GQ_4, finally, is the required apparent distance of the object OM as seen through the three lenses L_1, L_2 and L_3. We shall determine successively G_2Q_4, G_1Q_4 and GQ_4, and for this purpose we shall use the following notation:

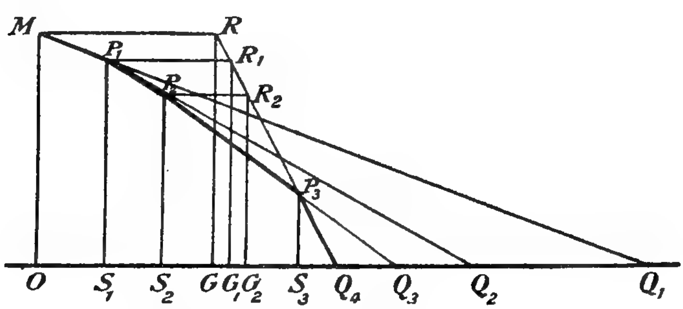

FIG. 42.

$OM = h$; $S_1P_1 = h_1$; $S_2P_2 = h_2$; $S_3P_3 = h_3$

Proof of Cotes' theorem.

$$S_3P_3 = h_3;\ S_2P_2 = G_2R_2 = h_2;\ S_1P_1 = G_1R_1 = h_1;\ OM = GR = h,$$

From the figure it follows that

$$h_2/h_3 = G_2Q_4/S_3Q_4 = S_2Q_3/S_3Q_3 ;$$

$$G_2Q_4 = S_3Q_4 \cdot \frac{S_2Q_3}{S_3Q_3} = S_3Q_4 \cdot \frac{S_2S_3 + S_3Q_3}{S_3Q_3} = S_3Q_4\left(1 + \frac{S_2S_3}{S_3Q_3}\right)$$

and also from equation (10), if we denote the focal lengths of the lenses L_1, L_2 and L_3 by f_1, f_2 and f_3:

$$\frac{1}{S_3Q_3} = \frac{1}{S_3Q_4} - \frac{1}{f_3} .$$

Substituting this value in the preceding equation, it follows that

$$G_2Q_4 = S_3Q_4 + S_2S_3 - \frac{S_3Q_4 \cdot S_2S_3}{f_3} = S_2Q_4 - \frac{S_2S_3 \cdot S_3Q_4}{f_3}$$

and similarly,

$$G_1Q_4 = G_2Q_4\left(1 + \frac{S_1S_2}{S_2Q_2}\right)$$

$$\frac{1}{S_2Q_2} = \frac{1}{S_2Q_3} - \frac{1}{f_2} = \frac{S_3Q_4}{G_2Q_4 \cdot S_3Q_3} - \frac{1}{f_2} = \frac{1}{G_2Q_4}\left(1 - \frac{S_3Q_4}{f_3}\right) - \frac{1}{f_2} .$$

$$G_1Q_4 = G_2Q_4 + S_1S_2 - \frac{S_1S_2 \cdot S_3Q_4}{f_3} - \frac{G_2Q_4 \cdot S_1S_2}{f_2} .$$

Substituting in these the value previously found for G_2Q_4, we obtain

$$G_1Q_4 = S_1Q_4 - \frac{S_1S_2 \cdot S_2Q_4}{f_2} - \frac{S_1S_3 \cdot S_3Q_4}{f_3} + \frac{S_1S_2 \cdot S_2S_3 \cdot S_3Q_4}{f_2f_3} .$$

Proceeding in the same manner from G_1Q_4 to GQ_4, it follows that

$$GQ_4 = OQ_4 - \frac{OS_1 \cdot S_1Q_4}{f_1} - \frac{OS_2 \cdot S_2Q_4}{f_2} - \frac{OS_3 \cdot S_3Q_4}{f_3}$$

$$+ \frac{OS_1 \cdot S_1S_2 \cdot S_2Q_4}{f_1f_2} + \frac{OS_1 \cdot S_1S_3 \cdot S_3Q_4}{f_1f_3} + \frac{OS_2 \cdot S_2S_3 \cdot S_3Q_4}{f_2f_3}$$

$$- \frac{OS_1 \cdot S_1S_2 \cdot S_2S_3 \cdot S_3Q_4}{f_1f_2f_3} .$$

This theorem, which holds good for any number of infinitely thin lenses, as was proved generally by Lagrange (*1*.), may be stated in the following words :—Let the distance between the object and the eye be divided up in every possible manner by the lenses into one, two, three or more segments. Form the products of the corresponding segments and divide these by the focal lengths or the products of the focal lengths of the dividing lenses. Regard these quotients as positive when their denominators contain an even number of factors, and as negative when they contain an odd number of factors. There is then obtained the apparent distance of the object from the eye, when the algebraical sum of these quotients is added to the true distance of the object.

Cotes' theorem is one of the first general theorems relating to lens systems, and is of considerable historical importance in view of the conclusions derived therefrom by Smith, the value of which was not fully appreciated until a later date.

The most important works which form part of the exceedingly extensive literature on the subject of this section have already been recorded in the preceding chapter. Matthiessen (*2*. *272*) has attempted to compile a synopsis, which is, however, very far from complete.

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2. REFRACTION OF OBLIQUELY INCIDENT ELEMENTARY PENCILS AT SPHERICAL SURFACES.

90. In the preceding pages we have dealt with the refraction of infinitely thin pencils, whose axes are infinitely near the common centre-line of the system. This is not the only case in which collinear image-formation arises. Equal importance in the theory of optical instruments attaches to the other case in which the image-forming pencils, while likewise infinitely thin, or elementary, are not infinitely near to the common centre-line, but are similarly related to another centre-line which traverses the system at a finite inclination to the axis. Though the abandonment of one of the restrictions previously imposed is associated with the introduction of new conditions in the investigation, it will be shown that the general theory still holds good, and that within modified limits, the resulting images are subject to the deductions previously established.

A.—Refraction of an Obliquely Incident Elementary Pencil of Rays at a Single Spherical Surface.

91. We can only briefly refer to the beautiful geometrical investigation by which Reusch (*1.3.*) and especially Lippich (*2.*) have developed the theory of this problem. To obtain expressions for the quantitative relations with which we are primarily concerned, we shall confine ourselves to their analytical proof.

In § 26 it was shown that an infinitely thin and initially homocentric pencil, when refracted at a surface of any form, becomes in general astigmatic; *i.e.*, the whole of the rays of the pencil pass after refraction, with an approximation of the first order, through two distinct straight lines which are at right angles to each other and to a mean principal ray. These are known as *Sturm's focal lines* or *image lines.* If we suppose the wave-front of the refracted rays to be constructed at any point B of the refracted principal ray, the focal lines will lie in the plane of the principal curvature of this wave-front which corresponds to the point B.

Now, if the pencil falls obliquely upon a spherical surface, it is easy to determine one of the planes of curvature. If we suppose the luminous point O to be joined by a straight line to the centre C of the sphere, the incident wave-front and the refracting spherical surface and hence also, by symmetry, the refracted wave-front, will be surfaces of revolution about OC as axis. The meridian sections of a surface of revolution are principal planes of curvature, so that the plane of incidence of the principal ray is a meridian section of the refracted wave-front and hence also one of its principal planes of curvature. It follows from this that one of the focal lines must be situated in it. The other focal line lies in the second plane of curvature, which passes through the principal ray (from its nature itself the normal to the wave-front) and is normal to the plane of incidence. Since the focal lines are normal to the principal ray, their directions are completely determined. It will be seen that they depend solely upon the position of the principal ray and not upon the position of the point O. The respective sets of primary and secondary focal lines of all points situated on the principal ray are accordingly parallel.

Having thus ascertained the direction of the focal lines, it remains to determine their position by defining the points where they intersect the principal ray. In Chapter I it was shown that these points are the points of intersection of the rays contained in the principal plane of curvature of the refracted wave-front. To derive their positions from the position of the incident pencil, the first step

is to determine which rays of this pencil correspond to the rays of the principal planes of curvature. In what follows we shall describe the plane of incidence as the **tangential plane, meridional plane,** or **primary principal section** of the incident and refracted pencils. The planes at right angles to it and passing through the principal axes of the two pencils will be called the **sagittal sections, equatorial sections** or **secondary principal sections** of these pencils. In the absence of special notation, the quantities relating to the tangential or sagittal sections respectively, will be distinguished by the suffixes t and f. The principal sections of the refracted pencil are evidently identical with the principal plane of curvature considered above, so that we have to find the rays in the first medium which correspond to the rays contained in the principal sections of the refracted pencil. It will readily be seen that these are the rays contained in the principal sections of the first medium. The rays of the tangential sections are in strict correspondence since they are in the plane of incidence. Those of the sagittal sections correspond with one another if we neglect small quantities of the third order, since the planes of the sagittal section are tangential to the circular cones generated by the revolution of the incident and refracted rays about OC as axis, and from what has been said in § 73, it follows that the rays contained within these cones are conjugate throughout their entire extent. From this it now follows that the rays of the infinitely thin pencil contained within the sagittal section and proceeding from O_f intersect at a point O_f' where the refracted ray meets the straight line O_fC. We shall call this point the sagittal image-point conjugate to O_f. It is also called the *secondary image-point* of the refracted pencil to distinguish it from the point of union of the tangential rays, which is similarly called the *primary image-point*. (This conforms to the normal nomenclature, whilst the terms are reversed by Lippich.)

From this it will be seen that the union of the rays at O_f' is of the second order of approximation. This follows, moreover, from the fact that the sagittal section is symmetrical relatively to the plane of incidence, and that, accordingly, those rays which proceed from O_f and are equally inclined to this plane on one and the other side of it intersect one another strictly on the refracted ray.

The notation which will be adopted in the succeeding pages is indicated in Figs. 43 and 44 and conforms to that given in § 28, § 44 and § 34.

92. The Sagittal Section.—To establish the relations subsisting between conjugate points on the principal rays of the incident and refracted sagittal pencils, let perpendiculars from

O_f and O_f' (Fig. 43) be drawn to the radius vector CB through the point of incidence B and let R, R' be the feet of these perpendiculars.

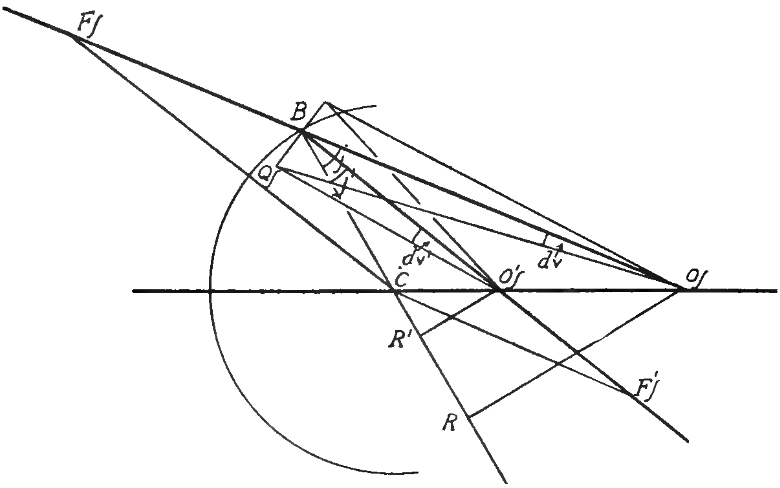

FIG. 43.

$BC = r$; $BO_f = f$; $BO_f' = f'$; $O_f\hat{B}C = j$; $O_f'\hat{B}C = j'$.

Refraction at a sagittal section (The straight line BQ_f is at right angles to the plane of the paper).

Then

$$O_f'R' \,/\, O_fR = CR' \,/\, CR\,,$$

that is

$$f' \sin j' \,/\, f \sin j = (f' \cos j' - r) \,/\, (f \cos j - r)\,,$$

and, applying the fundamental equation $n' \sin j' = n \sin j$, this becomes

$$\frac{n \cos j}{r} - \frac{n}{f} = \frac{n' \cos j'}{r} - \frac{n'}{f'}$$

or

$$\frac{n'}{f'} - \frac{n}{f} = \frac{n' \cos j' - n \cos j}{r} \quad \ldots \quad \ldots \quad (14)$$

or

$$\frac{n'}{f'} - \frac{n}{f} = \frac{n' \sin (j - j')}{r \sin j} = \frac{n \sin (j - j')}{r \sin j'} \quad \ldots \quad (14a)$$

or in accordance with the previous notation

$$\Delta \left(\frac{n}{f}\right) = \frac{1}{r} \Delta (n \cos j)\,. \quad \ldots \quad \ldots \quad (14b)$$

This is the required relation between conjugate intercepts of the sagittal section.

The ratio of the conjugate inclinations of the rays with respect to the axis of the pencil, *i.e.*, the angular magnification at conjugate points follows in a simple manner from the fact that for a point Q_f' on the sagittal line of intersection, *i.e.*, the line along which the sagittal section intersects the refracting surface, we have the relation :

$$BQ_f = f\,dv = f'\,dv' ;$$

hence
$$\frac{dv'}{dv} = \frac{f}{f'} \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

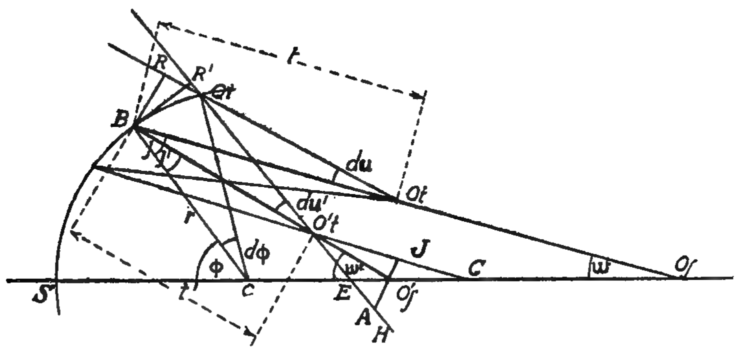

FIG. 44.

Refraction at the tangential section.

$BC = r$; $BO_t = t$; $BO_t' = t'$; $O_tBC = j$; $O_t'BC = j'$.

93. The Tangential Section.—Let B again be the point of incidence of the principal ray, and let Q_t (Fig. 44) in the tangential section be a point on the sphere in close proximity to B. Let $d\phi$, dj, dj', du and du' be the increments which the angle ϕ subtended at the centre of the refracting spherical surface, the angle of incidence j, the angle of refraction j', and the angles of inclination u, and u′, receive as the ray changes from its position of incidence at B to that at Q.

To derive the relations between the quantities which determine the positions of the vertices O_t and O_t' of the incident and refracted tangential pencils, we may proceed from the invariant $n \sin j = n' \sin j'$, which holds good for all rays of the pencil. For the ray incident at Q_t

$$n \sin (j + dj) = n' \sin (j' + dj'),$$

or, expanding into a series, while neglecting terms of the second order, and eliminating the terms $n \sin j = n' \sin j'$,

$$n \cos j \, dj = n' \cos j' \, dj' \quad \ldots \quad \ldots \quad \text{(i)}$$

Now,

$$j = \phi - w, \quad j' = \phi - w'$$

Hence

$$dj = d\phi - du\,; \quad dj' = d\phi - du'.$$

From B drop perpendiculars BR and BR' on $O_t Q_t$ and $O_t' Q_t$. Then, owing to the smallness of the arc BQ_t, which does not sensibly differ from a straight line, $RB = BQ_t \cos RBQ_t = r \, d\phi \cos j$; and $RB = t \, du$, hence

$$t \, du = r \, d\phi \cos j \ . \quad \ldots \quad \ldots \quad \text{(ii)}$$

Therefore, again neglecting quantities of the second order,

$$dj = d\phi \left(1 - \frac{r \cos j}{t}\right),$$

and, similarily,

$$dj' = d\phi \left(1 - \frac{r \cos j'}{t'}\right).$$

Introducing these expressions in the above equations and dividing by $d\phi$,

$$n \cos j \left(1 - \frac{r \cos j}{t}\right) = n' \cos j' \left(1 - \frac{r \cos j'}{t'}\right),$$

or

$$\Delta \left[n \cos j \left(1 - \frac{r \cos j}{t}\right)\right] = 0 \ . \quad \ldots \quad \text{(iii)}$$

Had we retained the terms of the second order $d\phi^2$ the expression would have contained, after division by $d\phi$, a term of the order $d\phi$. The equation accordingly is only applicable when quantities of this order are neglected.*

---

* If in the expansion of the expression the first terms to be neglected had been of the order $d\phi^3$ onwards, we should have obtained the following equation:

$$\Delta \left[n \cos j \left(1 - \frac{r \cos j}{t}\right)\right] = \frac{3}{2} \, rJ \, d\phi \Delta \left[\frac{r \cos^2 j}{t^2} - \frac{\cos j}{t}\right]$$

Czapski (*3* . *73*) gives a somewhat different expression, but he does not there take into consideration the fact that in the evaluation of $\frac{dj}{d\phi}$ the terms of the order $d\phi$ are not negligible. In another part of his book (*3* . *116*) he uses the correct expression.

Rayleigh (*2* . art. 7) arrives at this equation in a very simple manner by the principle of the shortest path.

The equation enables us to calculate t' from t when the principal ray has been determined trigonometrically, and j and j' are accordingly known. Since, however, the equation holds good so long only as we neglect $d\phi$, rays proceeding from O_t at different angles have corresponding to them values of t' which differ by quantities of the order of the intercepted elementary arc $rd\phi$; in other words, the union of the rays at O'_t is only one of the first order.

We may also write the equation (iii) in the following forms:

$$\Delta\left(\frac{n \cos^2 j}{t}\right) = \frac{1}{r}\,\Delta\,(n \cos j), \qquad \ldots \qquad \text{(iv)}$$

or

$$\frac{n' \cos^2 j'}{t'} - \frac{n \cos^2 j}{t} = \frac{n' \cos j' - n \cos j}{r} \qquad \ldots \qquad (16)$$

or

$$\frac{n' \cos^2 j'}{t'} - \frac{n \cos^2 j}{t} = \frac{n' \sin(j - j')}{r \sin j} = \frac{n \sin(j - j')}{r \sin j'}. \qquad \text{(16a)}$$

The angular magnification in the tangential section follows from the relation (ii) which we have used above, viz.:

$$t\,du = rd\phi \cos j \qquad \text{or} \qquad du = \frac{rd\phi \cos j}{t}$$

and from the analogous equation for the refracted ray:

$$du' = \frac{rd\phi \cos j'}{t'}$$

hence

$$\gamma_t = \frac{du'}{du} = \frac{\cos j'}{t'}\,\frac{t}{\cos j}\,. \qquad \ldots \qquad \ldots \qquad (17)$$

In the sagittal section the distances of the principal foci from the point of incidence B are obtained from the equations (14) and (14a):

$$\left.\begin{aligned} f_F &= S = -\frac{nr}{n' \cos j' - n \cos j} = -\frac{r \sin j'}{\sin(j - j')} \\ f_F' &= S' = \frac{n'r}{n' \cos j' - n \cos j} = \frac{r \sin j}{\sin(j - j')} \end{aligned}\right\} \qquad (18)$$

In the tangential section according to equations (16) and (16a)

$$\left.\begin{aligned} t_F &= T = -\frac{nr \cos^2 j}{n' \cos j' - n \cos j} = -\frac{r \sin j' \cos^2 j}{\sin(j - j')} \\ t_F' &= T' = \frac{n'r \cos^2 j'}{n' \cos j' - n \cos j} = \frac{r \sin j \cos^2 j'}{\sin(j - j')} \end{aligned}\right\} \qquad (19)$$

With the aid of these values we may write the equations (14) and (16) in the abbreviated forms :—

$$\frac{S'}{f'} + \frac{S}{f} = 1 \qquad \ldots \qquad \ldots \qquad (20)$$

$$\frac{T'}{t'} + \frac{T}{t} = 1 \qquad \ldots \qquad \ldots \qquad (21)$$

By an easily proved theorem in co-ordinate geometry* all straight lines joining two points O_f and O_f' (or O_t and O_t', as the case may be) situated respectively on a straight line and having abscissæ f, f' (or t, t' respectively) which satisfy an equation of this form, meet at a point whose co-ordinates referred to the principal rays BO, BO_f' (or BO_t, BO_t' respectively) are S, S' (or T, T' respectively), and conversely. In geometry of position these points are known as the **perspective centres** of the series of points O_f, O_f' (or O_t, O_t' respectively).

The perspective centre of the sagittal section is already known; it is C, the centre of the sphere.

The perspective centre K of corresponding points of the tangential plane may be found by determining the positions of two corresponding pairs of points O_t, O_t' and Q_t, Q_t'. The straight lines O_tO_t' and Q_tQ_t' joining them intersect at K. In Fig. 45, let two circles with radii $r\,\frac{n'}{n}$ and $r\,\frac{n}{n'}$, be described about C, as in Young's graphic method given in § 71. The aplanatic points O_t and O_t' where the incident and refracted rays intersect these circles are conjugate to one another. (It can also be shown that the values of t and t', appropriate to these points, satisfy equation (16)). The perspective centre lies accordingly on the straight line O_tO_t'.

We may obtain a second pair of points Q_tQ_t' if from the centre we drop perpendiculars CQ_t and CQ_t' upon the incident and refracted rays respectively. The feet Q_t and Q_t' of these perpendiculars correspond to one another, since their co-ordinates $t = r\cos j$ and $t' = r\cos j'$ satisfy equation (16). The straight line Q_tQ_t' passes, therefore, likewise through the perspective centre K.

The straight lines O_tO_t' and Q_tQ_t' are at right angles to one another. For, if on BC as diameter we describe a semicircle, the

* A proof of the theorem is given in text-books on co-ordinate geometry, *e.g.*, Smith's Conic Sections, § 47.

latter will pass through Q_t and Q_t', and we shall have the following relations :

$$\text{the angle } KQ_t\,C = \text{the angle } Q_t'BC = j'$$

$$\text{the angle } KQ_t'C = \text{the angle } Q_tBC = j\,.$$

When explaining Young's graphic method, we proved that the angle $BO_tC = j'$ and $BO_t'C = j$. Hence the angle

$$\begin{aligned} Q_tKO_t &= 180^\circ - KQ_tO_t - Q_tO_tK \\ &= 180^\circ - (90^\circ - j') - j' = 90^\circ. \end{aligned}$$

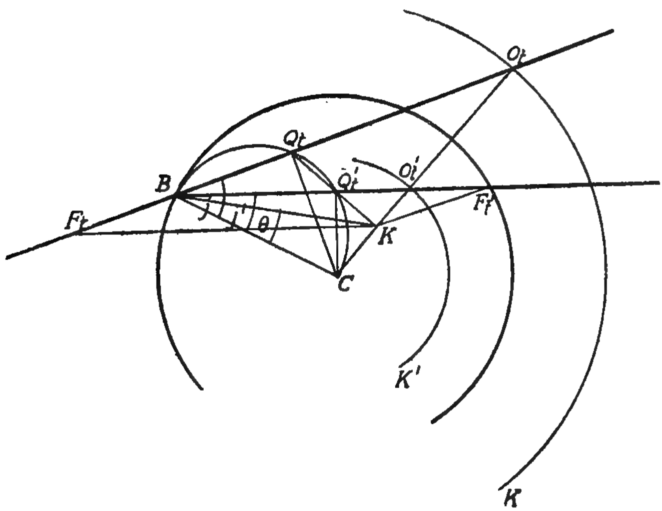

FIG. 45.

$$BC = r\,;\ O_tC = \frac{rn'}{n}\,;\ O_t'C = \frac{rn}{n'}$$

Graphic construction of the perspective centre and of the principal foci of the tangential section.

It is possible by the following graphic method, which was first given by Young, to find the position of the perspective centre K of the tangential rays. From the centre C of the refracting sphere drop perpendiculars CQ_t and CQ_t' upon the incident and refracted rays respectively, join their feet Q_t, Q_t' by a straight line, and from C drop a third perpendicular CK upon the line joining these points ; then the foot K of this perpendicular is the perspective centre of the tangential rays.

The distance between the two perspective centres C and K may be found from the triangle KQ_tC. It is

$$CK = CQ_t \sin j' = r \sin j \sin j'.$$

If now we draw BK and denote the angle KBC by θ,

$$\frac{CK}{r} = \frac{\sin\theta}{\sin(j - \theta + j')},$$

and

$$\tan\theta = \frac{CK \sin(j + j')}{r + CK\cos(j + j')} = \frac{\sin j \sin j' \sin(j + j')}{1 + \sin j \sin j' \cos(j + j')}. \quad (23)$$

The angle θ is thus shown to be independent of r. When the refractive indices are given it depends solely upon the angle of incidence.

It is very easy by means of the perspective centres C and K to find for any point on one or other principal ray the conjugate point with respect to the sagittal or tangential rays. It is only necessary to join this point with C and K respectively, and to produce this line CK until it meets the other axis. The point of intersection is the required conjugate point.

In particular, we can ascertain in this way the positions of F_f, F_f', and F_t, F_t' respectively, *i.e.*, the principal foci of the sagittal and tangential sections, by drawing parallels to the two principal rays which are respectively conjugate to the infinitely distant points on the two axes.

94. Astigmatism.—One and the same object-point O has corresponding to it two distinct conjugate points O_f', O_t', with respect to the sagittal and tangential sections. The portion lying between them, that is the *focal line*, is the central projection of the length CK from O outwards upon the refracted principal ray. This length obviously becomes equal to zero when the point O is situated on the straight line CK. This straight line is the line joining the aplanatic points on the sphere, which we have previously considered.

It is not the length of the focal line which serves as a measure of the astigmatism, but rather the magnitude of the expression $n'\left(\frac{1}{t'} - \frac{1}{f'}\right)$. To determine the change of this quantity caused by a single refraction, let the term $\cos^2 j$ in equation (16) be replaced by $1 - \sin^2 j$,

thus $$\frac{n'}{t'} - \frac{n'\sin^2 j'}{t'} - \frac{n}{t} + \frac{n \sin^2 j}{t} = \frac{n'\cos j' - n\cos j}{r}$$

By subtraction of equation (14), viz.

$$\frac{n'}{f'} - \frac{n}{f} = \frac{n' \cos j' - n \cos j}{r},$$

it follows that

$$n' \left(\frac{1}{t'} - \frac{1}{f'}\right) - n\left(\frac{1}{t} - \frac{1}{f}\right) = \frac{n \sin^2 j'}{t'} - \frac{n \sin^2 j}{t}$$

or, by the previous notation,

$$\Delta\, n \left(\frac{1}{t} - \frac{1}{f}\right) = J^2 \Delta \left(\frac{1}{nt}\right). \quad \dots \qquad \dots \qquad (24)$$

This equation shows the change of astigmatism, irrespective of the absence or presence of any astigmatism which may have been occasioned by a preceding refraction, in which case t would not be equal originally to f, whereas in a homocentrically incident pencil $f = t$.

The astigmatism does not experience any change, *i.e.*, a homocentrically incident pencil has corresponding to it a homocentrically refracted pencil, when $n't' = nt$.

By means of equation (16) this condition leads to the value

$$t = r \cos j + r \frac{n'}{n} \cos j',$$

which conforms to one of the aplanatic points. The other is given by the corresponding value of t'.

95. The Astigmatic Focal Lines.—The two equations (14) and (16) previously derived determine, as we have seen, the positions of the points O_f' and O_t' in the sagittal and tangential sections respectively. Let the pencil represented in Fig. 44 be a solid pencil rendered astigmatic by preceding refractions, and let the *homocentric pencil* be regarded as a special case of the *astigmatic pencil.* We may then consider the astigmatic pencil as if it were in two ways a plane pencil: First, as a tangential pencil, having all its axes passing through O_f; secondly, as a sagittal pencil having its axes passing through O_t. The axes of the tangential pencils, after refraction, pass through O_f, since these axes are identical with the rays of the principal sagittal pencil O_f. The axes of the sagittal pencils are the rays of the tangential section through the principal ray and accordingly intersect as such after the refraction at O_t'. The points of intersection of all plane tangential pencils constitute the primary focal line of the refracted pencil passing

through O_t', whilst those of the plane sagittal pencils form the secondary focal line through O_f'. It was shown in § 91 that one of the focal lines, viz., the secondary, is situated in the plane of incidence, whilst the other, viz., the primary, is at right angles to the plane of incidence. In the first chapter, in § 26, it was moreover shown that, within the limits of infinitely small quantities of the second order, both focal lines may be regarded as being normal to the principal ray.

For the secondary focal line situated in the plane of incidence and passing through O_f' there is not obtained directly a straight line normal to the principal ray $O_f'B$. The focal line obtained in the first instance is the portion EG of the centre-line (Fig. 44) lying between the two extreme rays of the principal tangential pencil. The whole of the rays of the pencil pass through this portion; for, if we imagine the plane of Fig. 44 to be rotated through a small angle forward or backward about the centre-line CO_f', the whole of the rays of the pencil will be passed through, whilst EG remains unchanged in magnitude and position and is therefore common to all. But the straight line HJ, which is normal to BO_f', has previously been shown to be such that none of the refracted rays deviate from it by quantities above the second order. It is therefore more convenient for most purposes.

The fact that the confluence of the rays at EG is more exact is of no value for our purpose, since we have shown that the intersection of the rays at the first principal foci O_t' is only of the first order, (*cf.* Bouasse (*1.*) and the papers of Schultén discussed in the historical part).

B. The Formation of Images of Extended Objects by Astigmatic Pencils.

96. It has been shown above that the refraction of oblique pencils does not, in general, result in the homocentric convergence of the rays. The image of a point due to a general oblique pencil may therefore be regarded as made up of the two focal lines of Sturm. The question so frequently raised, as to which of these two lines appears to the eye as the image proper, cannot be answered in general terms. The problem has been studied very thoroughly by Leroy (*1.*), who has investigated the formation of the images of extended objects by pencils having their principal rays inclined at any angle to refracting surfaces of double curvature. A special case has been considered by Gartenschlæger (*1.*).

Under certain conditions we are nevertheless called upon to give preference to one or other of these focal lines. To show how this may arise, let the object be a short straight line g normal to a mean obliquely incident principal ray. Suppose rays, all very

near to the mean principal ray, to proceed from the points of this line (for instance, let all these rays intersect the refracting surface within the same small surface element, so that the point of incidence B of the mean principal ray is also the point of incidence of all the other principal rays). Then all points on g will be subject to much the same conditions applicable to points lying on h. Although the principal rays of the pencils proceeding from the individual points of g are inclined at slightly varying angles to the normal of the spherical segment at B and thereby give rise to small differences in the astigmatism of the refracted pencils, the focal lines of either kind will nevertheless be very approximately parallel amongst themselves. Every point of the luminous straight line appears therefore in the image as a line similarly extended whether it is viewed in the primary or in the secondary focal plane. The image as a whole must therefore in both planes convey a blurred impression.

The case is different when the luminous straight line g lies within a tangential or sagittal section. To deal with a concrete case, let us suppose that the luminous straight line g is situated in the plane of incidence. All the principal rays proceeding from it will then likewise lie in the plane of incidence. The image points O_t' and O_f', corresponding to points on the straight line g, form two lines in the plane of incidence, which will be denoted by l'_t and l_f'. The primary focal lines b_1 of points on g pass through points on l_t' and are normal to the plane of incidence. In the aggregate they form accordingly a surface a_1 which is normal to the plane of incidence. The secondary focal lines b_2 pass through points on l_f' and lie in the plane of incidence. In the aggregate these form a surface a_2 situated in the plane of incidence. A screen at right angles to the principal ray at the position of the primary focal lines, receives the cross-section of the pencil converging towards a_1 and is composed of rays which are inclined at an infinitely small angle to the principal ray. If, on the contrary, the screen is placed at the position of the secondary focal lines, the cross-section of the intercepted rays proceeding to a_2 will be a straight line.

An eye situated on the principal ray and accommodated for the line b_1 will see a surface ; when accommodated for b_2 it will see a line, since the prolongation of a_2 passes through the eye.

Conversely, when the luminous straight line g is situated in the sagittal section the eye will perceive a sharp image at the position of the primary focal plane, and a blurred image at that of the secondary focal plane.

When the object has the form of a rectangular cross or, better still, that of a "grille" whose lines are parallel and at right angles to the plane of incidence of the mean principal ray, the parallel lines will appear sharply defined at the focal points of the sagittal rays, whilst the lines at right angles to it will be sharply defined at the

focal points of the tangential rays and sometimes will alone be visible. Conversely, this appearance is a characteristic feature and an indication of the presence of astigmatism, as has been shown by Oertling (*I.*).

The sensitiveness of this indication may be demonstrated by the case of reflection at a spherical surface. In the sagittal section, when $\frac{n'}{n} = -1$, the intercepts are related as follows (as will be shown in § 102):

$$\frac{1}{\mathfrak{f}'} + \frac{1}{\mathfrak{f}} = \frac{2 \cos j}{r},$$

and in the tangential section

$$\frac{1}{t'} + \frac{1}{t} = \frac{2}{r \cos j},$$

hence, when $\mathfrak{f} = t$

$$\frac{1}{t'} - \frac{1}{\mathfrak{f}'} = \frac{2}{r \cos j} - \frac{2 \cos j}{r},$$

$$\frac{\mathfrak{f}' - t'}{\mathfrak{f}'t'} = \frac{de}{e^2} = \frac{2}{r} \sin j \tan j,$$

e being the mean distance of the image-points from the mirror.

The last expression shows most distinctly to what extent the astigmatism increases with the angle of incidence. In fact, the increase is so pronounced that, according to Schroeder (*I.* 7), it is possible with suitable means to observe on an artificial mercury horizon the astigmatism of reflected pencils due to the curvature of the earth. If we take the radius of the earth to be $r = 6{,}370{,}000$ metres, and $j = 80°$, then $\frac{de}{e^2} = 0{\cdot}00000175$ m.

We will suppose such a reflected image to be viewed by means of a telescope having an objective of 7·5 m. focus and an effective aperture of about 50 cm., dimensions which at the present time are by no means unusual.

In § 56 it was shown that, if we ignore the sign,

$$dx_s' = \frac{dx_s}{x_s^2} f^2.$$

If now the front principal focus is made to coincide approximately with the surface of the mercury, so that $e = x_s$, we may put $\frac{dx_s}{x_s^2} = \frac{de}{e^2}$ $= 0{\cdot}00000175$ m, as found above, so that $dx_s' = 0{\cdot}1$ mm., nearly.

Now, a displacement of 0·1 mm. would be quite noticeable in a telescope of the dimensions stated. In one of the modern giant telescopes, such as the refractor of the Lick Observatory, the

difference in the positions of the focal planes would amount to 0·7 mm. and would therefore be quite conspicuous. In such a test it would be obviously necessary to use a large mercury horizon and to ensure that the surface is perfectly calm.

C. The Limits of Collinear Image-formation by Oblique Refraction.

97. In § 48 we defined *collinear* or *homographic image-formation* as the relation between two spaces, such that every homocentric pencil of rays in one space has corresponding to it a homocentric pencil in the other. In order that collinear image-formation may be brought about it is therefore necessary to establish such conditions that all rays proceeding from a point O in the object-space may converge to one and the same point in the image-space. This condition is not realised in an elementary solid pencil refracted at a finite angle, and the rays in the image-space which correspond to a homocentric pencil in the object-space do not intersect at a point, but, as we have seen, they converge to two straight lines, the so-called focal lines or image-lines, which are normal to the principal ray. It will thus be seen that the condition of collineation is not even fulfilled within the elementary space containing the rays which are infinitely near to the incident principal ray and the corresponding refracted ray.

Our investigations have shown, on the other hand, in what manner the two spaces must be limited in order to establish under these conditions an identical relation between conjugate points. According to these investigations, points A_1, B_1, C_1 situated in the first principal section (the plane of incidence) infinitely near to a mean incident ray l, are transformed by *rays which traverse the same principal section* and are inclined at infinitely small angles to the ray l into one, and only one, set of points A_1', B_1', C_1' situated likewise in the first principal section and very near to the ray l' which is conjugate to the ray l.

On the other hand, the points A_2, B_2, C_2... situated *in the second principal section* (normal to the plane of incidence) and near to the ray l, are transformed by the *rays incident in this principal section* into one, and only one, set of points A_2', B_2', C_2' . . . situated in a plane through l' normal to the plane of incidence and quite near to l'. Within the infinitely narrow bundles so defined the collinear condition is accordingly fulfilled, that is, there is optical image-formation in the sense above described.

The object-space of the one system penetrates that of the other along the principal incident ray l, whilst the image-space of the one penetrates that of the other along the refracted principal ray l'. In other respects not only are these two systems entirely separated in space, but their relative dimensions are different, so that it is necessary to treat them separately.

Strictly considered, the boundaries of the two collinear systems do not extend equally far at right angles to the axes. For, whereas the homocentric union of sagittal, as well as paraxial pencils is of the second order of approximation, we know that in tangential pencils it is of the first order only. Whilst therefore the collinear formation of images by sagittal and paraxial pencils can be effected by so restricting their transverse dimensions that the squares only of the apertures of the pencils, or of their linear diameters near the refracting spheres, may be infinitely small relatively to the abscissæ of the pencils, a similar degree of homocentric union in the tangential pencils can only be achieved by so limiting them that those diameters themselves may be of the same degree of smallness.

The assumed mean ray—before or after refraction—is in either case the axis of the system, *i.e.*, the linear elements normal to the incident ray l are transformed by one and the other kind of pencils into linear elements normal to the refracted ray l'. With regard to straight lines in the second principal section—normal to the plane of incidence—the truth of the statement follows from the principle of symmetry; in fact, the image of a straight line normal to l can only be a curve whose centre of curvature is situated at a point in l and which accordingly coincides with a straight line normal to l' within small quantities of the second order. The image of a straight line normal to l in the plane of incidence may be, and generally is, a curve c inclined at a finite angle* to l', when this image is constructed point for point in accordance with the rules established above. If, however, at the points of intersection of l' and the curve c we describe in the plane of incidence a straight line g' normal to l', we may regard g' as the image of g, provided we neglect small quantities of the second order. Now we are perfectly justified here in neglecting small quantities of the second order since it has already become necessary to neglect infinitesimals of this order to ensure that in the tangential section rays proceeding from a point will again meet in one point. That the negligible values are really of the second order of magnitude may readily be seen if we consider that the infinitely narrow tangential pencil proceeding from c to a point, subtends an interval on g' which is an infinitesimal of the second order, since the aperture of the pencil and the distance of its cross-section from its apex are both infinitesimals of the first order. The cross-section may, therefore, be regarded as a point and, if we neglect infinitesimals of the second order, the image-forming rays converge at this point on g', which may thus take the place of the true image-point situated on c.

98. To obtain quantitative expressions for the image-formations in the tangential and sagittal sections we shall have to consider the

* Cf. Lippich (*2*.).

positions of the principal foci and the magnitudes of the focal lengths. The positions of the principal foci have already been determined by equations (18) and (19). Expressions for the focal lengths must now be obtained.

In the case of paraxial pencils refracted at a single surface, the focal lengths and the focal intercepts on the axis have the same absolute value, from which it must not, however, be inferred that they are so in this case, as is often tacitly assumed. We shall, as before, define the focal length as the quotient of the height, in one space, of a point where a ray, parallel to the axis, enters or leaves the system, divided by the trigonometrical tangent, reckoned as negative, of the angle at which the refracted ray is inclined to the axis in the other space. We shall then obtain the following relations for the tangential pencils (Fig. 46), observing that the trigonometrical tangent of the small angle du is equal to the arc du.

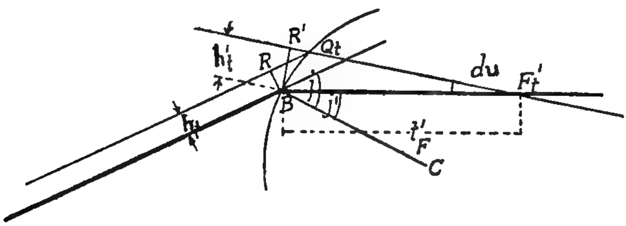

FIG. 46.

$BR = \mathrm{h}_t$; $BR' = \mathrm{h}_t'$; $\widehat{BF_t'Q_t} = d\mathrm{u}'$; $BF_t' = t_F'$.

Diagram relating to the focal lengths in tangential sections.

$$\mathrm{h}_t = BR = BQ_t \cos j\,; \qquad d\mathrm{u}' = BQ_t \frac{\cos j'}{BF_t'}\,;$$

similarly,

$$\mathrm{h}_t' = BR' = BQ_t \cos j'\,; \qquad d\mathrm{u} = BQ_t \frac{\cos j}{BF_t}\,;$$

hence, by § 54 (18) and § 93 (19),

$$\left.\begin{aligned} f_t &= -\frac{\mathrm{h}'}{d\mathrm{u}} = -BF_t \frac{\cos j'}{\cos j} = \frac{nr \cos j \cos j'}{n' \cos j' - n \cos j} \\ &= \frac{r}{2} \frac{\cos j \sin 2j'}{\sin (j - j')} \quad \ldots \quad \ldots \quad \ldots \\ \text{and} \\ f_t' &= -\frac{\mathrm{h}}{d\mathrm{u}'} = -BF_t' \frac{\cos j}{\cos j'} = -\frac{n'r \cos j \cos j'}{n' \cos j' - n \cos j} \\ &= -\frac{r}{2} \frac{\cos j' \sin 2j}{\sin (j - j')}, \quad \ldots \quad \ldots \end{aligned}\right\} \quad (25)$$

from which it will be seen that the focal lengths in the tangential sections are not equal to the negatively reckoned focal intercepts on the axis, though, as in the case of paraxial systems, they conform to the equation (iii) given in § 78 namely :

$$\frac{f_t}{f_t'} = -\frac{n}{n'}.$$

For sagittal pencils as in the case of paraxial systems, the equations are by (18)

$$\left.\begin{aligned} f_f &= -BF = \frac{nr}{n'\cos j' - n\cos j} = \frac{r\sin j'}{\sin(j-j')} \\ f_f' &= -BF_f' = -\frac{n'r}{n'\cos j' - n\cos j} = -\frac{r\sin j}{\sin(j-j')} \end{aligned}\right\} \quad (26)$$

and hence, as in the tangential section,

$$\frac{f_f}{f_f'} = -\frac{n}{n'}.$$

By means of these special values of the constants we are able to apply to either of the systems all the laws that have been proved to hold in general with respect to collinear image-formations. In particular the same relations between the quantities α, β, γ on the one hand, and f, f' on the other, are applicable here as before.

D. Collinear Formation of Images by the Oblique Refraction of Elementary Pencils which intersect the Axis.

99. The relations established for refraction at a spherical surface are not directly applicable to any system of such surfaces, since, in general, the principal sections in the image-space of the v^{th} surface do not coincide with the principal section of the object space of the $(v+1)^{th}$ surface. This circumstance which adds greatly to the difficulty of the entire problem, does not arise when the system is centred and when the axis of the incident pencil intersects the axis of the system. We shall confine our attention to this simple case. The plane of incidence now coincides under all conditions of refraction with the plane which contains the incident principal ray and the axis of the system. Whatever the position of the luminous point on the incident principal ray, the plane of incidence of this ray will always be a plane of symmetry of the incident wave-fronts and of all refracting surfaces, and hence also of all refracted wave-fronts. Consequently, this plane is likewise a principal plane of curvature of the refracted wave-front ; and

one of the focal lines of Sturm lies in it, whilst the other is situated in the corresponding sagittal section. This determines their position completely, since both are at right angles to the principal ray.

The laws governing the combination of several systems are directly applicable to the formation of images within tangential and sagittal sections, since now the image plane of the v^{th} surface is identical with the object-plane of the $(v + 1)^{\text{th}}$ surface. We are able in this way to derive the focal co-ordinates and the focal lengths of the composite system from the constants of the component systems.

Taking as object-point in the first medium an infinitely distant point on the principal ray, and determining with respect to it all values of t_v and f_v in accordance with the formulæ given in § 34 and § 44, and thence the positions of the principal foci F_t' and F_f', we can at once find the focal lengths without reference to the combination formula.

The focal length f_f' of the image-rays in the sagittal section, may be obtained exactly in the same manner as the focal length of paraxial object-rays. From the definition of the focal length (§ 54 (18))

$$f_f' = -\frac{\mathrm{h}_{1f}}{d\mathrm{v}_k'}$$

and by § 82

$$f_f' = -\frac{f_1' . f_2' \;.\;.\;.\;.\; f_k'}{f_2 \;.\;.\;.\;.\; f_k}\,. \tag{27}$$

To determine the focal length of the image-rays in the tangential section we make use of the relation established in the preceding article, viz.:

$$\frac{\mathrm{h}_t}{\mathrm{h}_t'} = \frac{\cos j}{\cos j'},$$

whence

$$f_t' = -\frac{\mathrm{h}_{1t}}{d\mathrm{u}_k'} = -\frac{\mathrm{h}_{1t}}{\mathrm{h}'_{1t}}\,\frac{\mathrm{h}'_{1t}}{\mathrm{h}_{2t}}\,\frac{\mathrm{h}_{2t}}{\mathrm{h}'_{2t}}\,\frac{\mathrm{h}'_{2t}}{\mathrm{h}_{3t}}\;.\;.\;.\;.\;\frac{\mathrm{h}_{kt}}{\mathrm{h}'_{kt}}\,\frac{\mathrm{h}'_{kt}}{d\mathrm{u}_k'}$$

$$= -\frac{\cos j_1}{\cos j_1'}\,\frac{\cos j_2}{\cos j_2'}\;.\;.\;.\;.\;\frac{\cos j_k}{\cos j_k'}\,.\,\frac{t_1' t_2' t_3' \;.\;.\;.\;.\; t_k'}{t_2 t_3 \;.\;.\;.\;.\; t_k}\,. \tag{28}$$

To determine the principal foci and the focal lengths of the object-rays we may trace in the reverse order through the system a ray parallel to the emerging ray in both sections. The focal length of the object-rays may also be found from the formulæ which hold for any number of surfaces, viz.:

$$f_f \,/\, f_f' = -n \,/\, n' \text{ and } f_t \,/\, f_t' = -n \,/\, n'.$$

100. Special Cases.—Plane surface.—We may regard the refracting plane as a sphere of infinitely great radius. If we put $r = \infty$, the abscissæ of the principal foci become infinitely great, according to formulæ (18) and (19), and similarly the focal lengths in either section. The system is thus telescopic. The ratio of the two focal lengths remains finite, and therefore, as before, by § 78 (iii),

$$f_f / f_f' = f_t / f_t' = -n / n'.$$

Equation (14) gives for the conjugate intercepts in the sagittal section :

$$\mathfrak{f}' = \frac{n'}{n} \mathfrak{f}, \quad \dots \quad \dots \quad \dots \quad \text{(i)}$$

Equation (16) gives for the tangential intercepts :

$$t' = \frac{n' \cos^2 j'}{n \cos^2 j} t, \quad \dots \quad \dots \quad \text{(ii)}$$

The angular magnification in the sagittal section becomes by equation (15):

$$\gamma_f = \frac{d\mathrm{v}'}{d\mathrm{v}} = \frac{\mathfrak{f}}{\mathfrak{f}'} = \frac{n}{n'}, \quad \dots \quad \dots \quad \text{(iii)}$$

From the angular magnification, in accordance with the general equation (22) of § 56, we obtain the magnification in the sagittal section, viz. :

$$\beta_f = -\frac{f_f}{f_f'} \cdot \frac{1}{\gamma_f} = 1, \quad \dots \quad \dots \quad \text{(iv)}$$

since $f_f / f_f' = -n/n'$.

In the tangential section the angular magnification according to formula (17) becomes

$$\gamma_t = \frac{\cos j'}{\cos j} \cdot \frac{t}{t'} = \frac{n \cos j}{n' \cos j'}, \quad \dots \quad \dots \quad \text{(v)}$$

hence

$$\beta_t = -\frac{f_t}{f_t'} \cdot \frac{1}{\gamma_t} = \frac{\cos j'}{\cos j}, \quad \dots \quad \dots \quad \text{(vi)}$$

from which it will be seen that the magnification is different in the two principal sections.

The perspective centre in the sagittal section is the centre of an infinitely great sphere, lying along the direction of the normal to the refracting surface.

The perspective centre K of the tangential section lies similarly at infinity, as may be inferred directly from the fact that both foci are at infinity. To determine its direction we must remember that the angle θ comprised between the rays BC and BK

(Fig. 45), proceeding to the centre of the sphere and the perspective centre respectively, is independent of the radius of the refracting sphere. The expression in terms of the angles of incidence and refraction, which was given in formula (23), viz.,

$$\tan \theta = \frac{\sin j \sin j' \sin (j + j')}{1 + \sin j \sin j' \cos (j + j')},$$

holds good also for the refracting plane. It determines the angle contained between the direction of the tangential rays proceeding towards the perspective centre and the normal to the refracting plane.

The direction in which the centre lies may be determined by a graphic method devised by Lippich (*2. 179*), as shown in Fig. 47 where the denser medium is supposed to be on the left. Let a circle be drawn in the plane of the tangential section touching the refracting plane at the point of incidence B of the principal ray,

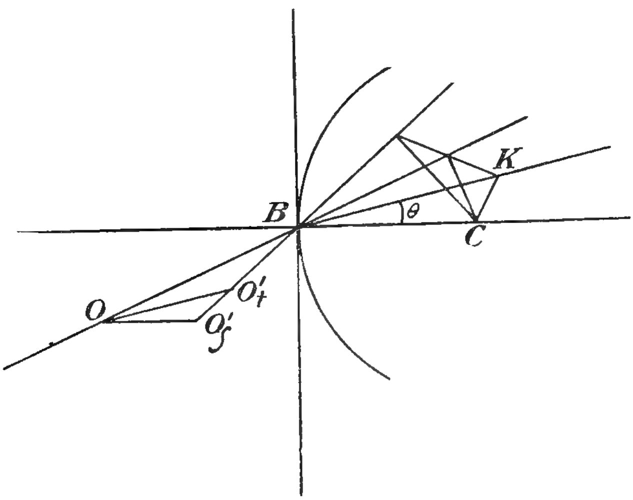

FIG. 47.

Astigmatic refraction at a plane. Graphic method of determining the perspective centre of the tangential rays.

and let the perspective centre K of this circle be determined by Young's method, as given in § 93. The direction of the straight line drawn from the point of incidence B to this centre passes through the perspective centre of the tangential section of the straight line in question.

Corresponding points $O\,O_s'$ of the sagittal section are situated on normals to the refracting plane. Corresponding points $O\,O_t'$ of the tangential section are on the lines which are parallel to the straight line BK.

101. Plane-parallel Plate.—Let n be the refractive index of the plate with respect to the surrounding medium, so that $\frac{n_1'}{n_1} = \frac{n_2}{n_2'} = n$. Let the thickness of the plate be d. Adopting our usual notation, $j_2 = j_1'$, $j_2' = j_1$; hence:

In the sagittal section, by (14),

$$f_1' = \frac{n_1'}{n_1} f_1 = nf_1 \qquad \ldots \qquad \ldots \qquad \text{(i)}$$

$$f_2 = f_1' - \frac{d}{\cos j_1'} \qquad \ldots \qquad \ldots \qquad \text{(ii)}$$

$$f_2' = \quad \frac{1}{n} f_2 \qquad \ldots \qquad \ldots \qquad \ldots \qquad \text{(iii)}$$

$$f_2' = f_1 - \frac{d}{n \cos j_1'} . \qquad \ldots \qquad \ldots \qquad \text{(iv)}$$

Similarly, in the tangential section, by (16)

$$t_2' = t_1 - \frac{d}{n} \frac{\cos^2 j_1}{\cos^3 j_1'} . \qquad \ldots \qquad \ldots \qquad \text{(v)}$$

The distance between the two image-points is

$$t_2' - f_2' = t_1 - f_1 + \frac{d}{n} \frac{1}{\cos j_1'} \left(1 - \frac{\cos^2 j_1}{\cos^2 j_1'}\right) . \qquad \text{(vi)}$$

Denoting by $t_1 = f_1$ the distance of the luminous point from the plate, the distance of the image-points will be

$$t_2' - f_2' = \frac{d}{n \cos j_1'} \left(1 - \frac{\cos^2 j_1}{\cos^2 j_1'}\right). \qquad \ldots \qquad \text{(vii)}$$

This astigmatic difference of the image-points is independent of the distance of the object-point from the plate, and remains accordingly finite when this distance becomes infinitely great. In the case of an infinitely distant point the astigmatic difference may accordingly be neglected.

When the plate is optically denser than the surrounding medium, so that $j_1' < j_1$ and $\cos j_1' > \cos j_1$, the astigmatic difference becomes positive. When t_2' and f_2' are positive the tangential point of intersection is farther away from the plate than the sagittal point of intersection.

The angular and linear magnifications are not affected by the transmission of the rays through the plate.

102. Refraction of a Pencil passing through the Centre of an Infinitely Thin Lens at a Finite Inclination.*—Let $n_1 = n_2' = n$, and $n_1' = n_2 = n'$. Also, since the principal ray which passes through the centre of the infinitely thin lens does not change its direction in passing through the lens, it follows that

$$j_1 = j_2' = j\,;\quad j_1' = j_2 = j'.$$

For brevity, let

$$N = n' \cos j' - n \cos j\,.$$

Then, by (18)

$$S_1 = -\frac{nr_1}{N}\,;\quad S_1' = \frac{n'r_1}{N}\,;\quad S_2 = \frac{n'r_2}{N}\,;\quad S_2' = -\frac{nr_2}{N}\,;$$

hence, by (20),

$$\frac{n'}{f_1'} - \frac{n}{f_1} = \frac{N}{r_1}$$

$$-\frac{n}{f_2'} + \frac{n'}{f_2} = \frac{N}{r_2}\,.$$

Subtracting the second equation from the first,

$$\frac{n}{f_2'} - \frac{n}{f_1} = N\left(\frac{1}{r_1} - \frac{1}{r_2}\right),$$

hence

$$\frac{1}{f_2'} - \frac{1}{f_1} = \frac{n'\cos j' - n\cos j}{n}\left(\frac{1}{r_1} - \frac{1}{r_2}\right). \quad \ldots \quad \text{(i)}$$

Similarly, we find for the tangential section by (19)

$$\frac{1}{t_2'} - \frac{1}{t_1} = \frac{n'\cos j' - n\cos j}{n\cos^2 j}\left(\frac{1}{r_1} - \frac{1}{r_2}\right). \quad \ldots \quad \text{(ii)}$$

* Hermann (*Z.I. 451*; *III. 293*) gives formulæ for the more general case in which the principal ray of the pencil passes excentrically through a thick lens. In this paper he also discusses stratified lenses and arrives at the conclusion that the increase of the optical density of the crystalline lens of the eye towards the centre has a beneficial effect upon the periscopic range.

By (27) the focal length is $f_s' = -\dfrac{f_1' f_2'}{f_2}$. Now since the lens is infinitely thin, $f_1' = f_2$, hence f_s' is plainly equal to the value which $-f_2'$ assumes when $f_1 = \infty$, or

$$f_s' = -\frac{n}{n' \cos j' - n \cos j} \cdot \frac{r_1 r_2}{r_2 - r_1} . \quad \ldots \quad \text{(iii)}$$

Since $f_s / f_s' = -n_1 / n_2' = -1$, then by § 78 (iii),

$$f_s = \frac{n}{n' \cos j' - n \cos j} \cdot \frac{r_1 r_2}{r_2 - r_1} . \quad \ldots \quad \text{(iv)}$$

In the tangential section, by (28),

$$f_t' = -\frac{\cos j_1 \cos j_2}{\cos j_1' \cos j_2'} \cdot \frac{t_1' t_2'}{t_2} = -\frac{\cos j \cos j'}{\cos j' \cos j} t_2' = -t_2', \quad \text{(v)}$$

hence

$$f = -f_t' = \frac{n \cos^2 j}{n' \cos j' - n \cos j} \cdot \frac{r_1 r_2}{r_2 - r_1} . \quad \ldots \quad \text{(vi)}$$

In both sections the focal lengths are equal to the negatively reckoned focal intercepts, so that in the sagittal, as well as the tangential sections the same formula applies to an infinitely thin pencil passing obliquely through the centre of the infinitely thin lens as was derived for normally incident pencils, viz: formula (10), which in this case assumes the forms

$$\left.\begin{aligned} \frac{1}{f_2'} - \frac{1}{f_1} &= \frac{1}{f_s} \\ \frac{1}{t_2'} - \frac{1}{t_1} &= \frac{1}{f_t} . \end{aligned}\right\} \quad \ldots \quad \ldots \quad \text{(vii)}$$

It should, however, be observed that in obliquely incident pencils the focal lengths depend upon the angle of incidence of the principal ray and, moreover, that they differ in the two sections.

Before the introduction of cylindrical lenses, this latter property was used for correcting the astigmatism of the eye. From the equation $f_t = f_s \cos^2 j$ it follows that the focal length within the tangential section is shorter and the lens accordingly more effective in the tangential azimuth than in the sagittal section.

103. Reflection at a Spherical Surface.—Taking the direction of the incident light throughout as the positive direction of the co-ordinates, we may derive the equations applicable to

reflection from those previously established for refraction by simply putting $n = -1$. Since in this case $j' = -j$, we have by equations (18) and (19)

$$S = S' = \frac{r}{2\cos j}; \qquad T = T' = \frac{r\cos j}{2},$$

and equations (20) and (21) assume the forms

$$\frac{1}{f} + \frac{1}{f'} = \frac{2\cos j}{r}$$

and

$$\frac{1}{t} + \frac{1}{t'} = \frac{2}{r\cos j}$$

or, since by (25) and (26)

$$f_f = f_f' = -\frac{r}{2\cos j}$$

and

$$f_t = f_t' = -\frac{r\cos j}{2},$$

we may also write

$$\frac{1}{f} + \frac{1}{f'} = -\frac{1}{f_f}$$

and

$$\frac{1}{t} + \frac{1}{t'} = -\frac{1}{f_t}.$$

This expression, it will be noticed, is analogous to that obtained in § 80 for a normally incident pencil, excepting that the two focal lengths are again different and that they depend upon the magnitude of the angle of incidence.

In the sagittal section the angular magnification becomes by (15)

$$\gamma_f = f\,/\,f',$$

and, since the Smith-Helmholtz equation (4) § 84 holds good in this case also, the magnification becomes

$$\beta_f = -f'\,/\,f.$$

In the tangential section the angular magnification is by (17)

$$\gamma_t = t\,/\,t'$$

and the magnification by (4)

$$\beta_t = -t'\,/\,t.$$

It will be understood that all these relations may also be deduced from general considerations.

The perspective centre of the sagittal section is, according to the invariable rule, the centre of the sphere. The perspective centre of the tangential section may be determined graphically by Young's method (Fig. 48), by drawing perpendiculars CM and CM' from the centre upon the incident and refracted rays

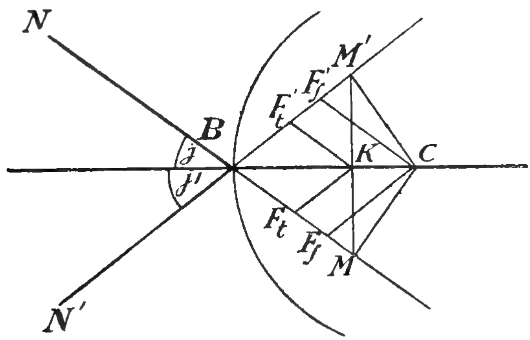

FIG. 48.

Reflection of an obliquely incident elementary pencil. Graphic method of determining the perspective centre and the principal foci.

respectively, joining their feet, and drawing from the centre C a third perpendicular upon the line joining M and M'. The foot K of this perpendicular is the perspective centre of the tangential rays, and, since the figure is symmetrical with respect to CB, the perspective centre must lie on this latter ray.

Parallels to the incident and refracted rays through C and K respectively, give us the positions of the principal foci, whose abscissæ confirm the values of S, S', T and T' as given above, which will be readily seen from the figure.

104. Reflection at a Plane.—In the extreme case when the sphere becomes a plane, we have the relations $f = -f'$; $t = -t'$; $\gamma_f = \gamma_t = -1$; $\beta_f = \beta_t = +1$.

These equations agree with the results obtained in § 81.

3.—REFRACTION THROUGH SURFACES OF DOUBLE CURVATURE.

105. When an infinitely thin pencil is incident on a surface of double curvature it becomes astigmatic, even when its principal axis is normal to the surface, as we shall presently see.

In the discussion of surfaces of double curvature we shall confine ourselves to the case in which the plane of incidence of the

principal ray is a principal section of the surface. The general case has hitherto been of no interest to the practical optician, so that with regard to this subject it will be sufficient to refer the reader to the treatise of Heath (**3**. art. *158*) and the original papers of Sturm (**1**. **2**. *1238*), Clerk Maxwell (**4**.), Neumann (**2**.) and Matthiessen (**10**.).*

Let that principal section of the surface which contains the principal ray a be called a **tangential section**, and let the term **sagittal section** be applied to the plane through a at right angles to the tangential section. It should be noted that this plane is not in general a principal section of the surface.

The tangential and sagittal sections of surfaces of double curvature are subject to the same formulæ and laws as the equivalent sections of the sphere, excepting that the radius of the sphere is now replaced by the radii of curvature of the surface at the respective lines of curvature.

We shall first consider the tangential section. Since it coincides with the plane of incidence it follows that the rays originally contained therein will remain in it after refraction ; *i.e.*, refraction occurs in one plane. This infinitely thin laminar space cuts the surface along a principal line of curvature. If we suppose the normals to be drawn to the surface along this curve, and if we confine ourselves to an infinitely small portion of the curve, all these normals will pass through the centre of curvature of the curve about the point of incidence B. The infinitely small portion

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* The problem investigated by these authors may be stated thus :—Let an infinitely thin pencil of rays, that is an infinitely thin pencil having a system of orthogonal surfaces, be defined by the positions of the principal ray and the two focal lines ; suppose further that it is refracted by a surface of double curvature. It is required to determine the positions of the refracted principal ray and its focal lines ; or as stated by Sturm : The radii of curvature of the principal sections of the incident wave-front being given, it is required to determine the radii of curvature of the refracted wave-front. Sturm determined, as a matter of fact, not only the positions of the radii of curvature but also those of the principal sections. The equations at which he arrived are identical with those of Clerk Maxwell and Neumann, who in all probability were not aware of his work. Sturm's proofs are somewhat discursive, especially so as he deals with theorems other than the one with which we are here concerned. Clerk Maxwell and Neumann attain their object by shorter methods. The former made use of the properties of Hamilton's characteristic function. His reasoning has been followed by Heath in his treatise in which are also discussed the other interesting investigations of Clerk Maxwell (**1**. **5**. **6**.). His graphic methods are, however, omitted by Heath. Neumann's proof is based upon the principle of least time. His formulæ are exceedingly well arranged. Whilst these authors, like ourselves, proceed upon the assumption that the focal lines are at right angles to the principal ray (which is permissible so long as we are concerned with small quantities of the second order), Matthiessen dispenses with this restriction. He states the inclination of the focal line with respect to the principal ray of the incident pencil and then computes it for the refracted pencil. Cf. also Leroy (**1**.).

of the curve which refracts the pencil coincides with a circle described through B about the centre of curvature. The refraction at the tangential section does not differ accordingly from the refraction at the analogous section of a sphere whose curvature is the same as the curvature at the tangential section.

Similar conditions are applicable to the sagittal section along the principal line of curvature at right angles to the tangential section. If now through B we suppose a sphere to be described about the centre of curvature of the element corresponding with the point B, this sphere will be intersected by the sagittal section along a circle which coincides with the actual line of curvature within the limits of the infinitely thin pencil. The normals to the refracting surface and to the sphere coincide along this curve. The refraction in the sagittal section of the surface is identical with the refraction in a sagittal section of the osculating sphere here considered.

It will thus be seen that an image is formed in the tangential and sagittal sections in precisely the same manner as in the case of a sphere, and that the formulæ established for the sphere hold likewise for a surface of double curvature.

Consider now an infinitely thin solid pencil having its vertex O on the principal axis a. To this pencil, like any other infinitely thin pencil, Sturm's theorem is applicable, *i.e.*, so long as we neglect infinitesimals of the third order all rays of the pencil may, after refraction, be considered as passing through the two focal lines of Sturm. On either line we know already one point. The first focal line passes through the tangential image-point O_t' which is conjugate to the vertex of the pencil O, whilst the second line passes through its sagittal image-point O_s'. In accordance with Sturm's theorem the focal lines are situated in the principal planes of curvature of the refracted wave-front. Now, one of these planes coincides with the tangential section, since the pencil of rays and the refracting surface about the point of incidence B are symmetrical with respect to this section. Also, since the principal planes of curvature are normal to one another, it follows that the sagittal section is the other of the two planes.

The primary focal line is accordingly normal to the tangential section at O_t', and the secondary is normal to the sagittal section at O_s'. The primary focal lines of all points on the principal axis are mutually parallel, as are similarly those of the secondary focal lines.

If now we consider in the aggregate the rays which are infinitely near to the principal ray a, a', it will be seen that the directions of the focal lines of these rays deviate by infinitely small amounts from the directions of the focal lines of the principal ray. The primary focal lines of all these rays will be approximately normal to the tangential section of the principal ray a', and the secondary focal

lines are normal to the sagittal section. This brings us to the following theorem which Lippich (. *176*) has formulated with respect to the sphere :

Let two conjugate rays g, g' infinitely near to a ray a, a' which traverses a principal section of the surface, be projected upon the tangential and sagittal sections respectively of this principal ray. Then the projections of this ray, viz., g_1, g_1', and g_2, g_2' respectively, will be mutually conjugate.

Let H_1 be the tangential section of the ray a, a', and let H_2 and H_2' be its sagittal sections. We shall first give the proof of the proposition for the projections g_1, g_1' upon the tangential section.

Since H_1 is the principal section of the surface and g_1 is infinitely near to a, it follows that one of the planes of the two principal sections passing through the point of incidence of g_1 coincides with H_1, and hence this ray conforms to the theorems which we have demonstrated above. The ray g lies in the sagittal section of g_1, and accordingly g' is in the sagittal section of the ray g_1' which is conjugate to g_1. Now this sagittal section is, however, normal to the tangential section H_1, and therefore g_1' is the projection of g' upon H_1.

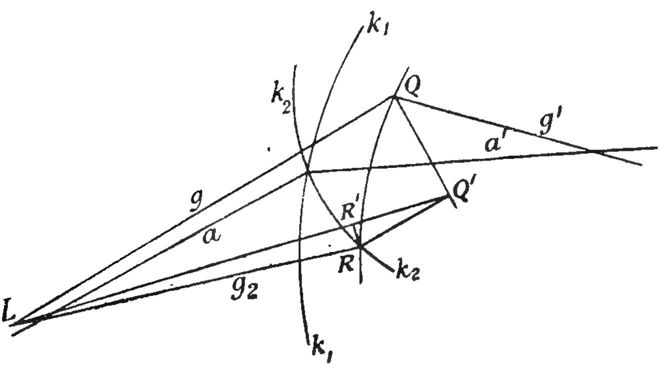

FIG. 49.

Proof of Lippich's proposition.

The projection of g_2 on the sagittal section of the incident ray g adjacent to the principal ray a.

As regards the projections upon the sagittal sections the proof is more difficult since the projection of the ray g upon this section does not lie in a principal section of the surface. Let k_1k_1 (Fig. 49, 50) be the principal line of curvature of the surface in the tangential section of a, and let k_2k_2 be that in the sagittal section. Through the point Q, where g intersects the surface, let the principal line of curvature be drawn parallel to k_1k_1 and produce it to meet k_2k_2 at R. Let L be the point where g

intersects the sagittal section H_2; and join L and R, then the straight line LR coincides with the projection g_2 of g upon H_2, if we neglect infinitesimals of the second order. For if, to obtain g_2, we drop a perpendicular QQ' upon H_2, then RQ' (Fig. 49) will be parallel to a, since the plane $QQ'R$ is parallel to the principal section H_1, from which it follows that the angle $LQ'R$ is infinitely small; and, since RQ' is also infinitely small, it follows that the perpendicular RR' from R upon LQ' is an infinitesimal of the second order. LR coincides accordingly with g_2. Since, now, g_2 lies in H_2, it follows that the conjugate refracted ray g_2' lies in H_2' (Fig. 50), and if g_2' and Q be joined by a plane this plane will be normal to H_2', since, neglecting infinitesimals of the second order, g_2' passes through the foot Q'' of the perpendicular from Q upon H_2. The

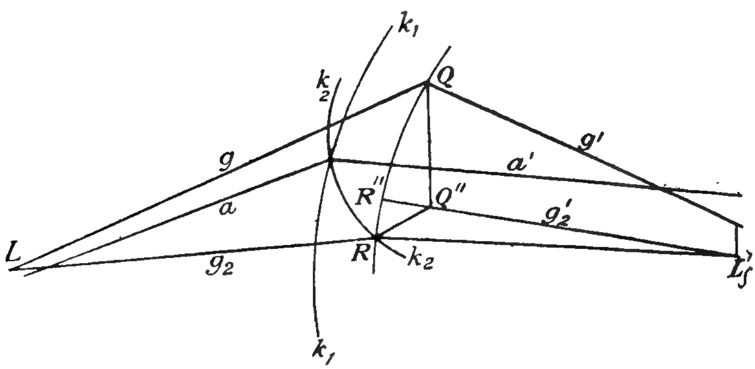

FIG. 50.

Proof of Lippich's proposition.

The projection of g'_2 on the sagittal section of the refracted ray g' conjugate to g.

plane Qg'_2 contains, however, the ray g' which corresponds to g; for, in the first place, g' passes through Q, and secondly, being a ray of the pencil L, it traverses the secondary focal line b_2 of the point L, which at a point of g_2' is normal to H_2' and hence lies in the plane Qg_2' normal to H_2'. Thus g_2' is the projection of g' upon H_2'. The projections of conjugate straight lines of the pencil upon the principal sections are therefore likewise conjugate.

With the aid of Lippich's theorem we can find the corresponding ray g' of any ray g which is infinitely near to the principal ray a if we know the optical constants in the sections H_1 and H_2, H_2'. It is only necessary to determine for g the corresponding projections g_1 and g_2, whilst in the tangential section H_1, and the sagittal sections H_2 and H_2', we need only find the rays g_1' and g_2' corresponding to g_1 and g_2 respectively. g_1' and g_2' are then the projections of g'. The manner of determining the optical constants follows from what has been said at the beginning of this article.

When the refracted principal ray a' falls upon a second refracting surface and lies in one of the principal sections of the point of incidence, images are formed likewise at this second surface within the tangential and sagittal sections; but the principal sections of the image-space of the first surface are not, in general, identical with the principal sections of the object-space of the second surface, and thus a direct transition from the first to the last medium is impracticable. We shall confine ourselves to the consideration of two special cases.

A.—Images formed by Surfaces of Double Curvature, all of which are at Right Angles to the Principal Ray, and the Principal Sections of which coincide.

106. In this case a direct transition from the first to the last medium is possible. Both principal sections (which are common to all the media) are tangential sections. We shall distinguish them arbitrarily as the first and second principal sections. As we have seen above, the curves in which the planes of principal sections intersect the surfaces, may be regarded as circles whose radii are equal to the radii of curvature of the surfaces, and which, therefore, in general, are of different magnitudes in the two principal sections. With the aid of the previous general formulæ of image-composition, the images due to these circles in either principal section, can be determined from a single relation; which indicates a direct transition from the first to the last medium. When this has been effected we may find for any ray g a conjugate ray g' in the last medium by determining the projections g'_1, g'_2, which are conjugate to its projections g_1, g_2 upon the two principal sections. Then g' will be the ray whose projections are g_1' and g_2'.

In a pencil radiating from a point O, if for every ray g through O we construct the corresponding projections g_1 and g_2, as well as the rays g_1' and g_2' corresponding to these rays in the first and second principal sections, it will be seen that all rays g_1 pass through the projection O_1 of O upon the first principal section. The corresponding rays g_1' pass through the image O_1' of O_1 in the first principal section, *i.e.*, all rays g' pass through the straight line which at O_1' is normal to the first principal section. Similarly, all rays g_2' pass through the image O_2' of the projection O_2 of O upon the second section, and hence the rays g' pass through the straight line which at the point O_2' is normal to the second principal section. The straight line normal to the first and second principal sections respectively, at the points O_1' and O_2' are Sturm's focal lines of the pencil proceeding from O, after its refraction through all the surfaces. In general, O_1' and O_2' are situated in two different planes normal to the axis, *i.e.*, the pencil is astigmatic. It becomes homocentric when O_1' and O_2' are situated in the same plane normal to the principal axis, for in

this case the two straight lines at O_1' and O_2' normal to the two principal sections intersect, and all rays of the pencil must pass through the point of intersection if they are to cut both straight lines.

A special case of this kind is the refraction of a pencil which is normally incident upon one or more cylindrical surfaces whose generating lines are situated in two planes passing through the axis and normal to one another, viz., the principal sections of the previous investigation. If we assume the cylindrical surfaces to be right circular cylinders, the radius of curvature in one principal section is infinitely great, whilst in the other it is equal to the radius of the cylinder. To comprehend the effect of the entire system in either principal section, all that is necessary in this case is to construct a system of straight lines and circles in either of the principal sections. By Lippich's theorem, the refraction of any straight line is reducible to the refraction of its projections upon the principal sections.

The investigation becomes much more complex when the principal sections of successive surfaces do not coincide, even though the principal ray still passes normally through the surfaces. Like others, we shall confine ourselves to the case of infinitely thin systems.

B. The Composition of Two Infinitely Thin Systems of Double Curvature whose Principal Sections do not coincide.*

107. We shall assume that the principal axis is at right angles to all the refracting surfaces whose vertices are all infinitely near to each other. We shall further assume that the surfaces of the first set have the same principal sections in common and that the same applies to the surfaces of the second set; but we shall assume that the principal sections of the surfaces of the first set are different from those of the second set. In either principal section, both of the first and second set of surfaces, we shall now assume all the successive images to be combined into a single image, thereby resolving the composite system into two infinitely thin systems in contact. The problem is now to combine these two resulting systems, assuming their composition to be practicable. Let f_1, f_1' be the focal lengths of the first system within what we arbitrarily fixed upon as the first principal section, and let g_1, g_1' be the focal lengths within the second principal section. Since the vertices of all the surfaces coincide it follows that their principal points will coincide likewise. The co-ordinates of the foci referred to the common vertex are accordingly equal to the negatively reckoned focal lengths. By the term "principal plane" we shall

* References :—Straubel (*l.*), Thompson (*l.*), van der Plaats (*l.*), Sowter (*l.*).

denote the plane which at the common vertex S of all the surfaces is normal to the principal axis. Points on this plane are common to the object and image spaces of both sets of surfaces.

We will introduce three systems of co-ordinates having their axes of x all coinciding with the principal axis of the pencil, whilst the axes of y and z differ in their directions. Let the vertex S be the origin for all systems; also let X, Y, Z be a system of reference to which the other systems are referred. The other two systems of co-ordinates are peculiar to the two optical systems in that their planes of xy and xz coincide with the principal sections of these systems. We will distinguish these auxiliary co-ordinate systems, like the optical systems, by the suffixes 1 and 2. A ray of light is defined in the system of reference by the co-ordinates Y, Z of the point at which it intersects the principal plane, and by the infinitely small angles U and V contained between the axis of x and the projections of the ray upon the planes of XY and XZ. Let y_1, z_1, u_1, v_1 be the corresponding quantities of the first auxiliary system, and let a_1 be the angle contained between the plane of reference XY and the plane of $x_1\, y_1$, the latter being the plane of the first principal section of the first system. We then have the following formulæ, which we may establish directly in the usual way by reference to Fig. 51.

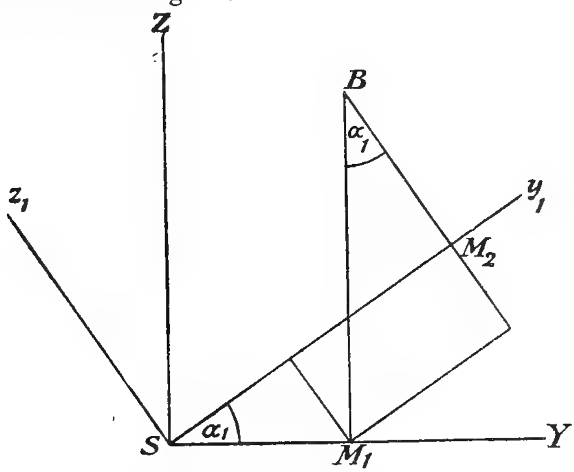

FIG. 51.

$SM_1 = Y$; $M_1B = Z$; $SM_2 = y_1$; $M_2B = z_1$.

Diagram connecting the primary and auxiliary axes.

$$\left.\begin{aligned} y_1 &= Y\cos a_1 + Z\sin a_1 \\ z_1 &= -Y\sin a_1 + Z\cos a_1 \end{aligned}\right\} \quad \dots \quad (29)$$

$$\left.\begin{aligned} Y &= y_1\cos a_1 - z_1\sin a_1 \\ Z &= y_1\sin a_1 + z_1\cos a_1 \end{aligned}\right\}. \quad \dots \quad (30)$$

Analogous formulæ hold for the angles U, V, u_1, v_1, since by assumption, these angles are infinitely small. To make this clear let a sphere of unit radius be described about the origin S and through its centre let a parallel be drawn to the ray. Let A be the point where this parallel meets the sphere. The segment of the sphere which contains all such points A may be regarded as a plane at right angles to the axis of x since the point A lies infinitely near the axis of x. It follows that U, V and u_1, v_1 respectively, are the rectangular co-ordinates of the point A referred to a plane system of co-ordinates whose origin is at the point where the axis of x intersects the sphere and whose axes are parallel to the axes of Y, Z and $y_1 z_1$, respectively. We may then obtain equations for the angles U, V, u_1, v_1 which are quite analogous to the equations (29) and (30), viz.:

$$\left.\begin{aligned} u_1 &= U_1 \cos a_1 + V_1 \sin a_1 \\ v_1 &= -U_1 \sin a_1 + V_1 \cos a_1 \end{aligned}\right\} \quad \ldots \quad \ldots \quad (31)$$

$$\left.\begin{aligned} U_1 &= u_1 \cos a_1 - v_1 \sin a_1 \\ V_1 &= u_1 \sin a_1 + v_1 \cos a_1 \end{aligned}\right\} \quad \ldots \quad \ldots \quad (32)$$

In these equations the values of the angles U and V before refraction at the first surface are denoted by U_1, V_1.

Since the planes of $x_1\, y_1$ and $x_1\, z_1$ are principal sections of the first system of surfaces, it follows that the general formulæ are applicable to the projections of the ray of light upon these planes. Let x_1, x_1' respectively be the x-co-ordinates of the point where the projection of the ray of light upon the plane of $x_1\, y_1$, before and after refraction through the system, intersects the axis of x. Then, by equation (29) in § 60, since both principal points coincide with the origin of co-ordinates,

$$\frac{f_1}{\mathrm{x}_1} + \frac{f_1'}{\mathrm{x}_1'} + 1 = 0\,.$$

Now,

$$u_1 = \frac{y_1}{\mathrm{x}_1}; \qquad u_1' = \frac{y_1}{\mathrm{x}_1'},$$

hence

$$f_1 u_1 + f_1' u_1' + y_1 = 0\,;$$

or, since by § 78 (iii),

$$f_1 / f_1' = -\, n_1 / n_1'$$

$$n_1' u_1' = n_1 u_1 + n_1 \frac{y_1}{f_1}\,.$$

For the second principal section we obtain in a similar manner

$$n_1' v_1' = n_1 v_1 + n_1 \frac{z_1}{g_1}\,.$$

We must now return to the system of reference X, Y, Z. The co-ordinates Y, Z of the point where the ray intersects the principal plane do not change when the pencil undergoes refraction. It is then only necessary to express the angles U_1', V_1' of the refracted ray in terms of u_1', v_1'. Several formulæ analogous to equations (32) are applicable to these angles.

$$\left.\begin{aligned} U_1' &= u_1' \cos a_1 - v_1' \sin a_1 \\ V_1' &= u_1' \sin a_1 + v_1' \cos a_1 \end{aligned}\right\} \quad \ldots \quad \ldots \quad (33)$$

Introducing in these equations the values just found for u_1' and v_1' and substituting for u_1, v_1, y_1 and z_1 by means of equations (31) and (29) the quantities U_1, V_1, Y, Z, we obtain the new equations:

$$\left.\begin{aligned} U_1' &= \frac{n_1}{n_1'}\left[U_1 + Y\left(\frac{\cos^2 a_1}{f_1} + \frac{\sin^2 a_1}{g_1}\right) + Z \sin a_1 \cos a_1 \left(\frac{1}{f_1} - \frac{1}{g_1}\right)\right] \\ V_1' &= \frac{n_1}{n_1'}\left[V_1 + Y \sin a_1 \cos a_1 \left(\frac{1}{f_1} - \frac{1}{g_1}\right) + Z\left(\frac{\sin^2 a_1}{f_1} + \frac{\cos^2 a_1}{g_1}\right)\right]. \end{aligned}\right\} (34)$$

These equations in conjunction with the co-ordinates Y, Z of the point where the ray intersects the principal plane completely determine a ray emerging from the first system.

The following analogous formulæ serve for tracing the path of the rays through the second optical system:

$$\left.\begin{aligned} U_2' &= \frac{n_2}{n_2'}\left[U_2 + Y\left(\frac{\cos^2 a_2}{f_2} + \frac{\sin^2 a_2}{g_2}\right) + Z \sin a_2 \cos a_2 \left(\frac{1}{f_2} - \frac{1}{g_2}\right)\right] \\ V_2' &= \frac{n_2}{n_2'}\left[V_2 + Y \sin a_2 \cos a_2 \left(\frac{1}{f_2} - \frac{1}{g_2}\right) + Z\left(\frac{\sin^2 a_2}{f_2} + \frac{\cos^2 a_2}{g_2}\right)\right]. \end{aligned}\right\} (35)$$

When a ray of light traverses both systems in succession, $U_2 = U_1'$, $V_2 = V_1'$, and if accordingly we substitute in the preceding equations the values obtained for U_1' and V_1' from equation (34), we have

$$\left.\begin{aligned} U_2' = {} & \frac{n_1}{n_2'} U_1 + \frac{n_1}{n_2'} Y\left(\frac{\cos^2 a_1}{f_1} + \frac{\sin^2 a_1}{g_2}\right) + \frac{n_1}{n_2'} Z \sin a_1 \cos a_1 \left(\frac{1}{f_1} - \frac{1}{g_1}\right) \\ & + \frac{n_2}{n_2'} Y\left(\frac{\cos^2 a_2}{f_2} + \frac{\sin^2 a_2}{g_2}\right) + \frac{n_2}{n_2'} Z \sin a_2 \cos a_2 \left(\frac{1}{f_2} - \frac{1}{g_2}\right) \\ V_2' = {} & \frac{n_1}{n_2'} V_1 + \frac{n_1}{n_2'} Y \sin a_1 \cos a_1 \left(\frac{1}{f_1} - \frac{1}{g_1}\right) + \frac{n_1}{n_2'} Z\left(\frac{\sin^2 a_1}{f_1} + \frac{\cos^2 a_1}{g_1}\right) \\ & + \frac{n_2}{n_2'} Y \sin a_2 \cos a_2 \left(\frac{1}{f_2} - \frac{1}{g_2}\right) + \frac{n_2}{n_2'} Z\left(\frac{\sin^2 a_2}{f_2} + \frac{\cos^2 a_2}{g_2}\right). \end{aligned}\right\} (36)$$

The question now arises as to whether it is possible to introduce an infinitely thin system between the media n_1 and n_2' which shall be equivalent to the two combined systems 1 and 2 in its effects on all rays, *i.e.*, for all values of Y, Z, U_1, V_1. Let f and g be the first focal lengths of this hypothetical system, and a the angle comprised between the first principal section and the plane of XY. For this system we accordingly have the following equations:

$$\left.\begin{aligned} U_2' &= \frac{n_1}{n_2'}\left[U_1 + Y\left(\frac{\cos^2 a}{f} + \frac{\sin^2 a}{g}\right) + Z \sin a \cos a\left(\frac{1}{f} - \frac{1}{g}\right)\right] \\ V_2' &= \frac{n_1}{n_2'}\left[V_1 + Y \sin a \cos a\left(\frac{1}{f} - \frac{1}{g}\right) + Z\left(\frac{\sin^2 a}{f} + \frac{\cos^2 a}{g}\right)\right] \end{aligned}\right\} \quad (37)$$

and, for all values of U_1, V_1, Y, Z, the sets of equations (36) and (37) should give the same values. This will be the case when the following three equations are satisfied:

$$n_1\left(\frac{\cos^2 a_1}{f_1} + \frac{\sin^2 a_1}{g_1}\right) + n_2\left(\frac{\cos^2 a_2}{f_2} + \frac{\sin^2 a_2}{g_2}\right)$$
$$= n_1\left(\frac{\cos^2 a}{f} + \frac{\sin^2 a}{g}\right), \quad \ldots \quad \ldots \quad \ldots \quad (38)$$

$$n_1 \sin a_1 \cos a_1\left(\frac{1}{f_1} - \frac{1}{g_1}\right) + n_2 \sin a_2 \cos a_2\left(\frac{1}{f_2} - \frac{1}{g_2}\right)$$
$$= n_1 \sin a \cos a\left(\frac{1}{f} - \frac{1}{g}\right), \quad \ldots \quad \ldots \quad (39)$$

$$n_1\left(\frac{\sin^2 a_1}{f_1} + \frac{\cos^2 a_1}{g_1}\right) + n_2\left(\frac{\sin^2 a_2}{f_2} + \frac{\cos^2 a_2}{g_2}\right)$$
$$= n_1\left(\frac{\sin^2 a}{f} + \frac{\cos^2 a}{g}\right). \quad \ldots \quad \ldots \quad \ldots \quad (40)$$

We have now three equations for determining the three unknown quantities a, f, g, so that in general it will be possible to solve the problem.

We shall use the following abbreviations:

$$\frac{n_1}{f_1} = \phi_1\,; \quad \frac{n_1}{g_1} = \psi_1\,; \quad \frac{n_2}{f_2} = \phi_2\,; \quad \frac{n_2}{g_2} = \psi_2\,; \quad \frac{n_1}{f} = \phi\,; \quad \frac{n_1}{g} = \psi,$$

and we shall call these quantities the **powers** of the systems in the respective principal sections. When the first medium of the three systems is air this definition is identical with that usually applied to lenses.

By the addition of (38) and (40) we shall have

$$\phi_1 + \psi_1 + \phi_2 + \psi_2 = \phi + \psi \quad \ldots \quad \ldots \quad (41)$$

and, by the subtraction of the same equations,

$$(\phi_1 - \psi_1) \cos 2\, a_1 + (\phi_2 - \psi_2) \cos 2\, a_2 = (\phi - \psi) \cos 2\, a. \quad (42)$$

By combining this equation with equation (39) written in the form

$$(\phi_1 - \psi_1) \sin 2\, a_1 + (\phi_2 - \psi_2) \sin 2\, a_2 = (\phi - \psi) \sin 2\, a\,,$$

squaring both equations and adding the resulting squares, we obtain the following equation:

$$(\phi_1 - \psi_1)^2 + (\phi_2 - \psi_2)^2 + 2\,(\phi_1 - \psi_1)\,(\phi_2 - \psi_2) \cos 2\,(a_1 - a_2) \\ = R^2 = (\phi - \psi)^2.$$

Since in this equation the left side is the result of the summation of two squares it is necessarily always positive. We shall therefore always obtain a real value $\pm\ R$ for $\phi - \psi$. From $\phi - \psi$ and $\phi + \psi$ we are at once able to derive the values of ϕ and ψ; and when these quantities are known we can find $2a$, viz :

$$\phi = \frac{1}{2}\,(\phi_1 + \psi_1 + \phi_2 + \psi_2 + R) \quad \ldots \quad \ldots \quad (43)$$

$$\psi = \frac{1}{2}\,(\phi_1 + \psi_1 + \phi_2 + \psi_2 - R) \quad \ldots \quad \ldots \quad (44)$$

$$\sin 2a = \frac{1}{R}\,[(\phi_1 - \psi_1) \sin 2a_1 + (\phi_2 - \psi_2) \sin 2a_2] \quad (45)$$

$$\cos 2a = \frac{1}{R}\,[(\phi_1 - \psi_1) \cos 2a_1 + (\phi_2 - \psi_2) \cos 2a_2]\,, \quad (46)$$

where

$$R^2 = (\phi_1 - \psi_1)^2 + (\phi_2 - \psi_2)^2 + 2\,(\phi_1 - \psi_1)\,(\phi_2 - \psi_2) \cos 2\,(a_1 - a_2). \quad (47)$$

If for R we put $-\ R$, then ϕ will assume the value which previously applied to ψ, and at the same time the magnitude of a

changes by 90°. This means that what was previously the first principal section now becomes the second principal section, and conversely, so that the solution is not new.

108. Both R and α are also readily obtainable by a graphic method (Fig. 52). In the plane of YZ draw a straight line making an angle $2\alpha_1$ with the axis of Y and make its length KM equal

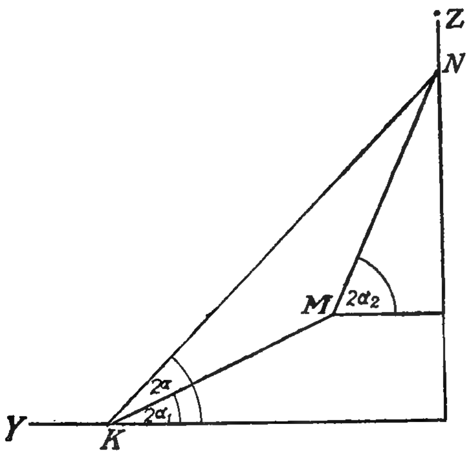

FIG. 52.

$$KM = \phi_1 - \psi_1\,;\ MN = \phi_2 - \psi_2 \quad KN = R$$

Graphic method of determining the cylindrical component and the angle at which it is inclined to the axis of Y.

to $(\phi_1 - \psi_1)$. At M draw another line the length of which is $(\phi_2 - \psi_2)$ making an angle $2\alpha_2$ with the axis of Y. Join the origin K with the terminal point of this second line. Then the length of the straight line joining K and N will be equal to R, and the angle which it makes with the axis of Y will be equal to 2α.

The principle of combining in this way two systems, having differently orientated principal planes of curvature, into a resultant system is applicable to the converse process of resolving any system into two such component systems. Naturally, this may be done in an infinite number of different ways. To emphasize the significance of the quantity R, let the system ϕ, ψ be resolved into a sphere and a cylindrical surface, which the result will show to be possible. Let the spherical surface have the power $\phi_1 = \psi_1 = \kappa$. Let the power of the cylindrical surface be zero in one principal

section and ζ in the other. We shall regard this latter section as the first principal section. Let it make an angle β with the plane XY. Introducing these values in equations (43) to (47), we have, beginning with R:

$$R = + \zeta ; \quad \alpha = \beta ; \quad \phi = \kappa + \zeta ; \quad \psi = \kappa \; \ldots \qquad (48)$$

(the solution $R = - \zeta$ is purely algebraical.)

Now, R in this case is no other than the power of the cylindrical surface. α is the angle between the plane of XY and the principal section of the cylindrical surface normal to the generating line. R may thus be regarded as the cylindrical component of the optical system, whilst α is the inclination to the axis of Y, and ψ the spherical component.

109. Stokes' lens.—For the better comprehension of the preceding formulæ we shall briefly discuss Stokes' lens.

This consists of two plano-cylindrical lenses of equal power, but of opposite sign, placed with their plane sides in contact. The optical axis will be assumed to be normal to all the surfaces of the system, so that it intersects the geometrical axis of the cylindrical surfaces at right angles. The cylindrical lens portions may be rotated relatively to each other about the optical axis. Let γ be the angle which the principal sections normal to the generator include in any given position of the component lenses. We shall regard both lenses as being infinitely thin, and we shall suppose that the distance between them is infinitely small. Let ϕ_1 be the power of the positive lens, $\phi_2 = - \phi_1$ that of the negative lens. ψ_1 and ψ_2 are both zero. We may place the plane of XY so that it bisects the angle γ. Then

$$\alpha_1 = \frac{\gamma}{2} ; \qquad \alpha_2 = - \frac{\gamma}{2}$$

Combining the two lenses by equations (43) to (46), we have

$$R = \pm 2 \phi_1 \sin \gamma ; \quad \alpha = \pm 45^\circ.$$

Also, by (48),

$$\kappa = \mp \phi_1 \sin \gamma.$$

It will thus be seen that Stokes' lens is equivalent to a lens having a spherical surface of power $\mp \phi_1 \sin \gamma$ and a cylindrical surface of power $\pm 2 \phi_1 \sin \gamma$. The azimuth of the cylindrical

component is inclined at $\pm 45°$ to the bisector of the angle between the two principal sections of the cylindrical lenses, and is therefore invariable when the bisector of the principal sections remains fixed in space. When the principal sections coincide, γ, R and κ are all zero, whilst as γ increases, the absolute value of the cylindrical component increases likewise, as well as that of the spherical component.

4. ANAMORPHOTIC IMAGES.

110. By anamorphotic images we understand a system in which the magnification is different according to the direction in which it is measured. Abbe (*7.*) has enunciated a series of theorems relating to optical images of this kind, some of which we shall prove, though not in such general terms as has been done by Abbe; for, whereas Abbe has shown the following theorems to hold with the same degree of generality as those of Sturm, we shall make our proof subject to the following limiting assumptions.

All rays which contribute to the formation of the image are assumed to be infinitely near to a given ray a. This ray, after traversing all the surfaces of the system, will be denoted by a'. Next, we shall assume that the two mutually perpendicular planes through a become two mutually perpendicular planes through a' in accordance with the laws established in Chapter I. Finally, we shall assume that the projections upon these planes of corresponding rays are themselves corresponding rays.

In § 105 it was shown that these assumptions are fulfilled in the case of refraction at a single surface of double curvature when the ray a lies in one of the principal sections of the surface corresponding to its point of incidence. When the ray a traverses several surfaces our assumption is fulfilled if it always remains in one plane and if this plane is a tangential section, the latter term being used in accordance with the definition given in § 105. Upon these assumptions the tangential and sagittal sections of the first medium are, by successive stages, ultimately transformed into tangential and sagittal sections of the last medium.

We shall describe as principal sections the two planes normal to one another in which an image is formed, distinguishing them by the suffixes 1 and 2. The first medium is called the object-space and the last medium the image-space. Letters referring to the image-space are distinguished by accents.

Let O be the vertex of a pencil of rays in the object-space. Let O_1 and O_2 be the projections of O upon the two principal sections, and let O_1' and O_2' be the images of these projections. In the image-space the first projections of all rays of the pencil pass through O_1', whilst the second projections pass through O_2'. All

rays of the pencil pass accordingly through two straight lines which at O_1' and O_2' are respectively perpendicular to the principal sections, these lines being the so-called focal lines of Sturm. In general, if O_1' and O_2', and hence also the two focal lines, are situated in two different planes normal to a', the pencil is astigmatic in the image-space. It becomes homocentric, or "stigmatic," when the two points O_1' and O_2' lie in the same plane normal to a', so that they may be regarded as the projections of a single point O'. In this case, and in this case only, a sharp image of the point O is formed at O'. Thus O and O' constitute a pair of stigmatic points.

It will be readily seen that when O and O' are stigmatic points, all other points situated on a plane normal to a at O will likewise have stigmatic images on a plane normal to a' at O'. Stigmatic points are accordingly situated in planes normal to the axes. These stigmatic planes alone give rise to sharp images. If, accordingly, we are alone concerned with sharply defined anamorphotic images, as in the present instance, it is first necessary to find the stigmatic planes, as it is in these alone that sharp images can be formed. It is sufficient for this purpose to determine the stigmatic points on the axes a and a'.

Let A, A' be a pair of such points. Let F_1 and F_1' be the object-focus and image-focus respectively of the first principal section, and F_2 and F_2' those in the second principal section. Let f_1, f_1', f_2, f_2' be the corresponding focal lengths.

We shall now put

$$F_1F_2 = \mathrm{d}\ ; \quad F_1'F_2' = \mathrm{d}'$$

$$F_1A = x_{s1}\ ; \quad F_1'A' = x_s'{}_1$$

$$F_2A = x_{s2}\ ; \quad F_2'A' = x_s'{}_2\,,$$

so that

$$x_{s2} = x_{s1} - \mathrm{d}$$

$$x_s'{}_2 = x_s'{}_1 - \mathrm{d}',$$

and in accordance with the laws of image-formation § 54 (19),

$$x_{s1}\ x_s'{}_1 = f_1\ f_1',$$

$$x_{s2}\ x_{s2}' = f_2\ f_2'.$$

If d and d′ do not vanish simultaneously, and provided one or both systems are not telescopic, these four equations serve to

determine the four unknown quantities x_{s1}, $x_s'{}_1$, x_{s2}, $x_s'{}_2$, in which case we have

$$x_{s1} = \frac{\mathrm{d}}{2}(A \pm \sqrt{C}),$$

$$x_s'{}_1 = \frac{\mathrm{d}'}{2}(A \mp \sqrt{C}),$$

$$x_{s2} = \frac{\mathrm{d}}{2}(B \pm \sqrt{C}),$$

$$x_s'{}_2 = \frac{\mathrm{d}'}{2}(B \mp \sqrt{C}),$$

where for simplification we put

$$A = m_1 - m_2 + 1,$$

$$B = m_1 - m_2 - 1,$$

$$C = (m_1 - m_2 + 1)^2 - 4m_1 \equiv (m_1 - m_2 - 1)^2 - 4m_2,$$

$$m_1 = \frac{f_1 f_1'}{\mathrm{d}\,\mathrm{d}'},$$

$$m_2 = \frac{f_2 f_2'}{\mathrm{d}\,\mathrm{d}'}.$$

According as C is positive, zero, or negative there are obtained two, one or no stigmatic points. It remains to investigate the case previously excluded, where $\mathrm{d} = \mathrm{d}' = 0$ (the cases $\mathrm{d} = 0$ or $\mathrm{d}' = 0$, as well as the case of telescopic images can be deduced from the general case). When $\mathrm{d} = \mathrm{d}' = 0$

$$x_{s2} = x_{s1}, \quad x'_{s2} = x'_{s1},$$

hence

$$x_{s2}\, x'_{s2} = x_{s1}\, x'_{s1}.$$

Since $x_{s1}\, x_s'{}_1 = f_1 f_1'$ and $x_{s2}\, x'_{s2} = f_2 f_2'$, stigmatic points can only arise when $f_1 f_1' = f_2 f_2'$. If this condition is satisfied all points will be stigmatic. The case $f_1 f_1' \gtrless f_2 f_2'$ leads to that already referred to above, where there are no stigmatic points. The remaining fourth case is that in which all points on the axis, and hence also all points in the elementary space under consideration are stigmatic points.

As a means of obtaining sharp images there are only three cases of stigmatic image-formation of importance. We shall see whether in these cases anamorphotic conditions arise. Let β_1 and β_2 be the magnifications of stigmatic pairs of points within the first and second principal sections respectively.

The ratio of these magnifications $\mu = \frac{\beta_2}{\beta_1}$ has been called by Abbe the **coefficient of distortion.** From the general laws of image-formation, § 54 (20), it follows that

$$\beta_2 = \frac{f_2}{x_{s\,2}}; \quad \beta_1 = \frac{f_1}{x_{s\,1}};$$

hence

$$\mu = \frac{\beta_2}{\beta_1} = \frac{f_2}{f_1} \cdot \frac{x_{s,1}}{x_{s,2}}.$$

In the case where d = d′ = 0 and $f_1 f_1' = f_2 f_2'$, and since by § 78 (iii) $f_1 / f_1' = f_2 / f_2' = -n / n'$,

$$f_1^2 = f_2^2;$$

hence $f_1 = \pm f_2$, and since, moreover, $x_{s1} = x_{s2}$, it follows that

$$\mu = \pm 1.$$

Hence there is no distortion, and the magnification is the same in all directions. This case conforms to the ordinary case of the formation of images by spherical surfaces arranged at right angles to the axis.

If d and d′ do not vanish simultaneously the values of x_{s1}, x'_{s1}, x_{s2}, $x_s{}'_2$ previously obtained are applicable, and there are formed two stigmatic points. (When $C = 0$ these points coincide. A negative value of C need not be considered since it is not consistent with the formation of a stigmatic image). The upper sign of $\sqrt{C}$ applies to one pair of points, and the lower sign to the other pair. Let μ_1 be the coefficient of distortion of the first pair, and μ_2 of the second pair, so that

$$\mu_1 = \frac{f_2}{f_1} \cdot \frac{A + \sqrt{C}}{B + \sqrt{C}}, \quad \mu_2 = \frac{f_2}{f_1} \cdot \frac{A - \sqrt{C}}{B - \sqrt{C}};$$

hence

$$\mu_1 \mu_2 = \frac{f_2^2}{f_1^2} \cdot \frac{A^2 - C}{B^2 - C} = \frac{f_2^2\, 4m_1}{f_1^2\, 4m_2} = \frac{f_1 f_1' f_2^2}{f_1^2 f_2 f_2'}.$$

Now, the value of this expression is unity since $f_1 f_1' / f_2 f_2' = f_1^2 / f_2^2$.

The two coefficients of distortion are therefore reciprocals. Hence, if for the one pair of stigmatic points the magnification is ν times greater in the direction of the first principal section than in that of the second principal section, the magnification for the other pair will be ν times smaller in the first principal section than in the second.

If $C = 0$, $\mu_1 = \mu_2$, *i.e.*, when there is only one pair of stigmatic points the magnification for this pair of points is identical in the two principal directions and hence in all directions.

Conversely, when $\mu_1 = \mu_2$, instead of two, there will be only one pair of stigmatic points (unless $\mathrm{d} = \mathrm{d}' = 0$ and there is an infinite number of stigmatic points). For, when $\mu_1 = \mu_2$ it follows from the above formulæ that

$$2\, A \sqrt{C} = 2\, B \sqrt{C}.$$

Since A is not equal to B this condition can only be satisfied when $C = 0$. The case of $C = \infty$, which is that of a telescopic system, may be deduced from the general case and should be investigated by means of the equations which have been established for these systems. We shall not consider this case, and will only note that the coefficient of distortion for a pair of stigmatic points whose image-point lies at infinity is given by the ratio

$$\frac{\tan u_2'}{\tan u_1'} = \frac{f_1'}{f_2'} = \frac{f_1}{f_2}.$$

When $C = 0$ there is only one pair of stigmatic points.

We may summarise these investigations as follows :—Stigmatic image-formation occurs either throughout the entire (elementary) space or only with respect to one or two distinct pairs of planes. In the first two cases the magnification is the same in all directions. In the last case there is always distortion. If the coefficient of distortion is ν for the one pair of stigmatic points it will be $\frac{1}{\nu}$ for the other pair.

5. HISTORICAL NOTES.

A.—On Smith-Helmholtz' Equation.

111. Rayleigh (**3.**) has pointed out that Robert Smith (*1.* i *111.* *3.* i *338*) was the first to formulate a special case of Helmholtz' equation. He derived it from Cotes' theorem, as given in § 89. From the symmetrical form of these expressions it follows at once that the apparent distance of the object from the eye remains unchanged when their positions are interchanged. From this first corollary Smith deduced with the aid of several propositions the following corollary to Cotes' theorem :—If an object be viewed through any number of lenses the width of the principal pencil where it meets the eye is to the aperture of the object-glass in the

same ratio as the apparent distance of the object (from the eye) is to its true distance from the object-glass; for example, in telescopes they are in the same ratio as the true size of the object to its apparent size.

By a **principal pencil** is to be understood that pencil which, proceeding from the point of the object on the axis, fills the entire object-glass. The true size of the object is measured by the angle under which the object appears to the unaided eye, whilst its apparent size is the angle under which it appears when viewed through the instrument.

The second part of the corollary is identical with the theorem of Lagrange, which caused Helmholtz to ascribe his own theorem to the latter. Lagrange (***1. 2.***) wrote two papers on the subject, the first in 1778, the second in 1803. The first paper was expressly written to establish the connection between Cotes' theorem, as communicated by Smith, (***1.*** i *111*; ***3.*** i *338*), together with other conclusions formulated by Euler. It is very probable that Lagrange had Smith's "Optics" before him, for he quotes the book accurately. Twenty-five years later he evidently reverted to his previous paper and elaborated it without recollecting that Smith before him had already formed what he now appears to have believed to be a new and independent conclusion. Priority undoubtedly belongs to Smith, and more especially so since it has been shown by Rayleigh that Smith had defined the scope of the corollary more completely than Lagrange. For, whereas the latter actually asserted that the light-transmitting power of a telescope was always equal to that of the unaided eye, Smith had already realised that when the pupil of the eye is larger than the exit pupil of the instrument, the light-transmitting power of the telescope is to that of the unaided eye as the size of the exit pupil is to that of the pupil of the eye. The equation $nuy_s = n'u'y'_s$ should therefore rightly be called the Smith-Helmholtz formula, it was, however, Helmholtz who gave the formula its present commonly accepted form. Moreover, he proved its application to any pair of conjugate points, whereas both Smith and Lagrange had applied it to certain specified points only.

A further extension of Smith-Helmholtz' theorem forms the subject of the paper by Rayleigh already referred to.

B.—On Astigmatism.

112. Barrow (***2.*** v and xiii) investigated the case of obliquely incident rays upon the plane and the sphere, and determined the corresponding refracted rays by a graphic method, and he endeavoured to find the image-points corresponding to the object-points on the incident rays. His whole investigation is restricted to rays contained in the plane of incidence, with respect to which he deals

with the subject in a very comprehensive manner. It is interesting to note the process by which Barrow finds the image-point. In the case of refraction at a plane he supposes the centre of the eye to lie on a certain refracted ray, which we shall call the principal ray, whilst the pupil is assumed to be at right angles to this ray. He then proceeds to ascertain the points where the principal ray is cut by the rays of the refracted pencil meeting the pupil on either side of the principal ray, and he shows that this occurs within two segments situated on either side of a point Z, whose position he determines graphically.

In view of the smallness of the pupil he concludes that this point Z, in the neighbourhood of which all rays entering the eye meet the principal ray, may be regarded as the image-point. He shows that the image-point does not coincide with the point K where the ray at right angles to the refracting plane meets the principal ray. He argues, however, that there is no reason why this point should be regarded as the image-point, as had been done previously, for among all the rays proceeding from it and its neighbourhood there is a single ray, viz., the principal ray, which alone enters the eye, unless the latter happens to be exactly on the incident principal ray, whereas all other rays which actually enter the eye proceed from the neighbourhood of the point Z. Extending the investigation to the case of the circle, he was able to identify the point Z as the point of intersection of two initially adjacent rays (*duorum incidentium sibi quam proximorum concipiuntur refracti*, etc.). The rule which Barrow gives for the determination of the point of intersection, translated in terms of our notation, furnishes an equation which is equivalent to formula (16) § 93. In addition to this rule, Barrow devised two graphic methods of determining the image-point, one of which was given him by a friend, presumably Newton, who assisted him in the preparation and publication of his work on optics. I have dwelt at some little length on the achievements of Barrow since his efforts have been insufficiently appreciated, notably by Wilde (*1.*). That he had an original mind is clearly shown by the manner in which, undeterred by the then ruling authority of Alhazen, he considered only those rays which actually enter the pupil of the eye. We shall now pass on to his great disciple.

To Newton's extensive list of great discoveries we must add that of the principles of astigmatism.* He considered infinitely thin solid pencils of rays incident obliquely on a plane (*2.* Prop. viii) and a sphere (xxxii), and he showed that, apart from the rays situated in the plane of incidence, only those situated on a cone, as defined in § 73, intersect at a single point of the refracted

* Prof. Tscherning, of Paris, kindly drew the author's attention to the early contributions of Newton and Young to the study of the principles of astigmatism.

principal ray. All other rays approach the principal ray somewhere between these two boundaries. With the aid of an hypothesis, to which, however, Newton himself attached little importance, he investigated which point on the image-line should be regarded as the image proper.

L'Hospital (*1. 106* and *133*) sought to determine the caustic curve by ascertaining the positions of the image-points in a tangential section both for reflection and refraction. The formula which he deduced in the case of refraction agrees with our formula (16). As we have pointed out above, this formula is already implied in the rule enunciated by Barrow for the determination of image-points. L'Hospital's formula cannot therefore be regarded as an essentially new contribution. He, however, succeeded in greatly simplifying the proof by introducing the differential calculus.

Smith has treated obliquely incident rays in a separate chapter (*1.* i *160* et seq.; *3.* i *415* et seq.). In the body of the chapter he confines himself to the plane of incidence. He relegates to a note (*1.* ii *82*; *3.* i *447*) the subject of astigmatism, regarding it more as a curiosity. In this note he reproduces Newton's observations respecting a plane surface with the added remark that a similar condition exists in the case of the sphere. Within the plane of incidence the investigations of Smith present, however, a notable advance. Smith determined the principal foci of the object and image-rays due to oblique refraction at one as well as at several surfaces, and he showed that the product of the focal distances of the object and image-rays (which we have denoted by x_s, x_s'), is constant when refraction occurs at several surfaces.

The process by which he determined the value of this product is essentially identical with the method of composition of two optical systems, as explained in § 63. Smith traces in either direction from one to the other extreme medium, a ray which proceeds in an intermediate medium m in a direction parallel to the principal ray. In this way he finds the extreme principal foci F_1 F_2' of two systems adjacent within the medium m. If we denote by F and F' the principal foci of the compound system previously ascertained by him, and by O and O' a pair of conjugate points, Smith showed in effect that the product $FO \cdot F'O'$ is equal to the product $FF_1 \cdot F'F_2'$. Now, $FF_1 \cdot F'F_2'$ or $\sigma\sigma'$ is equal to ff', as has been shown in § 63. Smith showed also that for the compound system $x_s x_s' = ff'$, though in this case the factors of the product ff' are not identical with ours, which is, however, of no consequence so long as we are only concerned with points on the principal ray.

The proof that Newton's formula $x_s \cdot x_s' = ff'$ is applicable also to the case of oblique refraction at several surfaces, represents an important advance, even though at that time its validity was restricted to the plane of incidence.

Young deals on two occasions with the case of oblique pencils incident upon a sphere. In 1801 (*1. 27* et seq.)* he emphasized the importance of Newton's discovery of astigmatism; he described the appearance of a number of cross sections of an obliquely refracted pencil of originally cylindrical cross section; he stated the formula enunciated by L'Hospital for determining the tangential intercepts t'; he determined the position of the tangential and sagittal foci F_t' and F_s'; he discovered the perspective centres K and C of the tangential and sagittal sections respectively, and was the first to describe the astigmatism of the eye. The tangential centre was determined by him with the aid of the graphic method outlined by us, which was subsequently rediscovered by Cornu (*1.*) in 1863, and by Lippich (*2. 171*) in 1878.

In 1807 Young (*3.*) extended his formulæ to include the sagittal intercepts f'; and, as he had already done to some extent in 1801, he applied the formulæ to various special cases, in particular to the case of infinitely thin lenses, and computed for equal-sided bi-convex lenses the axial curvature of the image surfaces corresponding to an infinitely distant object within the tangential and sagittal sections. He was also probably the first to show the existence of the two caustic lines. If we identify with the term "axis" the ray joining the luminous point with the centre of the refracting sphere, the primary focal line or caustic will be the circle described by the tangential point of intersection O_t', when rotated about the axis, whilst the secondary focal line will coincide with the axis. Young realised that certain lines appear most clearly defined at the primary focal line, and others at the secondary focal line; but his description did not distinctly define these lines. Finally, Young's papers show that he was cognisant of the aplanatic points of the circle, for he refers to them in the first tract, and states in the second that they are situated on the circle described with $(n'/n)\,r$ as radius about the centre of the refracting sphere.

With the discovery of the astigmatism of the eye the theory of astigmatism, which prior to the time of Young had been regarded more or less as an optical curiosity, entered into the domain of practical optics.

An important advance in this direction is due to Airy. Having ascertained that one of his eyes was affected with astigmatism (*2.*), he studied this defect in detail without having any knowledge of Young's previous work. For use with his own eye, which was short-sighted as well as astigmatic, he had made a sphero-cylindrical lens which completely removed the defect. He then investigated theoretically (*3.*) the influence of astigmatism of the eye upon the

* In Tscherning's translation of Young's tract into French he gives all formulæ in modern notation, and elucidates Young's arguments, which are often involved and difficult to follow, by numerous notes.

performance of various types of eye-pieces and upon the objective used in a camera obscura. When he wrote his paper it is most probable that he knew nothing of Young's previous work; he arrived at formulæ which did not involve trigonometrical functions, in that he expanded the expressions found by him directly in terms of ascending powers of the aperture. Apart from distortion, Airy determined the amount of the astigmatism and curvature for various types of eye-pieces, and he discussed with the aid of numerical examples the advantages and disadvantages of those eye-pieces. His formulæ are not very clearly arranged; they remained, however, for a long time the best that were available on the subject of eye-pieces.

Young's and Airy's investigations served as the foundation for Coddington's treatment of optics in his celebrated work (*1.*) Airy's achievements are, however, discussed at much greater length than those of Young and Smith. Coddington's merit lies in his lucid presentation of the subject, which is greatly enhanced by a very clear notation and excellent printing. From Coddington's work the theory of astigmatism passed into the other prominent English textbooks, whilst on the Continent it remained practically unnoticed. With Airy we come to the close of what seems to have been the first period in the history of astigmatism, during which investigation was practically restricted to the refraction and reflection of pencils at a spherical surface. (I have treated this section at some length. In what follows I shall content myself with a brief survey of the literature on the subject, since I have not been able to study all the literature of the period so thoroughly as its importance warrants.)

The second group of literature commences prior to Airy's time with Malus (*1.*) in 1808 and (*2.*) in 1811. In this group are to be included the writings of Hamilton (*1.* to *4.*) 1824–1837, Schultén (*1.*–*4.*) 1830–1838, Sturm (*1.*) 1838, and (*2.*) 1845, and lastly Kummer (*1.*) 1860. A significant feature of this period is the strikingly general character of the investigations. They relate to pencils refracted through any number of surfaces of any form, or still more generally, to pencils which are merely defined by their analytical properties and the laws which govern them. In this period it is the mathematical interest which outweighs the optical. The results of this period have been dealt with in the first chapter of this book. Some of the writings here referred to do not deal directly with astigmatism; but they are nevertheless essential to its proper comprehension. Later investigations benefited greatly by the very clearly propounded principles of the constitution of infinitely thin pencils in general which we owe to Sturm (*2.* *554*). His exposition has found its way into all textbooks of importance, whereas the more profound but far less readily accessible investigations of Hamilton and the valuable writings of Schultén have received but scant attention.

These writings were succeeded in the course of the second half of the century by the following :—

1857–1867, Helmholtz (*1*. *238*): General theorems relating, in particular, to the principle of the optical path and its application to prisms;

1857 and 1867, Reusch (*1*.) and (*3*.): A lucid presentation of the most important properties of oblique pencils;

1862, Quincke (*1*.): Verification of Kummer's propositions by experiments on a simple lens;

1862, Mœbius (*5*.): Geometrical investigation of Kummer's theorem regarding infinitely thin pencils of rays;

1863, Cornu (*1*.): Simple proof of Young's propositions on the properties of the perspective centre, as rediscovered by him;

1868, Reusch (*4*.): A clearly visualised theory of cylindrical lenses, the study of which is facilitated by figures giving horizontal and vertical projections;

1873 and 1874, Clerk Maxwell (*4*.) and (*6*.): *See* Note in § 105;

1874, 1878, and 1882, Hermann (*1*.) and (*2*.): *See* Note in § 102.

1877, Lippich (*2*.): An important investigation by means of which he showed that collinear images are formed when pencils undergo oblique refraction at a spherical surface within a tangential and sagittal section. The chief properties of these pencils are derived from the laws of collinear images, after the manner of Mœbius (*4*.), who in 1855 had deduced from these principles the properties of central pencils.

About the same time as Lippich, Abbe used the formation of images within the tangential and sagittal sections as the foundation for his theory of oblique pencils. This theory, which he expounded in the course of his university lectures during the period after 1870 and which was subsequently published by Czapski (*3*.), assumed a greatly simplified form as a result of an assumption which places the secondary axis of the system at right angles to the principal axis. (*See* § 97.)

1879, Lippich (*3*.): Elegant investigations on the subject of infinitely thin pencils which, entering a sphere, undergo repeated refraction at its surface, together with their applications to the rainbow.

1880, Neumann (*2.*). *See* Note in § 105.

1880, Leroy (*1.*) : Refraction at surfaces of double curvature based upon refraction at spherical surfaces. He investigated the points in space which are transformed stigmatically by an infinitely small element of a single surface of this kind. All pencils situated without the principal sections of the surface element (or more exactly its centre) invariably give rise to astigmatic images. Stigmatic points occur in the principal sections only. The author established the conditions which they are required to fulfill. He discusses in detail the astigmatic image-formation of curves situated in planes normal to the principal ray, especially the case of straight lines. Finally he applies these investigations to the eye.

1883–1888, Matthiessen (*4.5.6.7.9.10.*): The subject matter of these papers is sufficiently indicated by their titles as given in the bibliography appended to this book. Respecting (*10.*) see the note in § 105.

1886, Anderson (*1.*) :

1887, Heath (*1.2.*), who discusses the theory of thin pencils as propounded by Kummer and Clerk Maxwell (The author had access to the second edition only).

1888, Gartenschlaeger (*1.*) : *See* § 96.

1888, Fraenkel (*1.*): Eight stereoscopic pictures to demonstrate the path of the rays in an astigmatic eye.

1888, Gleichen (*1.*) : Refraction at a curve of a plane pencil which does not follow Snell's law.

1889, Gleichen (*2.*) : A neat exposition of oblique refraction at a sphere and plane, together with applications to plane-parallel plates and infinitely thin pencils.

1891, Czapski (*2.*) : A paper in support of the perpendicularity to the principal ray of Sturm's focal lines previously assailed by Matthiessen (*5.*) and (*10.*).

1893, Czapski (*3. 69-81*) : Abbe's theory of astigmatic refraction.

1895, Foussereau (*1.*), who determined the tangential and sagittal intercepts of a pencil passing through the centre of an infinitely thin lens at a finite angle of inclination, and established the equations of the tangential and sagittal image surfaces corresponding to a plane normal to the axis, and computed their curvatures on the principal axis.

1898, Straubel (*1.*): Solution of the following problem:—An infinitely small element of a wave-front of double curvature meets two infinitely thin cylindrical lenses. To ascertain the position and form of the wave-front after having traversed the lenses. It is assumed that the principal normal to the incident wave-front is in a straight line with the optical axis of the cylindrical lenses. With the aid of the resulting data the author has investigated the following problems: (i) Under what circumstances are the principal planes of curvature of the wave-front parallel to each other before and after traversing the cylindrical lenses?—(ii) What are the conditions which must be satisfied in order that the emerging wave-front may be symmetrical with respect to the axis?—In conclusion Straubel shows in what manner a system of two cylindrical lenses may be employed in the examination of two regularly reflecting surfaces.

1900, Silvanus Thompson (*1.*): Composition of two obliquely crossed cylindrical lenses into a single optical system, together with a simple graphical construction.

1901, van der Plaats (*1.*): Experimental and theoretical investigation of the image of a straight line at right angles to the optical axis formed by a cylindrical lens. Displacement of the image due to rotation of the object or the lenses about the optical axis. Composition of obliquely crossed cylindrical lenses.

1901, Sowter (*1.*): A new method of determining the composition of crossed cylindrical lenses.

1902, Bouasse (*1.*) shows that, neglecting infinitely small quantities of the second order, the whole of the rays constituting a thin originally homocentric pencil may be regarded as passing through the two focal lines of Sturm, the latter being normal to the principal ray. There are, however, an infinite number of straight lines having the same property. It is therefore more correct to state the case in the following manner:—In either principal section π_1 and π_2 of the pencil there is an infinitely small surface element which is traversed by all the rays of the pencil. The surface element situated at π_1 surrounds the principal focus F_2 of the rays contained in π_2, whilst that situated at π_2 encloses the principal focus F_1 of the rays contained within π_1.

As we shall see later, Schultén had previously described in this way the constitution of infinitely thin pencils.

This brief survey of the literature relating to astigmatism gives only an incomplete idea of the attention which this subject has received during the latter half of the nineteenth century; for it will be noticed that it takes no cognisance of, or refers only quite incidentally to, the applications of these investigations. They have indeed given rise to many useful devices for assisting the eye, and to great improvements in the construction of photographic lenses. Both the eye and these lenses receive principal rays at very considerable angles of inclination with respect to the axis, and exhibit accordingly large astigmatic defects unless special means are adopted for their elimination. References to the literature relating to the eye will be found in the extensive bibliography appended by Arthur Koenig to the second edition of Helmholtz' "Physiologische Optik." Matthiessen has reviewed current literature in Michel's Jahresb. üb. d. Fortschr. d. Ophth., 1879 *et seq.* (*see* Czapski, ***3.*** *81*). The literature on the astigmatism of photographic lenses has been discussed in detail by M. v. Rohr (***3.***).

C.—On Sharply Defined Anamorphotic Images.

113. In a comprehensive paper on refraction at surfaces of double curvature, Schultén (***4.***) has paid special attention to the case in which all surfaces are normal to the axis of the pencil and are so directed that their principal sections are parallel. He investigated (§ viii) the conditions which, under these circumstances, cause the rays proceeding from a point to converge homocentrically, and he formulated the following equation:

$$(A - Be)(G - He) - (C - De)(E - Fe) = 0,$$

where $A, B \, . \, . \, . \, . \, . \, H$ are constants depending upon the position and the curvature of the surfaces, whilst e is the distance of the object-plane normal to the axis from the origin of the co-ordinate system.

Schultén next determined for such a stigmatic system the magnifications in the two principal directions and found that in general they have different values.

When all the surfaces are spherical the fundamental equation is satisfied for all values of e. The magnifications in the two principal directions are in this case equal. This, however, is not the only case in which the fundamental equation is satisfied by all values of e. Moreover, if the equation can be satisfied by certain values of e, then sharp images are obtained only in certain positions of the object-plane. Schultén regarded these cases as too specialised to merit further attention.

It will be seen that Schultén came very near to enunciating Abbe's theorems.

In 1862 Farrenc (*1.*) took out a patent for an anamorphotic device consisting of two cylindrical lenses crossed at right angles to one another and arranged normally to their axes, without, however, going more fully into the theory. Lippich (*5.*) rediscovered this anamorphotic arrangement (as did Rudolph again after him) and employed it in 1884 as the eye-piece of a spectrum apparatus, noting that images formed by such a system are generally acentric and are stigmatic only in two planes. Another anamorphotic device composed of prisms was introduced by Anderton (*1.*) in 1889. Particulars respecting these anamorphotic systems will be found in M. v. Rohr's work on photographic objectives (*3. 393*).

Abbe's theory (*7.*) was published in 1897. It should be noted that only part of the contents of the publication has been included in the body of this chapter. Apart from the patent specification access was obtained to a sheet of formulæ by Abbe.

D.—On Schultén's Memoirs on the Constitution of Infinitely Thin Pencils.

114. Bouasse (*1.*) having in recent times more than once drawn attention to the constitution of infinitely thin pencils, it may be interesting to review briefly the work of Schultén, who was one of the first to investigate the subject.

His first paper on this subject was submitted to the St. Petersburg Academy in 1823 but was not published until 1830, whilst the first part of Hamilton's great work (*1.*), in which this subject was dealt with, appeared in a series of papers presented to the Royal Irish Academy in the years 1824 to 1827, which were published in 1828. In that paper Schultén (*1.*) considered an infinitely thin pencil refracted or reflected at several surfaces of any form. Any three rays of the pencil passing through one point in the first medium were supposed to be cut by a plane in the last medium. Schultén then proceeded to compute the area of the triangle formed by the three points where the rays intersect the plane. He showed that the area of the triangle vanished when the intersecting plane passed through two principal points on the axis of the pencil. Any cross-section of the pencil has at these two points only one instead of two dimensions. Schultén states that these points are those in which the principal ray is cut by its adjacent rays.

The points indicated are accordingly the foci of the pencil. The cross section of the pencil which is reduced to a single dimension conforms to Sturm's focal lines, whose existence had already been anticipated by Hamilton. Schultén did not assume, however, as was subsequently done by Sturm, that the planes are normal

to the axis of the pencil. He showed that the cross-section reduces to a line, whatever may be the inclination of the plane of intersection to the axis.

In his second paper (*2.*), which was submitted in 1836 and published in 1845, he investigated an infinitely thin pencil defined in terms of any analytical law. Those rays of the pencil which intersect the principal axis are situated in two planes, which Schultén called the focal planes of the pencil. All other rays of the pencil intersect either focal plane within an infinitely small surface element enclosing the focus of the pencil whose rays form the other focal plane. He computed the angle of the two focal planes and showed that it becomes a right angle when the rays of the pencil are regular, *i.e.*, when a system of orthogonal surfaces is in question.

In a supplementary paper (*3.*) he showed finally that the pencil may be split up into an infinite number of plane partial pencils, all of which proceed from the points of the surface elements referred to.

The infinitely small surface elements enclosing the foci are the "aires d'amincissements" of Bouasse (*1.*). It will be seen that Schultén's investigation anticipates in all essential points that of Bouasse. It shows that the focal lines may be given any finite inclination with respect to the principal axes, so long as the investigation remains confined to infinitely small quantities of the second order. This being so, the most convenient proceeding is obviously to assume that the focal lines are normal to the principal axis, as we have done in our investigations.

Chapter V.

THEORY OF SPHERICAL ABERRATION.

(A. Koenig and M. v. Rohr.)

1.—SEIDEL'S THEORY OF IMAGE-FORMATION.

(Definition of the Problem.)

115. The Gaussian theory of the formation of images, as expounded in the preceding pages, applies to an elementary space about the axis and was made subject to the reservation that in the expansions all powers above the first of the angle subtended at the centre of the spherical surface by the cross-section of the incident pencil, were to be regarded as negligibly small in comparison to unity.

We shall now take a further step and consider the higher powers of the angular apertures.

To this end we shall consider a centred system of spherical surfaces $S_1 \ldots\ldots S_k$, and at a certain point O of the axis let Pl_1 be a plane, namely the object-plane, normal to the axis, and let this plane contain the object-point O_w. We may now, without prejudice to the general character of the investigation, consider this point to be in the meridian plane (*i.e.*, in the plane of the paper). The two rectangular co-ordinates of the point O_w in the plane of the object, viz., the tangential and sagittal ordinates, are then denoted respectively by l and $L = 0$. At O' in the plane of the image these co-ordinates have corresponding to them in general the co-ordinates l', L'.

To define completely any ray $O_w Q$ proceeding from O_w two additional quantities are required. We may for this purpose choose either the rectangular co-ordinates m, M of the point Q where the ray intersects another plane Pl_{II}, or the corresponding angular

apertures u, v of the pencil of rays. As a rule, though not necessarily so, we shall so choose this plane that it intersects the axis at the point P where an eye situated in the place of the object would perceive the centre of what appears to be the smallest stop. The distance S_1P will be denoted by x. We shall accordingly refer to this plane Pl_{II} as the *aperture plane on the object side*, since the magnitude of the angular apertures u, v depends directly upon the position of the point Q. With the help of the Gaussian formula we can then find the conjugate plane through O' at a distance $s'=S_kO'$ from S_k along the axis, and likewise the plane Pl'_{II} conjugate to

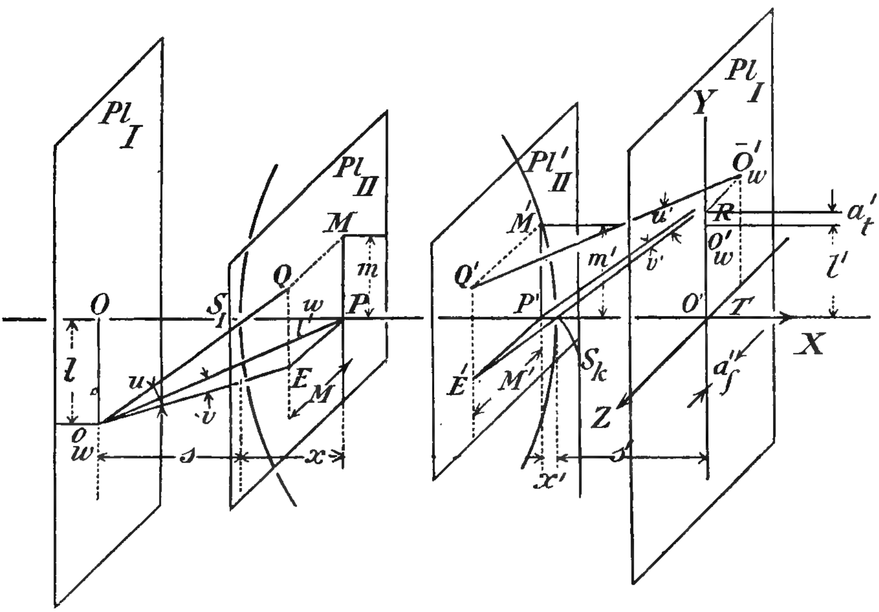

FIG 53.

$OO_w=l$; $PM=m$; $PE=M$; $O'O'_w=l'$; $P'M'=m'$; $P'E'=M'$

$S_1O=s$; $S_kO'=s'$; $S_1P=x$; $S_kP'=x'$; $O_w'R=a_t'$; $O'T=a'_f$.

Pl_I = object-plane; Pl'_I = Gaussian image-plane; Pl_{II} = aperture plane with respect to the object; Pl'_{II} = aperture plane with respect to the image.

Definition of the position and direction of the oblique ray $O_w\,Q \ldots\ldots Q'\overline{O}'_w$.

Pl_{II}, and passing through a point P' on the axis at an abscissa distance $x' = S_kP'$ from S_k. It should be noted also that in most cases the plane Pl'_{II} may be regarded as identical with the aperture plane on the image side.

Since, in a singly refracting optical system any monochromatic ray in the object-space has corresponding to it one, and only one, ray in the image-space, it follows that each of the four quantities by which the image-ray is defined, viz.:

$$l', L' \left\{ \begin{matrix} m' \\ \mathrm{u}' \end{matrix}, \begin{matrix} M' \\ \mathrm{v}' \end{matrix} \right\},$$

must be a single-value function of the four quantities which serve to define the object-ray, viz.

$$l, 0 \left\{ \begin{matrix} m \\ \mathrm{u} \end{matrix}, \begin{matrix} M \\ \mathrm{v} \end{matrix} \right\}.$$

As the accented quantities all change signs when the unaccented quantities assume the opposite values, it follows that in the expansion into a series these single-value functions can only contain odd powers, the first terms of which conform to the Gaussian expressions. Taken in this connection, these terms (*i.e.*, those containing the lowest powers of the expansion) may be looked upon as the principal values, whereas the remaining terms comprising the third, fifth and higher orders furnish certain supplementary values, which may be regarded as *deviations* or *aberrations* from the principal values.

In representing these conditions graphically, we shall call the **principal ray** the line $O_{\prime\prime}P$ in the meridian plane, which joins the object-point to the position where the aperture plane cuts the axis, and the angle contained between it and the axis will be called the **angle of inclination of the principal ray.** We then have obviously the relation

$$\tan w = \frac{l}{x - s}, \ldots \qquad \ldots \qquad \ldots \qquad \text{(i)}$$

and in the case of the angles w, which are so small that quantities of the third order may be disregarded we have

$$w = \frac{l}{x - s}.$$

Similarly, small angular apertures u and v are connected by the relations

$$\mathrm{u} = \frac{m}{s - x} \qquad \mathrm{v} = \frac{M}{s - x}; \ldots \qquad \ldots \qquad \text{(ii)}$$

since for small values of l it is obviously immaterial whether the angle is measured at $O_{\prime\prime}$ or at O.

Now the object-ray, defined in terms of m, M or u, v, has corresponding to it an image-ray defined in terms of m', M' or u′, v′, which intersects the Gauss image-plane at the point $\overline{O}'_w$; and we will suppose l' to be the principal value of the ordinate in the meridian plane and O'_w the corresponding image-point. From Fig. 53 it will now be seen that it is necessary to apply to these Gauss co-ordinates l', $L' = 0$, certain supplementary values a_t' and a'_f, whereby O'_w is transposed to $\overline{O}_w$. These supplementary values will be designated as the **tangential and sagittal aberrations in the Gauss image-plane.**

As regards these supplementary quantities a_t' and a_f' we know at least, that they are of an odd degree of magnitude with respect to the co-ordinates of the points where the two planes are cut by the object-rays. In accordance with Petzval's method we shall refer to the lowest order of coefficients occurring in these expressions as the **numerical order of the image.** Expressed in this way, the small area about O'_w which is the aggregate of all the points $\overline{O}'_w$, is an image of the point O_w of the third, fifth . . . $(2\,v + 1)^{th}$ order. We conclude accordingly, that the quality of the image may be raised to the fifth, seventh . . . $(2\,v + 1)^{th}$ order by eliminating the whole of the aberrational quantities up to the third, fifth . . . $(2\,v + 1)^{th}$ order.

We are here concerned with images of the fifth order of correction or which comes to the same thing, with the evaluation and removal of aberrations of the third order. In view of the fact that Seidel (*3.*) was the first to establish completely the expressions resulting from this theory, we shall refer to it as the Seidel system in order to distinguish it from the Gaussian system.

The evolution of these expressions has also been called the **theory of spherical aberrations of the first order** to distinguish it from the **theory of spherical aberrations of the second, third and higher orders.** We prefer to identify these investigations broadly with the **theory of spherical aberrations,** and to associate with it the **theory of primary, secondary and higher zonal terms.**

From the preceding remarks it follows that there are only ten possible aberrational terms which may occur in our investigation, viz.:

$$\begin{gathered} m^3,\ m^2M,\ mM^2,\ M^3, \\ lm^2,\ lmM,\ lM^2, \\ l^2m,\ l^2M, \\ l^3, \end{gathered}$$

and, as we shall see, they will all be found to occur in the expressions which will be established in the succeeding paragraphs.

We shall conduct our investigation in two ways :—In the first instance we shall investigate the various regions of the field of view as here defined, by the four different powers of l which occur in these expressions, following throughout **Abbe's method of invariants.** Now, whereas Seidel's theory, as explained at the conclusion of this chapter, indicates the total amount of the defect in the image with respect to the extra-axial point, so that its component defects must be determined by a separate investigation, Abbe's method cannot even be applied as a means of obtaining analytical expressions for the defects in the image without first ascertaining the nature of the component defects. Abbe's method however dispenses with any restriction to principal rays of small inclinations.

We shall discuss the evolution of Seidel's five image aberrations as modified by Kerber at the end of the chapter. This elegant and purely analytical investigation confirms the completeness of the preceding results.

2.—THE SPHERICAL ABERRATION OF AXIAL POINTS.

(INTRODUCTION TO ABBE'S METHOD OF INVARIANTS.)

116. Assuming that we are dealing with homogeneous light and spherical surfaces only, we shall, in accordance with the preceding remarks, include all deviations from complete union of the rays at a point in the general term **spherical aberration.** In the simplest case the assumption is made that the luminous object-point is situated on the axis of the centred system of spherical surfaces. There is in this case **spherical aberration of axial points** or **spherical aberration in a narrower sense,** which is frequently referred to, without further qualification, as **spherical aberration.**

A. Longitudinal Aberration of Axial Points.

117. We have shown above how the path of a ray proceeding from a point on the axis may be determined analytically. To obtain a concrete example we shall take the case of converging refraction, and consider rays meeting the first surface at different heights of incidence. We know that in this case the abscissæ of the points where the rays meet the axis are not constant. In fact, the rays proceeding from an object-point do not meet in a point, and the resulting deviation may be rightly described as **longitudinal aberration.**

118. Longitudinal Aberration of a System of Planes of Finite Aperture.—There is only one case in which it is practicable to investigate the longitudinal aberration due to pencils of any angular aperture. This case arises when the centred system is composed of surfaces all of which have infinite radii of curvature.

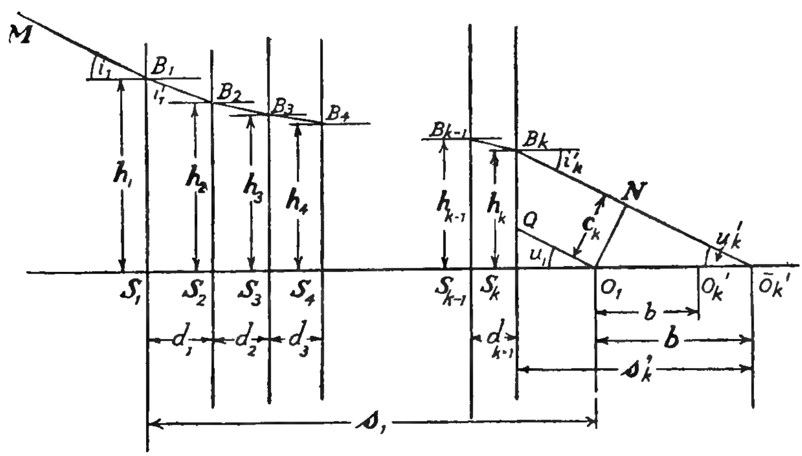

FIG. 54.

$S_1B_1 = \boldsymbol{h}_1$; $S_2B_2 = \boldsymbol{h}_2$; $S_3B_3 = \boldsymbol{h}_3$ . . . . . $S_{k-1}B_{k-1} = \boldsymbol{h}_{k-1}$; $S_kB_k = \boldsymbol{h}_k$
$S_1S_2 = d_1$; $S_2S_3 = d_2$; . . . . $S_{k-1}S_k = d_{k-1}$; MB_1 . . . . $QO_1 =$ incident ray ;
$B_k\bar{O}_k' =$ emerging ray ; $O_1\bar{O}_k' = \boldsymbol{b}$; $O_1O'_k = b$; $O_1N = c_k$; $S_1O_1 = \boldsymbol{s}_1$;
$S_k\bar{O}_k' = \boldsymbol{s}'_k$.

Longitudinal aberration due to a system of planes of finite aperture.

In the investigation of such a system of parallel planes normal to the axis we proceed from the identity

$$\boldsymbol{h}_k = \boldsymbol{h}_1 + \sum_{v=1}^{k-1} (\boldsymbol{h}_{v-1} - \boldsymbol{h}_v),$$

where, in accordance with our previous notation, the ordinates of the points of incidence, which alone occur here, are denoted by $\boldsymbol{h}_v$. Since, in the case of plane surfaces we have quite generally $u_v = -i_v$, we may, by introducing the angles of refraction, write this identity in the form

$$\boldsymbol{s}'_k \tan i'_k = \boldsymbol{s}_1 \tan i_1 - \sum_{v=1}^{k-1} d_v \tan i'_v.$$

Combining with this a further identity, in which b denotes the displacement of the point of intersection on the axis, viz.

$$s_1 = s_k' - b + \sum_{v=1}^{k-1} d_v ,$$

we obtain the final result

$$b = \frac{-\sum_{v=1}^{k-1} d_v (\tan i_v' - \tan i_k') + s_1 (\tan i_1 - \tan i_k')}{\tan i_k'} ,$$

which shows that, in general, this displacement is dependent upon the distance s_1 of the object.

When, however, the system of planes is situated in one and the same medium, in which case $i_1 = i_k'$, an expression is obtained which is independent of the distance of the object, viz.

$$b = \sum_{v=1}^{k-1} d_v \left(1 - \frac{\tan i_v'}{\tan i_1}\right) = \sum_{v=1}^{k-1} d_v \left(1 - \frac{\sin i_v'}{\sin i_1} \cdot \frac{\cos i_1}{\cos i_v'}\right)$$

$$= \sum_{v=1}^{k-1} d_v \left(1 - \frac{n_1}{n'_v} \cdot \frac{\cos i_1}{\cos i'_v}\right) .$$

If now we introduce the parallel displacement c_k of the rays corresponding to i_1, it will be seen from the figure that

$$b = \frac{c_k}{\sin i_k'} = \frac{c}{\sin i_1} ,$$

from which it follows that

$$c_k = \sum_{v=1}^{k-1} d_v \cdot \frac{\sin (i_1 - i'_v)}{\cos i_v'} .$$

To determine the longitudinal aberration it is necessary to consider, in addition, the elongation of the paraxial intercepts to which the system of planes gives rise. From the expression for b in terms of $\tan i_1$ and $\tan i_v'$ given above we have

$$b = \sum_{v=1}^{k-1} \frac{n_v' - n_v}{n_v'} d_v = \sum_{v=1}^{k-1} d_v \left(1 - \frac{\sin i_v'}{\sin i_1}\right) .$$

For the longitudinal aberration $\mathfrak{b} - b$ we obtain accordingly

$$\mathfrak{b} - b = \sum_{v=1}^{k-1} d_v \frac{\sin i_v'}{\sin i_1}\left(1 - \frac{\cos i_1}{\cos i_v'}\right) = \sum_{v=1}^{k-1} d_v \frac{n_1}{n_v'}\left(1 - \frac{\cos i_1}{\cos i_v'}\right),$$

or in the case of an optical system of planes in an external medium of air

$$\mathfrak{b} - b = \sum_{v=1}^{k-1} d_v \frac{\sin i_v'}{\sin i_1}\left(1 - \frac{\cos i_1}{\cos i_v'}\right) = \sum_{v=1}^{k-1} \frac{d_v}{n'_v}\left(1 - \frac{\cos i_1}{\cos i_v'}\right),$$

and since in a system of planes surrounded by air and consisting of media denser than air

$$i_v' < i_1\,, \quad \text{and hence} \quad \cos i_v' > \cos i_1\,,$$

it follows that for all values of i_1 and i_v', whether positive or negative,

$$\mathfrak{b} - b > 0\,,$$

i.e., the intercept of the rays which emerge at finite angles of inclination increases with the inclination of the rays.

119. The Axial Intercept in Terms of $\mathfrak{s} = s + \mathsf{A}\, u^2$. We know from observation that converging spherical surfaces cause the intercepts $\mathfrak{s}$ and $\mathfrak{s}'$ as a rule to diminish as the incidence height or the angle of the cone of rays increases. A simple consideration will show at once that the intercepts $\mathfrak{s}$ and $\mathfrak{s}'$ are independent of the sign of the ordinate of incidence or of the angle which it subtends at the centre of the spherical surface. Analytically this is expressed by the absence of odd powers in the expansion of the expressions for $\mathfrak{s}$ and $\mathfrak{s}'$ in terms of those quantities. It is immaterial which of these variables we choose for the expansion. In our present investigation we take the aperture angles u, u' comprised between the axis and the oblique ray. Hence, if we assume that we have ascertained the conditions after refraction at the v^{th} surface we then obtain two series of the form

$$\left.\begin{aligned} \mathfrak{s}_v &= s_v + \mathsf{A}_v u_v^2 + \mathsf{B}_v u_v^4 + \ldots\ldots. \\ \mathfrak{s}_v' &= s'_v + \mathsf{A}_v' u'^2_v + \mathsf{B}'_v u'^4_v + \ldots\ldots. \end{aligned}\right\} \quad \ldots \qquad \ldots \text{(i)}$$

To render our investigation quite general, we shall assume at the outset that the incident pencil is subject to spherical aberration.

This mode of representing these conditions will remain applicable to a centred system of several surfaces. As was indicated in § 28, the linear substitutions

$$u'_v = u_{v+1}\,; \quad \boldsymbol{s}'_v - d_v = \boldsymbol{s}_{v+1}\,,$$

enable us to connect the quantities resulting after refraction at a surface with the corresponding quantities preceding refraction. This gives us the following set of relations

$$\boldsymbol{s}_{v+1} = s_{v+1} + \mathsf{A}_{v+1}\, u^2_{v+1} + \mathsf{B}_{v+1}\, u^4_{v+1} + \ldots\ldots,$$

where

$$s_{v+1} = s'_v - d_v\,; \; \mathsf{A}_{v+1} = \mathsf{A}'_v\,; \; \mathsf{B}_{v+1} = \mathsf{B}'_v\,.$$

After the last refraction through a system consisting of k surfaces the expansion of the axial intercept will thus be

$$\boldsymbol{s}'_k = s'_k + \mathsf{A}'_k\, u'^2_k + \mathsf{B}'_k u'^4_k + \ldots\ldots \qquad \ldots \text{ (ii)}$$

In the entire infinite series, excluding the first constant term, if we consider only the second term in which the independent variable occurs in the second power, our computation of the spherical longitudinal aberration of points on the axis will be of the first order of approximation.

Our series is then reduced to the simple forms:

$$\left.\begin{aligned} \boldsymbol{s}_v &= s_v + \mathsf{A}_v\, u_v^2 \\ \boldsymbol{s}'_v &= s'_v + \mathsf{A}'_v u'^2_v \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \text{ (iii)}$$

The quantities A_v, A'_v are linear, being the differences of the intercepts of the paraxial ray and a ray of unit inclination, so that

$$\left.\begin{aligned} \mathsf{A}_v &= [\boldsymbol{s}_v]_{u=1} - s_v \\ \mathsf{A}'_v &= [\boldsymbol{s}'_v]_{u'=1} - s'_v \end{aligned}\right\} \quad \ldots \quad \ldots \quad .. \text{ (iv)}$$

The magnitudes of A and A' determine the amount of the longitudinal aberration $\boldsymbol{s}_v - s_v$, $\boldsymbol{s}'_v - s'_v$, which according to the above expression is proportional to the square of the angular aperture. Their algebraical sign indicates whether the intercepts $\boldsymbol{s}_v$, $\boldsymbol{s}'_v$ increase or decrease with increasing values of u_v, u'_v (or of the incidence height). A system of k surfaces is described as spherically corrected when $\mathsf{A}'_k = 0$, as in this case the intercept s'_k is independent of u'_k. In the case of spherically uncorrected systems we frequently meet with the terms **under-corrected,**

indicating that $A'_k < 0$, and **over-corrected**, indicating that $A'_k > 0$. These terms derive their origin from the fact that single thin lenses, which were the first to receive attention in early attempts to remove the effects of spherical aberration, invariably exhibited characteristic aberrations in the images of very distant objects. Now, when in attempting to remove these aberrations, or in other words, to devise means for their correction, the resulting aberrations were of the opposite character, they were appropriately described as *aberrations in the sense of over-correction.* The natural consequence of this terminology was that the defects in the images of very distant object-points produced by uncorrected converging lenses were looked upon as *aberrations in the sense of under-correction.*

Considering the intercepts, s_v, s'_v as conforming to the various values of u_v, u'_v and adhering to the usual convention that the light travels from left to right, then negative values of A_v, A_v' (*i.e.*, those occurring in under-corrected systems) have corresponding to them a series of points arranged in order from right to left, whilst positive values of A_v, A'_v (*i.e.*, those occurring in an over-corrected system) have corresponding to them points ranged in the order from left to right.

120. The Evaluation of the Coefficient A'_k according to Abbe's Method of Invariants in the case of a System of Centred Surfaces.—In § 72 we established an expression for the optical invariant of a ray of any inclination refracted at a surface. Dividing both sides by r, in order to establish agreement with the invariant of the zero rays as given in § 31, and neglecting the surface index for the present, we have

$$Q_s = \frac{n\,(s-r)}{p\,r} = \frac{n'\,(s'-r)}{p'r}. \qquad \ldots \qquad \text{(i)}$$

This invariant might be expanded directly in terms of one of the above-mentioned variables. We shall, however, in this case first find an expression for the invariant in terms of the angular aperture ϕ. The coefficients of similar powers of ϕ, that is in this case the constant terms, and the coefficients of ϕ^2, must be equal to one another, so that we shall have

$$\begin{aligned} Q_s &= Q_s + q_s\,\phi^2 = Q'_s + q'_s\,\phi^2 \qquad \ldots \qquad \text{(ii)} \\ Q &= Q'_s\,;\ q_s = q'_s\,. \end{aligned}$$

From the form of Q_s it will at once be seen that Q_s and Q'_s, q_s and q'_s differ only in that Q_s and q_s contain the elements of the ray before refraction, and Q'_s and q'_s those after refraction,

Q_s and Q'_s, as already stated, are identical with the zero invariant previously determined in the same way.

Writing now Q_s in a more easily computed form, viz.

$$Q_s = n\frac{\mathfrak{s}}{\mathfrak{p}}\left(\frac{1}{r} - \frac{1}{\mathfrak{s}}\right) = n'\frac{\mathfrak{s}'}{\mathfrak{p}'}\left(\frac{1}{r} - \frac{1}{\mathfrak{s}'}\right), \quad \ldots \quad \text{(iii)}$$

the first step is to expand $\mathfrak{p}$, $\mathfrak{p}'$ in terms of ϕ. Reverting to the expression given in § 73 and confining ourselves to terms containing ϕ^2, we obtain the expression

$$\mathfrak{p} = \mathfrak{s}\left(1 - \frac{1}{2}\frac{r^2}{ns}Q_s\,\phi^2\right); \quad \ldots \quad \ldots \quad \text{(iv)}$$

hence

$$\frac{\mathfrak{s}}{\mathfrak{p}} = 1 + \frac{1}{2}\frac{r^2}{ns}Q_s\,\phi^2.$$

and

$$\frac{\mathfrak{s}'}{\mathfrak{p}'} = 1 + \frac{1}{2}\frac{r^2}{n's'}Q_s\,\phi^2.$$

For $\mathfrak{s}$ and $\mathfrak{s}'$ we have the equations indicated above, viz.

$$\mathfrak{s} = s + \mathsf{A}u^2; \qquad \mathfrak{s}' = s' + \mathsf{A}'u'^2,$$

in which it remains for us to express u^2 and u'^2 in terms of ϕ^2. From § 28 it follows quite generally that

$$\sin u = \frac{r}{\mathfrak{p}}\sin\phi; \qquad \sin u' = \frac{r}{\mathfrak{p}'}\sin\phi,$$

and, neglecting the third powers of the expansion of the sine, we may write

$$u = \frac{r}{\mathfrak{p}}\phi; \qquad u' = \frac{r}{\mathfrak{p}'}\phi,$$

and also, neglecting all terms of a higher order than the second,

$$u^2 = \frac{r^2}{s^2}\phi^2; \qquad u'^2 = \frac{r^2}{s'^2}\phi^2;$$

hence

$$\mathfrak{s} = s + \mathsf{A}\frac{r^2}{s^2}\phi^2 = s\left(1 + \mathsf{A}\frac{r^2}{s^3}\phi^2\right); \qquad \mathfrak{s}' = s'\left(1 + \frac{\mathsf{A}'}{s'^3}r^2\phi^2\right) \quad \text{(v)}$$

and

$$\frac{1}{\mathfrak{s}} = \frac{1}{s} - \mathsf{A}\frac{r^2}{s^4}\phi^2; \qquad \frac{1}{\mathfrak{s}'} = \frac{1}{s'} - \mathsf{A}'\frac{r^2}{s'^4}\phi^2. \quad \ldots \quad \text{(vi)}$$

Substituting the above values of $\frac{s}{p}$, $\frac{s'}{p'}$, and $\frac{1}{s}$, $\frac{1}{s'}$ in the last expression for Q_s and simplifying, we obtain :

$$\left.\begin{aligned} Q_s &= n\left(\frac{1}{r} - \frac{1}{s}\right) + \left[\frac{1}{2}\frac{Q^2{}_s}{n}\frac{r^2}{s} + n\,\mathsf{A}\,\frac{r^2}{s^4}\right]\phi^2 \\ Q_s &= n'\left(\frac{1}{r} - \frac{1}{s'}\right) + \left[\frac{1}{2}\frac{Q^2{}_s}{n'}\frac{r^2}{s'} + n'\mathsf{A}'\frac{r^2}{s'^4}\right]\phi^2 \end{aligned}\right\} \quad \ldots \quad \text{(vii)}$$

and since $q_s = q'_s$ it follows that

$$\frac{2n\,\mathsf{A}}{s^4} + \frac{Q_s^2}{ns} = \frac{2n'\,\mathsf{A}'}{s'^4} + \frac{Q_s^2}{n's'} \quad \ldots \quad \ldots \quad \text{(viii)}$$

or adopting our previous notation in which the surface index again appears,

$$\underset{v}{\Delta}\frac{2n\,\mathsf{A}}{s^4} = -\,Q^2{}_{vs}\underset{v}{\Delta}\frac{1}{ns}\,. \quad \ldots \quad \ldots \quad \text{(ix)}$$

When there is a single surface this completes the solution of the following problem :—Having given the spherical aberration (by its coefficient A_v) and the constituents of the paraxial ray, it is required to determine the spherical aberration of the emerging ray (*i.e.*, its coefficient A'_v).

For the investigation of a series of k surfaces we have established a recurrence formula by the aid of which we are able to express the coefficient of aberration, resulting after refraction at any v^{th} surface, in terms of the constituents of the paraxial rays through this surface, and the coefficient of aberration A_v before refraction. Now, from our previous investigation, it follows in general that $\mathsf{A}_v = \mathsf{A}'_{v-1}$. Clearly, we are thus enabled to express the coefficient of aberration A_k', which results after refraction at the last surface, in terms of the constituents of the paraxial ray and the coefficient of aberration A_1 which prevails before refraction at the first surface. In fact, by carrying out the summation and noting that $n'_{v-1} = n_v$, $\mathsf{A}'_{v-1} = \mathsf{A}_v$, and, by § 46 and § 83, $\frac{s'_{v-1}}{s_v} = \frac{h_{v-1}}{h_v}$ we obtain the formula

$$\mathsf{A}'_k = -\frac{s_k'^4}{2n'_k}\sum_{v=1}^{k}\left(\frac{h_v}{h_k}\right)^4 Q^2{}_{v,s}\underset{v}{\Delta}\frac{1}{ns} + \left(\frac{h_1}{h_k}\right)^4\frac{n_1}{n'_k}\left(\frac{s'_k}{s_1}\right)^4\mathsf{A}_1 \ldots \quad \text{(x)}$$

The term $\frac{h_v}{h_k}$ in the series may be readily computed with the aid of the relation

$$h_v \,/\, h_{v+1} = s'_v \,/\, s_{v+1}\,,$$

and accordingly we obtain

$$\frac{h_v}{h_k} = \frac{s'_v}{s_{v+1}} \quad \frac{s_v'{}_{+1}}{s_{v+2}} \cdots\cdots \frac{s'_{k-1}}{s_k}\,. \qquad \ldots \qquad \text{(xi)}$$

Generally, the last term of this formula vanishes, since, as a rule, the pencil of rays at incidence at the first surface is free from spherical aberration, so that $\mathsf{A}_1 = 0$.

121. Aberration at a System of Plane Surfaces.—In the special case of a system of planes this formula can be simplified. The general expression

$$Q_{vs} = n_v \left(\frac{1}{r_v} - \frac{1}{s_v}\right) = n'_v \left(\frac{1}{r_v} - \frac{1}{s_v}\right)$$

then assumes the special form

$$Q_{vs} = \frac{n_v}{s_v} = \frac{n'_v}{s'_v}\,, \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

so that if we place $\left(\frac{h_1}{h_k}\right)^4$ in front of the summation, which will be done as a rule, the general term of the sum becomes

$$\left(\frac{h_v}{h_1}\right)^4 Q^2_{vs} \underset{v}{\Delta} \frac{1}{ns} = \left(\frac{h_v}{h_1}\right)^4 \left(\frac{n'_v}{s'^3_v} - \frac{n_v}{s_v^3}\right).$$

Combining the second part of this term with the first part of the preceding term and noting that

$$n_v = n'_{v-1}\,,$$

we obtain the equation

$$\left(\frac{h_{v-1}}{h_1}\right)^4 \frac{n'_{v-1}}{s'^3_{v-1}} - \left(\frac{h_v}{h_1}\right)^4 \frac{n_v}{s_v^3} = n'_{v-1}\left(\frac{h_{v-1}}{h_1}\right)^4 \left[\frac{1}{s'^3_{v-1}} - \left(\frac{h_v}{h_{v-1}}\right)^4 \frac{1}{s_v^3}\right],$$

and hence, with the aid of the relations,

$$\frac{h_v}{h_{v-1}} = \frac{s_v}{s'_{v-1}} \quad \text{and} \quad s_v = s'_{v-1} - d_{v-1}\,,$$

$$n'_{v-1}\left(\frac{h_{v-1}}{h_1}\right)^4\left(\frac{1}{s'^3_{v-1}} - \frac{s'_{v-1} - d_{v-1}}{s'^4_{v-1}}\right) = n'_{v-1}\left(\frac{h_{v-1}}{h_1}\right)^4 \frac{d_{v-1}}{s'^4_{v-1}}\,.$$

From the constancy of the invariant throughout the entire system, viz:

$$I_v = n_v' \sin i'_v = n_v' \sin u_v' = n'_v \frac{h_v}{s_v'}$$

it follows that

$$\frac{n'_{v-1}\, h_{v-1}}{s'_{v-1}} = \frac{n_1\, h_1}{s_1},$$

so that finally

$$\left(\frac{h_{v-1}}{h_1}\right)^4 \cdot \frac{n'_{v-1}}{s'^3_{v-1}} - \left(\frac{h_v}{h_1}\right)^4 \cdot \frac{n_v}{s_v^3} = \frac{n_1^4}{s_1^4} \cdot \frac{d_{v-1}}{n'^3_{v-1}}.$$

For the entire sum we obtain the following expression,

$$\sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^4 Q^2_{vs} \underset{v}{\Delta} \frac{1}{ns} = \left(\frac{h_k}{h_1}\right)^4 \frac{n'_k}{s'^3_k} - \frac{n_1}{s_1^3} + \frac{n_1^4}{s_1^4} \sum_{v=1}^{k-1} \frac{d_v}{n'^3_v}$$

$$= \frac{n_1^4}{s_1^4}\left(\frac{s'_k}{n'^3_k} - \frac{s_1}{n_1^3} + \sum_{v=1}^{k-1} \frac{d_v}{n'^3_v}\right),$$

which may be simplified still further by the introduction of the relation

$$\frac{s'_k}{n'_k} = \frac{s_1}{n_1} - \sum_{v=1}^{k-1} \frac{d_v}{n'_v}, \qquad \ldots \qquad \ldots \qquad \text{(ii)}$$

as follows:

$$\sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^4 Q^2_{vs} \underset{v}{\Delta} \frac{1}{ns} = \frac{n_1^3}{s_1^3}\left(\frac{1}{n'^2_k} - \frac{1}{n_1^2}\right) + \frac{n_1^4}{s_1^4} \sum_{v=1}^{k-1} \frac{d_v}{n'_v}\left(\frac{1}{n'^2_v} - \frac{1}{n'^2_k}\right). \text{(iii)}$$

The first term of this expression vanishes when the system is situated in one and the same medium.

In this case, when $n_1 = n_k' = 1$, the expression becomes

$$\sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^4 Q^2_{vs} \underset{v}{\Delta} \frac{1}{ns} = \frac{1}{s_1^4} \sum_{v=1}^{k-1} \frac{d_v}{n'_v}\left(\frac{1}{n'^2_v} - 1\right), \qquad \ldots \qquad \text{(iv)}$$

which is essentially a negative quantity since every term which does not vanish is negative on account of $n'_v \geqq 1$.

It will thus be seen that when an image of an object-point is formed by a system of planes surrounded by air, A'_k assumes an essentially positive value. The longitudinal aberration produced in the case of small angles u_1 is accordingly always of the nature of an over-correction, which conforms to the result obtained in the case of finite angles u_1 (§ 118).

B.—The Circle of Confusion due to Spherical Aberration.

122. The **tangential aberration** l'_k in the Gauss image-plane, due to the spherical aberration of a ray having an angular aperture u'_k pertaining to the image, may be found by reference to Fig. 55, if we consider only the first powers of the angles, by multiplying the amount of the longitudinal aberration $\mathbf{s}'_k - s'_k$ by the angular aperture, thus

$$l'_k = (\mathbf{s}'_k - s'_k)\,u'_k = \mathsf{A}'_k\,u'^{3}_k \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

This tangential line of confusion is also called the **radius of the circle of confusion** due to the spherical aberration of a point on the axis, and in what follows we shall use this expression.

The diameter of the circle of confusion has, of course, double the magnitude of the radius.

If now for A'_k we substitute its value as found above, confining ourselves to the case in which the pencil on entering the system is free from initial aberration, we shall find that

$$l'_k = -\frac{s'^{4}_k}{2n'_k}\,u'^{3}_k \sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^4 Q_{vs}^{2}\, \underset{v}{\Delta} \frac{1}{ns}\,.$$

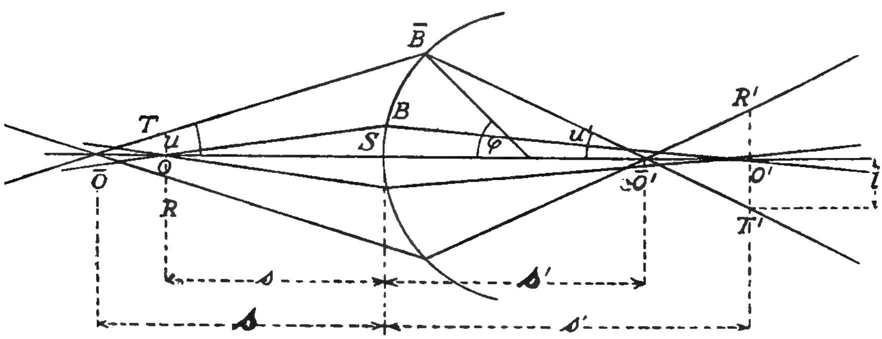

FIG. 55.

$SO = s$; $S\bar{O} = \mathbf{s}$; $SO' = s'$; $S\bar{O}' = \mathbf{s}'$; $O'T' = l'$.

Circle of confusion due to spherical aberration of an axial point.

By the introduction of the identity

$$\frac{h_v}{h_k} = \frac{h_1}{h_k}\,\frac{h}{h_1}\,,$$

it follows that

$$l'_k = -\frac{s'^{4}_k}{2\,n'_k}\,u'^{3}_k \left(\frac{h_1}{h_k}\right)^4 \sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^4 Q_{vs}^{2}\, \underset{v}{\Delta} \frac{1}{ns}\,. \quad \ldots \quad \text{(ii)}$$

The condition therefore that a system of k surfaces may be spherically corrected is that the summation should vanish, so that

$$\sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^4 Q^2_{vs} \underset{v}{\Delta} \frac{1}{ns} = 0 . \quad \ldots \quad \ldots \quad \text{(iii)}$$

The great advantage of this formula is that it furnishes a ready means of calculation, in that it contains a distinct expression for the aberration due to each component surface.

In the investigation of the spherical aberration of a system it is not so much the absolute magnitude of the circle of confusion of the image which is of importance, as the influence which the aberration exercises upon the distinctness of the details of the image of an object.

We may obtain a concrete measure of the degree of indistinctness, due to the aberration, by ascertaining the magnitude of the linear portion of the object whose image is exactly equal to the radius of the circle of confusion, due to the aberrational effect of the given k refracting surfaces.

According to the Smith-Helmholtz formula for axial rays §§ 84 and 111,

$$n' \, u' \, y'_s = n \, u \, y_s ,$$

This equation holds quite generally for any number of refractions if n, u, y_s refer to the pencil before the first refraction, and $n' \, u' \, y'_s$ to the pencil after the last refraction. When applied to the lateral aberration, or to the radius of the circle of confusion, the equation assumes the form—

$$n'_k \, u'_k \, l'_k = n_1 \, u_1 \, l^{(k)} , \quad \ldots \quad \ldots \quad \text{(iv)}$$

where the index (k) indicates the number of surfaces in the system.

We obtain accordingly the radius $l^{(k)}$ of the circle of confusion projected into the object by putting generally, $h_v = s_v u_v = s'_v u'_v$, as we are at liberty to do within the limits imposed by our present investigation, so that

$$l^{(k)} = \frac{n'_k \, u'_k \, l'_k}{n_1 \, u_1} = - \frac{s_1^4}{2 \, n_1^4} (n_1 \, u_1)^3 \sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^4 Q^2_{vs} \underset{v}{\Delta} \frac{1}{ns} \ldots \quad \text{(v)}$$

This expression may be given a different form by introducing the co-ordinates of the aperture by means of the equation

$$u_1 = \frac{m_1}{s_1 - x_1} .$$

The resulting expression

$$\frac{n_1 \, l^{(k)}}{s_1} = \frac{m_1^3 \, s_1^3}{2 \, (x_1 - s_1)^3} \sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^4 Q^2_{vs} \underset{v}{\Delta} \frac{1}{ns} \quad \ldots \quad \text{(vi)}$$

is one which may be easily identified with that which will be derived later in a different manner.

We shall now introduce a quantity which will be dealt with later and which is of the utmost importance in the theory of optical instruments. This quantity $n \sin u$ is called by Abbe the **numerical aperture**, viz., the product of the refractive index into the sine of the appropriate angular aperture. It will be clear that in the case of very small angular apertures u between a ray and the axis the numerical aperture is given by the expression

$$[\mathrm{NA}] = nu \qquad \ldots \qquad \ldots \qquad \ldots \qquad \text{(vii)}$$

Substituting this value in the formula for the radius of the circle of confusion we have

$$l^{(k)} = -\frac{s_1^4}{2\,n_1^4}[\mathrm{NA}]^3 \sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^4 Q^2_{vs}\,\Delta_v \frac{1}{ns}\,. \qquad \ldots \qquad \text{(viii)}$$

From this form of the expression for $l^{(k)}$ it follows, at once, that the influence of the spherical aberration on the distinctness of the image increases as the third power of the numerical aperture of the incident pencil.

In the case of an infinitely distant object it is evident that the deterioration of the image cannot be stated in terms of a linear quantity. It must be expressed in angular measure, and accordingly the angular equivalent of the radius of the circle of confusion is

$$\lambda^{(k)} = \frac{l^{(k)}}{s_1}\,,$$

and, further, if we consider that when $s_1 = \infty$, $u_1\, s_1 = h_1$ and that in this case $n_1 = 1$, the expression becomes

$$\lambda^{(k)} = -\frac{h_1^3}{2}\sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^4 Q^2_{vs}\,\Delta_v \frac{1}{ns}. \qquad \ldots \qquad \text{(ix)}$$

In a system arranged for telescopic vision the influence of the spherical aberration within the limits of the first degree of approximation is accordingly proportional to the cube of the linear aperture of the system.

The sum on the right side of the expression for $\lambda^{(k)}$ is inversely proportional to the cube of the focal length of the system, provided that the system is not in itself a telescopic system, whilst the statement is entirely applicable to parts thereof, such as the objective of a telescope. If by K we denote the value of this sum corresponding to the focal length $f = 1$, the expression assumes the form

$$\lambda^{(k)} = -\frac{1}{2}\left(\frac{h_1}{f}\right)^3 K = -\frac{1}{16}\frac{K}{k^3} \qquad \ldots \qquad \ldots \qquad \text{(x)}$$

Denoting by $\frac{2h_1}{f}=\frac{1}{k}$ the **relative aperture** of a system for the observation of an object at a great distance, this equation may be interpreted as follows : The confusion in the image of an infinitely distant object due to the first term of the spherical aberration is proportional to the third power of the relative aperture of the system. The factor K depends upon the composition of the system, viz., upon the values of the curvatures, refractive indices, and distances between the component surfaces. Now, in any instrument of a given type K, and relative aperture $\frac{1}{k}=\frac{2\,h_1}{f}$, it follows that the angular value $\lambda^{(k)}$ is independent of the magnitude of the focal length itself.

C. Spherical Aberration in Simple Special Cases.

123. The Spherical Aberration of a Single Spherical Surface. — The expression obtained in the preceding article becomes in this case

$$l^{(k)}=-\frac{s^4}{2n^4}\,[\mathrm{NA}]^3 n^2\left(\frac{1}{r}-\frac{1}{s}\right)^2\left(\frac{1}{n's'}-\frac{1}{ns}\right). \quad \ldots \qquad \text{(i)}$$

There are three cases only in which it can vanish, the spherical aberration being then independent of the aperture of the pencil. These cases arise

(1) when $s=0$, *i.e.*, when the object-point under consideration coincides with its image at the vertex of the surface ;

(2) when $s=r$, *i.e.*, when the object-point coincides with its image at the centre of the surface ;

(3) when $n's'=ns$. This relation is by § 74 identical with

$$s=r+\frac{n'}{n}r\,; \qquad s'=r+\frac{n}{n'}r\,; \qquad \ldots \qquad \text{(ii)}$$

that is to say the object-point and its image coincide with one of the aplanatic points of the surface (*see* § 23).

In all other cases the coefficient in the expression of the spherical aberration, viz.

$$A=n^2\left(\frac{1}{r}-\frac{1}{s}\right)^2\left(\frac{1}{n's'}-\frac{1}{ns}\right) \qquad \ldots \qquad \text{(iii)}$$

has a finite value other than zero, whose sign is governed by that of the last factor.

From $$\frac{1}{n's'}-\frac{1}{ns}=\frac{n'-n}{n'^2}\left(\frac{1}{r}-\frac{n'+n}{n}\,\frac{1}{s}\right) \qquad \ldots \qquad \text{(iv)}$$

which follows from the transformation of

$$\frac{n'}{s'} - \frac{n}{s} = \frac{n'-n}{r},$$

it will be seen that the sign depends in general upon the values of s and that only when $s = \infty$ is it determined by the quantity $\frac{n'-n}{r}$.

When $\frac{n'-n}{r} > 0$, and the surface accordingly has a converging effect, $A > 0$, and therefore $A' < 0$, that is to say, there is under-correction, whereas in the case of a diverging surface in which $\frac{n'-n}{r} < 0$, the image of a distant object is always subject to over-correction.

124. Spherical Aberration of a Single Thin Lens.— The next case for investigation is that of two adjacent surfaces, the first and last medium being identical. We shall assume the latter to be air and shall accordingly put $n_1 = n'_2 = 1$ and $n'_1 = n_2 = n$. As in the last case, we have the expression

$$l^{(k)} = -\frac{s_1^4}{2}[\mathrm{NA}]^3 A. \quad \ldots \quad \ldots \quad \text{(i)}$$

Since the case in which the circle of confusion of the spherical aberration vanishes when the object-point and with it the image are moved into the position of the vertex of the lens offers no further interest, we shall only consider the coefficient

$$A = \left(\frac{1}{r_1} - \frac{1}{s_1}\right)^2 \left(\frac{1}{ns_1'} - \frac{1}{s_1}\right) + \left(\frac{1}{r_2} - \frac{1}{s_2'}\right)^2 \left(\frac{1}{s_2'} - \frac{1}{ns_2}\right),$$

noting that in the case of an infinitely thin lens $h_2 = h_1$.

The redundant variables contained in the expression for A may be eliminated by means of the following equations, which readily follow from § 88 (9), (10) and § 78 (ii)

$$\frac{1}{r_1} - \frac{1}{r_2} = \frac{1}{n-1}\frac{1}{f}; \quad \frac{1}{s_2'} = \frac{1}{s_1} + \frac{1}{f}; \quad \frac{1}{ns_1'} = \frac{1}{n^2 s_1} + \frac{n-1}{n^2} \cdot \frac{1}{r_1}$$

and by introducing at the same time the new symbols

$$\frac{1}{r_1} = \rho; \quad \frac{1}{s_1} = \sigma; \quad \frac{1}{f} = \phi.$$

It should be observed that by reason of the first of these substitutions the curvature ρ of the front surface may be given any

value we please, without affecting the power of the lens, since the curvature of the back surface changes accordingly. In what follows this conjoint alteration of the curvature of both surfaces without change of power will be termed **Coflexure.***

The coefficient A will accordingly assume the form

$$A = \left(\frac{n}{n-1}\right)^2 \phi^3 + \frac{3n+1}{n-1}\sigma\phi^2 + \frac{3n+2}{n}\sigma^2\phi - \frac{2n+1}{n-1}\rho\phi^2$$
$$-\frac{4(n+1)}{n}\rho\sigma\phi + \frac{n+2}{n}\rho^2\phi \quad \ldots \quad \text{(ii)}$$

If we regard this expression as a function of the principal variable ρ it may be reduced to the form

$$A = A_0 - A_1\rho + A_2\rho^2.$$

This function becomes a minimum with respect to ρ when

$$\frac{\partial A}{\partial \rho} = -A_1 + 2A_2\rho = 0;$$

i.e., when

$$\rho_{min} = \frac{A_1}{2A_2} = \frac{(2n+1)\,n}{2(n-1)(n+2)}\phi + \frac{2(n+1)}{n+2}\sigma \ldots \quad \text{(iii)}$$

and hence

$$A_{min} = A_0 - \frac{A_1^2}{4A_2} = \frac{(4n-1)\,n}{4(n-1)^2(n+2)}\phi^3 - \frac{n}{n+2}\sigma\phi(\sigma+\phi). \quad \text{(iv)}$$

The minimum value of the coefficient of aberration so found enables us to express the general coefficient in the following form:

$$A = A_{min} + \frac{A^2_1}{4A_2^2}A_2 - A_1\rho + A_2\rho^2 = A_{min} + A_2\rho^2_{min} - 2\rho\rho_{min} + A_2\rho^2,$$

so that

$$\frac{A}{\phi^3} = \frac{A_{min}}{\phi^3} + \frac{n+2}{n}\mathrm{P}^2, \text{ where } \mathrm{P} = \frac{\rho}{\phi} - \frac{\rho_{min}}{\phi} \quad . \text{ (v)}$$

It will be seen that the minimum value $\dfrac{A_{min}}{\phi^3}$, as well as $\dfrac{\rho_{min}}{\phi}$,

* The process of varying the radii of curvature of the two lens surfaces without change of power is known in German as "Durchbiegung." The term "coflexure," as the equivalent of "Durchbiegung," has been suggested by Mr. Dennis Taylor. Trans.

depends solely upon $\frac{\sigma}{\phi}$. Also, it follows from the equation that the spherical aberration can only be removed in connection with negative values of A; on the other hand, it can always be done for two values P_1, P_2 associated with coflexure.

Replacing A_{min} by $-a_{min}$ to emphasize its negative value, the two roots $\frac{\sigma}{\phi}$ of the quadratic equation containing A_{min}, viz.

$$\frac{\sigma}{\phi} = -\frac{1}{2} \pm \sqrt{\frac{n+2}{n}\left(\frac{a_{min}}{\phi^3} + \frac{n^2}{4(n-1)^2}\right)} \qquad \text{(vi)}$$

will obviously have corresponding to them two roots $\frac{\rho_{min}}{\phi}$, viz.:

$$\frac{\rho_{min}}{\phi} = \frac{1}{2(n-1)} \pm \frac{2(n+1)}{n+2}\sqrt{\frac{n+2}{n}\left(\frac{a_{min}}{\phi^3} + \frac{n^2}{4(n-1)^2}\right)}. \qquad \text{(vii)}$$

From these formulæ it follows that

$$\sigma_I + \sigma = -\phi; \quad \rho_{I,min} + \rho_{II,min} = \frac{1}{n-1}\phi.$$

Since, conjointly with these, we have the general relations $\frac{1}{s'} - \frac{1}{s} = \frac{1}{f}$ by § 88 (10), and $\frac{1}{r_1} - \frac{1}{r_2} = \frac{1}{(n-1)f}$ by § 88 (9), *i.e.*, in our case

$$\sigma'_I - \sigma_I = \phi; \quad \rho_{I,min} - \rho_2 = \frac{1}{n-1}\phi,$$

it follows that

$$\sigma_{II} = -\sigma'_I; \quad \rho_{II,min} = -\rho_2;$$

in other words, the second case in which A_{min} assumes a certain prescribed value $-a_{min}$ becomes identical with the first case if we reverse the lens and at the same time interchange the object and the image. Excluding this case from consideration as being self-evident, it may be stated that in every thin lens there is only one position of the object (σ) and a corresponding coflexure (ρ_{min}) which will cause A_{min} to assume a given value $-a_{min}$. In order that σ may have a real value, a_{min} may be given any positive value whatever, but its negative values are restricted to the condition that $-a_{min} \leqq \frac{n^2\phi^3}{4(n-1)^2}$. Expressed in another way, this conclusion may be stated thus:

By an appropriate choice of the values of σ and ρ the minimum value of A in the expression for the spherical aberration may be given any negative value when the focal length of the lens in question is positive, and, conversely, it may be given any positive

value when the focal length is negative ; but when A_{min} has the same sign as the focal length its limiting value is $\frac{n^2\phi^3}{4\,(n-1)^2}$, which it reaches when $\sigma = -\frac{\phi}{2}$ and $\rho_{min} = \frac{\phi}{2\,(n-1)}$.

We can put the result of our investigation more clearly if we express the value $\frac{A}{\phi^3}$ of the coefficient of the spherical aberration as a function of the two variables $\frac{\sigma}{\phi}$ and $\frac{\rho}{\phi}$. $\frac{A}{\phi^3}$ will then figure as the parameter to the equation

$$a_{1\,1}\left(\frac{\rho}{\phi}\right)^2 + 2a_{1\,2}\left(\frac{\rho}{\phi}\right)\left(\frac{\sigma}{\phi}\right) + a_{22}\left(\frac{\sigma}{\phi}\right)^2 + 2a_{13}\left(\frac{\rho}{\phi}\right)$$
$$+ 2a_{2\,3}\left(\frac{\sigma}{\phi}\right) + \left(\frac{n}{n-1}\right)^2 - \frac{A}{\phi^3} = 0\,,$$

where the values $a_{\nu\,k}$ can be obtained directly from the expression for A, as given above.

It may easily be shown by co-ordinate geometry that the curve represented by this equation is for all values of A a hyperbola with the same pair of asymptotes, which in the special case when $\frac{A}{\phi^3} = \frac{n^2}{4\,(n-1)^2}$ degenerates into a pair of straight lines, viz., this pair of asymptotes.

The parallel displacement of the system of co-ordinates is

$$\frac{\rho}{\phi} = \frac{\rho'}{\phi} + \frac{1}{2\,(n-1)}\,; \qquad \frac{\sigma}{\phi} = \frac{\sigma'}{\phi} - \frac{1}{2}\,,$$

and the angle α through which the axes are turned is

$$\tan 2\alpha = \frac{2\,(n+1)}{n}\,,$$

whilst the entire equation, referred to two diameters at right angles, assumes the form

$$\lambda_1\, r_s^2 - \lambda_2\, s_s^2 = \frac{A}{\phi^3} - \frac{n^2}{4\,(n-1)^2}\,;$$

$$\lambda_{1\,2} = \frac{2\,n + 2 \pm \sqrt{4\,(n+1)^2 + n^2}}{n}\,.$$

Considering $\frac{A}{\phi^3}$ as the third co-ordinate in solid space, the equation is that of a hyperbolic paraboloid.

The general expression of the coefficient of aberration of a lens as given above, may also be arranged as a function of σ, thus

$$A = \mathsf{A}_0 + \mathsf{A}_1\sigma + \mathsf{A}_2\sigma^2, \quad \ldots \quad \ldots \quad \text{(viii)}$$

which signifies that a lens which is spherically corrected for a given distance of the object will, in general, remain so for a second value of $\sigma = \frac{1}{s}$. It is easy to see that the presence of any number of thin lenses does not add to this number of object-distances for which the spherical aberration can be made to vanish.

If we indicate the numerical order of the lenses, the coefficient of the total aberration of a system of k thin lenses arranged close together, will take the form

$$A = \sum_{v=1}^{k} A^{(v)} = \sum_{v=1}^{k} \mathsf{A}_{0v} + \sum_{v=1}^{k} \mathsf{A}_{1v}\ \sigma_v + \sum_{v=1}^{k} \mathsf{A}_{2v}\,\sigma_v^2.$$

Since in the case of thin lenses we may always put

$$\sigma_v = \sigma_1 + \sum_{\lambda=1}^{v-1} \phi_\lambda$$

we have finally

$$A = \sum_{v=1}^{k} A^{(v)} = \sum_{v=1}^{k} \left(\mathsf{A}_{0v} + \mathsf{A}_{1v} \sum_{\lambda=1}^{v-1} \phi_\lambda + \mathsf{A}_{2v} \left[\sum_{\lambda=1}^{v-1} \phi_\lambda\right]^2\right)$$
$$+ \sigma_1 \sum_{v=1}^{k} \left(\mathsf{A}_{1v} + 2\,\mathsf{A}_{2v} \sum_{\lambda=1}^{v-1} \phi_\lambda\right) + \sigma_1^2 \sum_{v=1}^{k} \mathsf{A}_{2v},$$

from which it will be seen that in general, unless every coefficient vanishes separately, the spherical aberration in a system composed of any number of thin lenses can vanish only for two or one or finally for no real object-point at all.

We shall now discuss two quite special cases in which the spherical aberration can be removed in a system of two or more lenses.

125. Spherically Corrected Systems consisting of a Positive and a Negative Lens.—In the case of two adjacent thin lenses, if we introduce the minimum values indicated in § 124 (v) the coefficient of spherical aberration assumes this form

$$A = A^{(1)} + A^{(2)} = A^{(1)}_{min} + \frac{n_1 + 2}{n_2}\,\phi_1{}^3\,\mathrm{P}^2{}_1 + A^{(2)}_{min} + \frac{n_2 + 2}{n_2}\,\phi_2{}^3\,\mathrm{P}^2{}_3,$$

where the quantities $A^{(1)}_{min}$ and $A^{(2)}_{min}$ can be readily calculated from the formula § 124 (iv), if we put $\sigma_2 = \sigma_1 + \phi_1$.

The simplest case in which A can be made to vanish is that where ϕ_1 and ϕ_2 have opposite signs. Let the focal lengths of the lenses have any absolute values, but let ϕ_1 be positive and ϕ_2 negative. The value of $A^{(1)}_{min} + A^{(2)}_{min}$ will then provide a measure of the permissible amount of coflexure in one of the two lenses. For, when $A^{(1)}_{min} + A^{(2)}_{min}$ is equal to the intrinsically positive value $a^2 \frac{n_2 + 2}{n_2} \left| \phi_2 \right|$ then, in order that A may vanish for all real values of P_1, we must have $P_3 \geqq a$. But when $A^{(1)}_{min} + A^{(2)}_{min}$ is equal to the intrinsically negative value $-\beta^2 \frac{n_1 + 2}{n_1} \left| \phi_1 \right|$ the condition that A may vanish for real values of P_3 will be $P_1 \geqq \beta$.

When these limiting conditions are satisfied, any value of P_3 will have corresponding to it two real values $P_{1\,1}$ and $P_{1\,2}$ and, similarly, two values $P_{3\,1}$, $P_{3\,2}$ will correspond to P_1.

We are thus able, while retaining a free choice of the focal lengths of the two lenses, to satisfy a further condition respecting the radii of curvature. A condition which enters freely into the scheme is that generally known as **Herschel's condition**,* which requires the value of A to vanish, not only for the object-distance σ, but also for the adjacent object-distance. Accordingly

$$A^{(1)} + A^{(2)} = 0 \text{ and } \frac{dA^{(1)}}{d\sigma_1} + \frac{dA^{(2)}}{d\sigma_2} \cdot \frac{d\sigma_2}{d\sigma_1} = 0.$$

In view of the relation $\sigma_2 = \sigma_1 + \phi_1$ the latter equation becomes

$$\frac{dA^{(1)}}{d\sigma_1} + \frac{dA^{(2)}}{d\sigma_2} = 0,$$

and, again noting that $\sigma_2 = \sigma_1 + \phi_1$, the differentiation of equation § 124 (ii) with respect to σ_1 gives us the following result

$$2\sigma_1 \left(\frac{3n_1 + 2}{n_1} \phi_1 + \frac{3n_2 + 2}{n_2} \phi_2 \right) = 4 \left(\frac{n_1 + 1}{n_1} \rho_1\phi_1 + \frac{n_2 + 1}{n_2} \rho_3\phi_2 \right)$$

$$- \frac{3\,n_1 + 1}{n_1 - 1} \phi_1{}^2 - \frac{3\,n_2 + 1}{n_2 - 1} \phi_2{}^2 - 2\,\frac{3\,n_2 + 2}{n_2} \phi_1 \phi_2 .$$

We thus obtain a linear expression connecting ρ_1 and ρ_3, which, in conjunction with the quadratic equation $A^{(1)} + A^{(2)} = 0$, furnishes two roots.

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* This condition will be discussed more fully in § 165.

If we impose the further condition that the value of A is to be entirely independent of the distance of the object-point, it follows by analogy that

$$\frac{d^2A^{(1)}}{d\sigma_1^2} + \frac{d^2A^{(2)}}{d\sigma_2^2} = 0.$$

Differentiating the last expression, we thus obtain

$$\frac{3\,n_1 + 2}{n_1}\,\phi_1 + \frac{3\,n_2 + 2}{n_2}\,\phi_2 = 0.$$

When this condition is satisfied in conjunction with the two preceding conditions, the calculation furnishes a system consisting of a converging lens and a diverging lens which are spherically corrected for objects at any distance. The refractive indices n of the available glasses are such that extremely strong curvatures would be involved.

126. Spherically Corrected Systems consisting of Lenses having Focal Lengths of the same Sign.—It is interesting to investigate whether it is possible to satisfy the condition $A = 0$ by a combination of thin lenses all having focal lengths of the same sign. In this case the smallest value which can be realised occurs when $\sum_{v=1}^{k} A^{(v)}$ is a minimum.

Proceeding from a combination of two lenses we shall first determine the amount D by which the minimum value of the spherical aberration can be diminished by replacing a lens of focal length ϕ by two others of focal lengths ϕ_1 and ϕ_2 of identical sign and optical material, and having jointly the same power as ϕ, so that $\phi = \phi_1 + \phi_2$.

Then, by § 124 (iv),

$$D = A_{min} - \left(A^{(1)}_{min} + A^{(2)}_{min}\right) = \frac{(4\,n - 1)\,n}{4\,(n - 1)^2\,(n + 2)}\,(\phi_1 + \phi_2)^3$$

$$-\frac{n}{n + 2}\,\sigma\,(\phi_1 + \phi_2)\,(\sigma + \phi_1 + \phi_2) - \frac{(4\,n - 1)\,n}{4\,(n - 1)^2\,(n + 2)}\,(\phi_1^3 + \phi_2^3)$$

$$+\frac{n}{n + 2}\,\sigma\phi_1\,(\sigma + \phi_1) + \frac{n}{n + 2}\,(\sigma + \phi_1)\,\phi_2\,(\sigma + \phi_1 + \phi_2)$$

$$= \frac{(2\,n + 1)^2\,n}{4\,(n - 1)^2\,(n + 2)}\,\phi\,\phi_1\,\phi_2.$$

By giving ϕ_1 and ϕ_2 different signs, D might be made of any magnitude we please and of the opposite sign to ϕ. This case is, however, excluded by our present assumption. It will now be easily seen that the value of D reaches its maximum at the same time that $A^{(1)}_{min} + A^{(2)}_{min}$ becomes a minimum when $\phi_1 = \phi_2 = \frac{\phi}{2}$. If we compare this minimum value of

$$A^{(1)}_{min} + A^{(2)}_{min} = \frac{(4\,n - 1)\,n}{4\,(n - 1)^2\,(n + 2)}\frac{\phi^3}{4} - \frac{n}{n + 2}\,\phi\left\{\sigma\,(\sigma + \phi) + \frac{\phi^2}{4}\right\}$$

with

$$A = \frac{(4n - 1)\,n}{4\,(n - 1)^2\,(n + 2)}\,\phi^3 - \frac{n}{n + 2}\,\phi\,(\sigma + \phi),$$

it will be seen that the positive part has diminished considerably, whilst the negative part has even increased. This justifies the conclusion that by successively augmenting the number, and proportionately diminishing the power of the component lenses, spherical correction may be achieved even for $\sigma = 0$, where otherwise there is invariably a coefficient of aberration having the same sign as ϕ. The accuracy of this conclusion is demonstrated in the simplest manner by the following numerical examples :

Minimum aberration $\sum\limits_{v=1}^{k} A^{(v)}$:

for one lens (ϕ)

$$\phi^3\,\frac{(4\,n - 1)\,n}{4\,(n - 1)^2\,(n + 2)};$$

two lenses (each $\frac{\phi}{2}$)

$$\frac{\phi^3}{4}\cdot\frac{n\,(12\,n - 4\,n^2 - 5)}{4\,(n - 1)^2\,(n + 2)};$$

three lenses (each $\frac{\phi}{3}$)

$$\frac{\phi^3}{27}\cdot\frac{n\,(76\,n - 32\,n^2 - 35)}{4\,(n - 1)^2\,(n + 2)};$$

four lenses (each $\frac{\phi}{4}$)

$$\frac{\phi^3}{16}\cdot\frac{n\,(44\,n - 20\,n^2 - 21)}{4\,(n - 1)^2\,(n + 2)}$$

or expressed as multiples of A,

two lenses

$$\frac{(2\,n - 5)\,(3 - 6\,n)}{2^2\,(4\,n - 1)}\,\frac{A}{3},$$

three lenses

$$\frac{(4\,n - 7)\,(5 - 8\,n)}{3^2\,(4\,n - 1)}\,\frac{A}{3},$$

four lenses

$$\frac{(6\,n - 9)\,(7 - 10\,n)}{4^2\,(4\,n - 1)}\,\frac{A}{3};$$

i.e.

one lens	two lenses	three lenses	four lenses
for $n = 1{\cdot}5$			
$2{\cdot}143\,\phi^3$	$0{\cdot}429\,\phi^3$	$0{\cdot}111\,\phi^3$	0
for $n = 1{\cdot}75$			
$1{\cdot}245\,\phi^3$	$0{\cdot}194\,\phi^3$	0	negative ϕ^3
for $n = 2{\cdot}5$			
$0{\cdot}555\,\phi^3$	0	negative ϕ^3	negative ϕ^3

D. The Primary Zonal Term of the Spherical Aberration.

127. If in the expansion of the axial intercept enunciated in § 119, viz.:

$$\mathbf{s}_k = s_k + \mathsf{A}_k\, u^2 + \mathsf{B}_k\, u^4 + \ldots\ldots,$$

we include in our consideration the third term, which contains the fourth power of the independent variable u, and if we suppress the surface indices the series assumes the form

$$\mathbf{s} = s + \mathsf{A}\, u^2 + \mathsf{B}\, u^4.$$

128. The General Character of the Aberrations. A and B are linear quantities whose signs and amounts govern the character of the longitudinal aberration. To ensure spherical correction over a given angular aperture $u = \bar{u}$, the condition

$$\bar{u}^2\, (\mathsf{A} + \bar{u}^2) = 0,$$

must clearly be satisfied. It will be seen that A and B must have different signs or must vanish separately if, for an angular aperture other than zero, the resulting axial intercept $s = \mathbf{s}_{\bar{u}}$ is to be identical with that corresponding to an axial ray. When this is the case the system is said to be **spherically corrected for an angular aperture** $u = \bar{u}$. From the form of the expression for $\mathbf{s}$ it is apparent that spherical correction is obtained for all angles $u < \bar{u}$ so long only as $\mathsf{A} = 0 = \mathsf{B}$, *i.e.*, when both coefficients vanish separately. Systems of this kind are said to be **free from spherical aberration for a particular angular aperture** $u = \bar{u}$. On the other hand, if A and B differ from zero, and if for the moment v be substituted for u^2, then the differentials

$$\frac{d\mathbf{s}}{dv} = \mathsf{A} + 2\,\mathsf{B}v = \mathsf{A} + 2\,\mathsf{B}u^2\,; \qquad \frac{d^2\mathbf{s}}{dv^2} = 2\,\mathsf{B}$$

enable us to trace the function $\mathbf{s}$. The value $\bar{\bar{u}}^2 = -\dfrac{\mathsf{A}}{2\,\mathsf{B}}$ has clearly corresponding to it a stationary value of the function $\mathbf{s}$, and from the second differential quotient it will be seen that this is a minimum when $\mathsf{B} > 0$ and a maximum when $\mathsf{B} < 0$. When $u = \bar{\bar{u}}$, $\mathbf{s}_{\bar{\bar{u}}} = s - \dfrac{\mathsf{A}^2}{4\mathsf{B}}$ which confirms the above result.

The nature of the spherical aberration, in so far as it is represented by an expansion of the above form with non-vanishing values of A and B, may accordingly be defined by the following general statement :—Correction of the spherical aberration can only be accomplished if the two coefficients of aberration are finite and

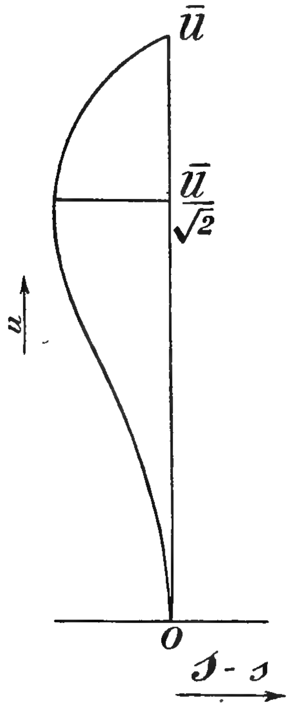

FIG. 56.

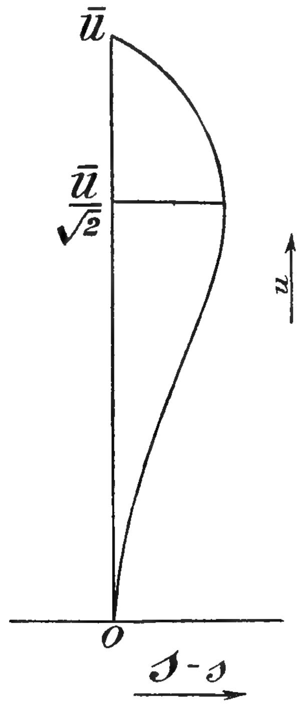

FIG. 57.

The two types of zones which may occur in spherically corrected systems.

$\mathfrak{s} - s = - \mathsf{A}u^2 + \mathsf{B}u^4$ (Standard zonal type). $\mathfrak{s} - s = \mathsf{A}u^2 - \mathsf{B}u^4$ (Abnormal zonal type).

of different sign. When this is the case $\bar{\bar{u}} = \sqrt{-\frac{\mathsf{A}}{2\mathsf{B}}}$ indicates the point where the deviation from the zero value of s is greatest, whilst smaller variations of the angular aperture do not affect the amount of this aberration. When $\bar{u} = \sqrt{-\frac{\mathsf{A}}{\mathsf{B}}} = \sqrt{2}\cdot\bar{\bar{u}}$ the initial value s is again obtained, and thereby the spherical aberration is removed. Figs. 56 and 57 show side by side the two possible curves of the function $\mathfrak{s} - s$, the abscissæ being the values of $\mathfrak{s} - s$ and the ordinates those of u.

In accordance with the practice of opticians in the case of spherically corrected systems, we shall denote by the term **primary zones** the finite deviation from $s = s_{\bar{u}}$, the maximum value of which is $\frac{A^2}{4B}$, and we shall make a distinction between **negative** and **positive primary zones.** From what has been said before we are justified in referring to the additive term Bu^4 as the first **zonal term.**

129. The Evaluation of the Coefficient B' by Abbe's Method of Invariants.—To obtain an analytical expression for the zonal term we shall revert to the invariant of the refraction at a surface, as established in § 120 (i), viz.:

$$Q_s = \frac{n\,(s - r)}{pr} = \frac{n'\,(s' - r)}{p'r}.$$

This must then be written in the form § 120 (ii)

$$Q_s = Q_s + q_{1s}\,\phi^2 + q_{2s}\,\phi^4 = Q'_s + q'_{1s}\,\phi^2 + q'_{2s}\,\phi^4, \qquad \text{(i)}$$

whence it follows that

$$q_{2s} = q'_{2s}.$$

In conformity with our previous investigation (§ 120, iii) we may write this in the form

$$Q_s = \frac{ns}{p}\left(\frac{1}{r} - \frac{1}{s}\right) = \frac{n's'}{p'}\left(\frac{1}{r} - \frac{1}{s'}\right),$$

and then proceed to expand separately the individual factors in ascending powers of ϕ up to the fourth.

From the equation established in § 73, if we confine ourselves to the terms containing powers not exceeding the fourth, we derive the following equation:

$$\frac{s}{p} = 1 + \left(\frac{r}{2s} - \frac{r^2}{2s^2}\right)\phi^2 - \left(\frac{1}{3}\,\frac{r^2}{ns}\,Q_s - \frac{3}{4}\,\frac{r^4}{n^2s^2}\,Q_s^2\right)\frac{\phi^4}{8}. \qquad \text{(ii)}$$

Since we are only concerned with the coefficient of ϕ^2 no sensible error will be introduced if we carry the expansion of $\frac{1}{s}$ only as far as the terms containing ϕ^2, and since in § 120 (vi) it

was shown that $\frac{1}{\boldsymbol{s}} = \frac{1}{s} - \frac{\mathsf{A}\,r^2}{s^4}\,\phi^2$, we can at once substitute this expression for $\frac{1}{\boldsymbol{s}}$ in the coefficient of ϕ, and we obtain accordingly

$$\frac{\boldsymbol{s}}{p} = 1 + \frac{r^2}{2\,ns} Q_s \phi^2 - \left(\frac{1}{2}\frac{r^2}{ns^4} Q_s - \frac{1}{2}\frac{r^2}{s^5}\right) \mathsf{A}\, r^2 \phi^4 - \left(\frac{1}{3}\frac{r^2}{ns} Q_s - \frac{3}{4}\frac{r^4}{n^2 s^2} Q_s^2\right) \frac{\phi^4}{8} \quad \text{(iii)}$$

From

$$\sin u = \frac{r}{p} \sin\phi$$

and

$$\boldsymbol{s} = s + \mathsf{A}\, u^2 + \mathsf{B}\, u^4,$$

we derive the following equations

$$u^2 = \frac{r^2\phi^2}{s^2}\left\{1 - \phi^2\left(\frac{r^2}{3n^2} Q_s^2 - \frac{r^2 Q_s}{3ns} + \mathsf{A}\,\frac{r^2}{s^3}\right)\right\}$$

and

$$\frac{1}{\boldsymbol{s}} = \frac{1}{s} - \frac{\mathsf{A}\, r^2\phi^2}{s^4} + \left\{\frac{\mathsf{A}\, r^2}{s^4}\left(\frac{r^2}{3n^2}\, Q_s^2 - \frac{r^2}{3ns}\, Q_s\right) + \frac{3\,\mathsf{A}^2 r^4}{s^7} - \frac{\mathsf{B} r^4}{s^6}\right\}\phi^4,$$

hence

$$\boldsymbol{Q} = \frac{\boldsymbol{s}}{p}\left[Q_s + \frac{n\mathsf{A}r^2}{s^4}\,\phi^2 - \left\{\frac{n\mathsf{A}r^2}{s^4}\left(\frac{r^2}{3n^2}\, Q_s^2 - \frac{r^2}{3ns}\, Q_s\right) + \frac{3n\mathsf{A}^2 r^4}{s^7} - \frac{n\mathsf{B}r^4}{s^6}\right\}\phi^4\right]. \quad \text{(iv)}$$

After multiplication this may be reduced to

$$\boldsymbol{Q}_s = Q_s + \frac{r^2\phi^2}{2}\left\{\frac{2n\mathsf{A}}{s^4} + \frac{Q_s^2}{ns}\right\}$$

$$-\frac{r^2\phi^4}{24}\left\{\frac{20\mathsf{A}r^2 Q_s^2}{ns^4} - \frac{32\mathsf{A}r^2 Q_s}{s^5} + \frac{72n\mathsf{A}^2 r^2}{s^7} - \frac{24n\mathsf{B}r^2}{s^6} + \frac{Q_s^2}{ns} - \frac{9r^2 Q_s^3}{n^2 s^2}\right\}. \quad \text{(v)}$$

The first two terms are already known from the investigation in § 120, and from the relation

$$q_{2\,s} = q_{2\,s}',$$

employing our previous notation, we obtain the equation

$$\Delta \frac{n\mathsf{B}}{s^6} = 3\,\Delta \frac{n\mathsf{A}^2}{s^7} + \frac{5}{6}\,Q_s^2\,\Delta \frac{\mathsf{A}}{ns^4} - \frac{4}{3}\,Q_s\,\Delta \frac{\mathsf{A}}{s^5} - \frac{3}{8}\,Q_s^3\,\Delta \frac{1}{n^2 s^2}$$

$$+ \frac{1}{24}\,\frac{Q_s^2}{r^2}\,\Delta \frac{1}{ns} \quad \ldots \quad \ldots \quad \ldots \qquad \text{(vi)}$$

From this it would be easy to derive a recurrence formula for B'_k, which would then have to contain terms embodying A'^2, A', A^2 and A. To facilitate calculation we shall introduce the new variable $\omega = \dfrac{s'}{s}$, with the aid of which we may obtain the following expressions:

From § 31 (ii) $$\frac{1}{r} = \frac{n'}{n'-n}\cdot\frac{1}{s'}\left(1 - \frac{n}{n'}\,\omega\right),$$

from § 31 (i) $$Q_s = \frac{nn'}{n'-n}\cdot\frac{1}{s'}\left(1 - \omega\right),$$

from § 120 (viii) $$2n'\mathsf{A}' - 2n\mathsf{A}\omega^4 = s'\,\frac{nn'^2}{(n'-n)^2}\,(\omega - 1)^2\left(\omega - \frac{n}{n'}\right).$$

Substituting these values in equation (vi) and multiplying both sides by s^6 we finally obtain the simplified expression

$$\left.\begin{aligned}
&24\,(n'\mathsf{B}' - n\mathsf{B}\omega^6) = 72\,\frac{n}{n'}\,\mathsf{A}^2\omega^7\,\frac{n-n'}{r} + 72\,\frac{n^2n'\mathsf{A}}{(n'-n)^2}\,\omega^4 f_2(\omega)\\
&\qquad + 18\,\frac{n^2n'^3}{(n'-n)^4}\,s'\,(\omega - 1)^2\left(\omega - \frac{n}{n'}\right) f_3(\omega),\\
&\text{where } f_2(\omega) \text{ and } f_3(\omega) \text{ have the values}\\
&f_2(\omega) = \omega^2 - \left(\frac{13}{9}\,N - \frac{8}{9}\right)\omega + \left(\frac{5}{9}\,N - \frac{1}{9}\right)\\
&f_3(\omega) = \omega^3 - \left(N + \frac{1}{2}\right)\omega^2 + N\omega - \left(\frac{1}{9}\,N + \frac{5}{18}\right)\\
&N = \frac{(n+n')^2}{2nn'}.
\end{aligned}\right\}\qquad \text{(vii)}$$

These equations enable us to determine for each surface the values of A' and B' corresponding to given values of s', s, A and B; each set of values serving as the initial data for the next surface.

With regard to the coefficients $f_2(\omega)$ and $f_3(\omega)$, it should be noted that they are symmetrical with respect to n and n', and that they do not change their values when n and n' are interchanged. Moreover, $f_3(\omega)$ has three roots within the range of $\frac{n}{n'}$, viz. 1 to 1.65, as shown in the subjoined table.

Table showing the Roots of $f_3(\omega)$

$\frac{n}{n'}$	ω_1	ω_2	ω_3
1·1	0·497512	0·971682	1·035350
1·2	0·491059	0·951210	1·074398
1·25	0·486835	0·943120	1·095046
1·3	0·482100	0·936212	1·116302
1·4	0·471733	0·925071	1·160338
1·5	0·460672	0·916593	1·206067
1·6	0·449402	0·910238	1·252860
1·64	0.444944	0·908051	1·271887

From the form of the expression for B′, it follows that in the special case of an object-point without aberration and a single refracting surface, the zonal term vanishes for the same three distances in connection with which the first term of the spherical aberration is reduced to zero, as explained in § 123. This occurs, when $\omega = 1$ for the vertex of the surface and its centre, and when $\omega = n/n'$ for the aplanatic points of the surface. Apart from these points, the zonal term vanishes in our special case only for the three real values of ω, as defined by the equation $f_3(\omega) = 0$. For the object-distances corresponding to the values of ω given above, the resulting spherical aberration is of the first order only when the conditions are such that terms of the fifth and higher orders may be neglected.

The significance of this pair of aplanatic points and of the centre of the sphere becomes apparent, as it naturally must, in the zonal term.

We may use these points in the construction of an aplanatic converging lens of finite thickness, which for pencils of any

angular aperture will form a virtual image of an object-point free from aberration. Lenses of this kind are employed in the construction of microscope objectives of high power.

With the object distance $s_1 = r_1$ as radius, let a spherical surface be described about the object-point, whereby the pencil proceeding from an object-point in air passes without aberration into the medium n. When the thickness d is introduced we shall have $s_2 = r_1 - d$, and we have then only to solve the equation considered in § 123 under case (3), which now assumes the form

$$r_1 - d = \frac{n+1}{n} r_2, \quad \ldots \quad \ldots \quad \text{(viii)}$$

whence we obtain

$$r_2 = \frac{n(r_1 - d)}{n+1} \text{ and } s'_2 = n(r_1 - d). \quad \ldots \quad \text{(ix)}$$

Also, from § 82 we may find at once the magnification, viz.

$$\beta = n \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(x)}$$

3. The Aberration of Extra-axial Points in Terms of the Inclination of the Principal Ray only.

(DISTORTION.)

130. In the preceding pages we have considered the aberrations of axial points in terms only of the aperture u of the pencils of rays, and we shall now investigate in the same way solely the aberrations due to the principal rays. We shall confine our investigations to pencils of an infinitely narrow aperture, and, in tracing them through the system, we shall make the reservation that rays may pass only through the centre P of the aperture plane.

A. Distortion in terms of the Spherical Aberration of the Centre of the Diaphragm and of the Inclination of the Principal Ray.

131. To render the investigation as general as possible, we shall assume that the system comprises a front combination associated with a back combination, separated by a diaphragm I, which in our case is supposed to have an extremely narrow aperture. Any interior angle $\overline{w}$ will then have corresponding to it the two angles of inclination w and w', which are conjugate to it in respect of the two components and also mutually conjugate in respect to the entire system. To obviate the inconvenience of having to

determine for a given extra-axial object-point the inclination w of the principal ray which, after refraction, passes through the given centre of the diaphragm, we shall take O_w as the point in the object-plane at a distance l from the axis where the principal ray inclined at an angle w to the axis intersects the object-plane. The position of the conjugate image point $\bar{O}'_w$ at a distance l' from the axis will then be defined by the point where the principal ray intersects the Gauss image-plane.

The necessary condition that there shall be no deviation from the value given by the Gaussian system, namely $\gamma = l' / l$, or, as we may now say, that there shall be **no distortion**, can obviously be expressed by the ratios :

$$\frac{l'}{l} = \frac{l'}{l} = \beta .$$

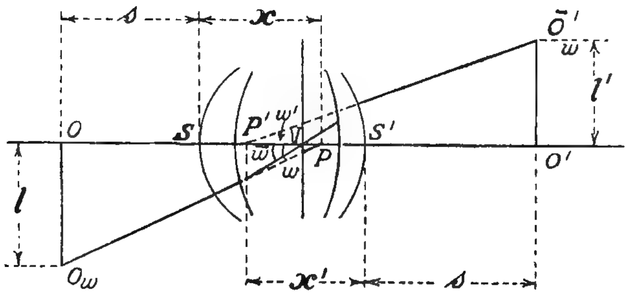

FIG. 58.

$OO_w = l$; $O'\bar{O}'_w = l'$.

Distortion in terms of the spherical aberration of the middle of the diaphragm and the inclination of the principal ray.

If now we express the quantities l' and l in terms of the axial abscissæ and the angles of inclination we shall have

$$\frac{l'}{l} = \frac{O'P'}{OP}\frac{\tan w'}{\tan w} = \frac{P'O'}{PO}\frac{\tan w'}{\tan w} = \frac{P'S' + S'O'}{PS + SO} \cdot \frac{\tan w'}{\tan w}$$

$$= \frac{-\ x' + s'}{-\ x\ + s}\frac{\tan w'}{\tan w},$$

and we thus see that, generally speaking, the distortion is governed by two factors, viz., the ratio of the tangents, $\tan w' / \tan w$, and also by the aberrations $\delta x' = x' - x'$ and $\delta x = x - x$ in the images

P, P' of the middle of the diaphragm I due to the components of the system.

In the event of the object or image being at infinity the linear magnitude of the object or image should be replaced by the angular magnitude $\frac{l}{s-x} = \frac{l}{s-x}$ or $\frac{l'}{s'-x'} = \frac{l'}{s'-x'}$, and in this case the finite aberration $\delta x'$ or δx of the diaphragm with respect to the object or the image, ceases to be of any moment.

In the case of an astronomical telescope, where the object and image distances are both infinite, there is obtained what is known as the **Airy tangent condition**, viz.:

$$\frac{l'}{s'-x'} \Big/ \frac{l}{s-x} = \frac{\tan w'}{\tan w} = constant.$$

The variability of the tangent ratio with changing values of w, is in this case the only factor which determines the amount and nature of the distortion, and the condition for the absence of distortion is here identical with the requirement that this ratio shall be constant or that the Airy tangent condition shall be satisfied.

This holds good, even in such cases where $x' \equiv x'$, due say, to the absence of the back component, whilst at the same time the object recedes to infinity; for in this case it is obvious that on the right side of the equation

$$l' \Big/ \frac{l}{s-x} = (s'-x') \frac{\tan w'}{\tan w}$$

it is $\frac{\tan w'}{\tan w}$ which alone determines the magnitude of the distortion. This case is found in photographic lenses having a back stop. An analogous position arises when δx vanishes identically and the image is at infinity, a condition which may occur in practice in the case of an eye-piece adjusted for an eye accommodated for infinity.

When none of these special cases arise, and when the tangent ratio is to be the only criterion of the magnitude of the distortion, it will be necessary to ensure the spherical correction of the middle of the diaphragm with respect to either component of the system. This is known as the **Bow-Sutton condition.**

In actual practice it is possible to correct distortion at all distances of the object by satisfying simultaneously the conditions of Airy and Bow-Sutton.

Two optical systems may be instanced as embodying these principles. These are the truly concentric combinations of the type of Sutton's *panoramic lenses*, and the magnifying "globe"

lenses of Schroeder and A. Steinheil as representatives of one class, and a class of hemi-symmetrical systems in which an aplanatic image of the middle of the diaphragm is formed on the object side and on the image side by components containing each at least one aplanatic converging lens of the kind described at the end of § 129.

B. Expression for the Line of Confusion due to Distortion at Small Inclinations of the Principal Ray.

132. If we now proceed to determine the amount of distortion within the first degree of approximation it is clear that we may include third powers of the co-ordinate l, and that the aberrations are solely tangential, since every element of the principal pencil is contained within a meridian plane.

Under the conditions of the Gaussian system $l'/l = \beta$, since there is no initial distortion, and since $L' = l' + \delta l'$, it follows that

$$\frac{L'}{l} = \beta + \frac{\delta l'}{l} = \beta + \delta\beta\,.$$

In other words: If we regard the position of the object-point as the point where the principal ray intersects the Gauss image-plane and if, in the expansion of the angle of inclination w of the principal ray, fifth and higher powers may be regarded as negligible, there may arise tangential aberration which becomes apparent by a change of the lateral magnification β.

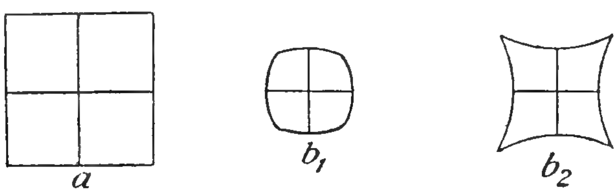

FIG. 59.

a; Object b_1; Image showing barrel-shaped distortion;
b_2; Image showing cushion-shaped distortion.

Where the relative magnification decreases, *i.e.*, where $\delta\beta < 0$, we find that in the case of a rectangular object placed at the centre of the field of view the distortion takes the form of a figure which bulges out from the centre and is then known as **barrel-shaped distortion.** When the ratio of magnification increases,

so that $\delta\beta > 0$, the sides of the resulting quadrilateral in the image are convex towards the centre and present the appearance known as **cushion-shaped distortion.**

We will now proceed to investigate the distortion which occurs at the individual surfaces. Let the principal ray, inclined at an angle w, intersect the corresponding spherical surface at B and the object-plane at $\bar{O}_w$. We shall then have the proportion:

$$HB \mid O\bar{O}_w = HP \mid OP$$

$$= (HS + SP) \mid (OS + SP)$$

or

$$r \sin\phi \mid \mathfrak{l} = (-2r\sin^2\frac{\phi}{2} + \mathfrak{x}) \mid (-s + \mathfrak{x})\,; \qquad \text{(i)}$$

or, if we neglect the fifth and higher powers and substitute

$$\mathfrak{l} = l\phi - l_s\frac{\phi^3}{6} \text{ and } \mathfrak{x} = x + \frac{\mathsf{E}r^2}{x^2}\phi^2, \qquad \dots \qquad \text{(ii)}$$

where the relationship of E to x and ϕ is the same as that of A to s and ϕ, we obtain

$$\frac{l}{r}\left[1 - \frac{\phi^2}{6}\left(\frac{l_s}{l} - 1\right)\right] = \frac{x-s}{x}\left[1 + \phi^2\left(\frac{\mathsf{E}r^2}{x^2(x-s)} - \frac{\mathsf{E}r^2}{x^3} + \frac{r}{2x}\right)\right].$$

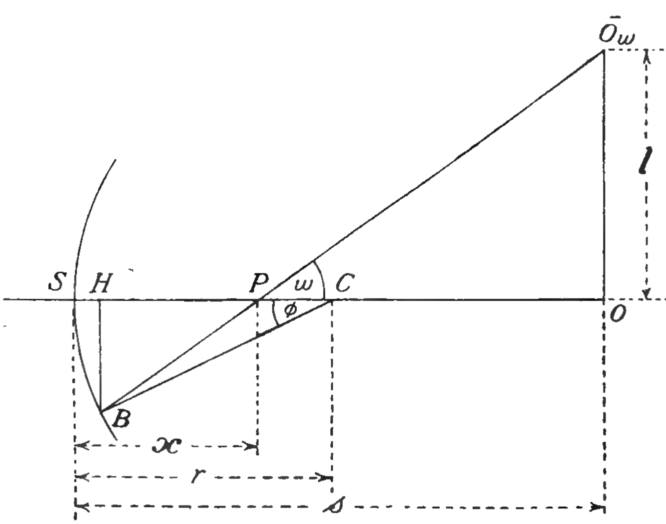

FIG. 60.

$SC = r\,;\ SO = s\,;\ SP = \mathfrak{x}\,;\ O\bar{O}_w = \mathfrak{l}$

Diagram descriptive of the tangential aberration due to distortion.

From this equation may be derived the value of l and l_s thus :

$$l = \frac{x - s}{x} r$$

and

$$l_s = \frac{x - s}{x} r^3 \left(\frac{1}{r^2} - \frac{6\mathsf{E}s}{x^3 (x - s)} - \frac{3}{rx} \right)$$

whence

$$\frac{nl}{s} = (Q_x - Q_s) r\phi \left[1 - \frac{r^2\phi^2}{6} \left(\frac{1}{r^2} - \frac{6n\mathsf{E}}{x^4 (Q_x - Q_s)} - \frac{3}{rx} \right) \right]$$

$$\frac{n'l'}{s'} = (Q_x - Q_s) r\phi \left[1 - \frac{r^2\phi^2}{6} \left(\frac{1}{r^2} - \frac{6n'\mathsf{E}'}{x'^4 (Q_x - Q_s)} - \frac{3}{rx'} \right) \right]$$

therefore

$$\frac{n'l's}{nls'} = 1 + \frac{r^2\phi^2}{2} \left(\frac{1}{Q_x - Q_s} \Delta \frac{2n\mathsf{E}}{x^4} + \frac{1}{r} \Delta \frac{1}{x} \right).$$

By § 120 (ix)

$$\Delta \frac{2n\mathsf{E}}{x^4} = - Q_x^2 \Delta \frac{1}{nx}$$

and by § 75 (i)

$$\Delta \frac{1}{x} = - Q_x \Delta \frac{1}{n};$$

therefore

$$\frac{n'l's}{nls'} = 1 - \frac{r^2\phi^2}{2} \left(\frac{Q_x^2}{Q_x - Q_s} \Delta \frac{1}{nx} + Q_x \frac{1}{r} \Delta \frac{1}{n} \right).$$

Introducing the surface index

$$l'_v = l_{v+1}$$

and

$$r_v \phi_v = y_v .$$

further according to the formula of Smith-Helmholtz [§ 84(4)]

$$\frac{n_1 l_1}{n'_k l'_k} = \gamma_k$$

and by § 82 (vi)

$$\gamma_k = \prod_{v=1}^{k} \frac{s_v}{s'_v},$$

hence

$$\frac{n'_k l'_k}{n_1 l_1} = \prod_{v=1}^{k} \frac{s'_v}{s_v}.$$

Also,

$$l_1 = l_1 .$$

Hence, forming the product of all the expressions $\frac{n'_v l'_v s_v}{n_v l_v s'_v}$,

$$\frac{n'_k l'_k}{n_1 l_1} \prod_{v=1}^{k} \frac{s_v}{s'_v} = \frac{l'_k}{l'_k} = 1 - \frac{r_1^2 \phi_1^2}{2} \sum_{v=1}^{k} \left(\frac{y_v}{y_1}\right)^2 \left(\frac{Q^2_{vx}}{Q_{vx} - Q_{vs}} \Delta_v \frac{1}{nx} + Q_{vx} \frac{1}{r_v} \Delta_v \frac{1}{n}\right),$$

and hence we obtain for the tangential aberration $l'_{ok} = l'_k - l'_k$ the following equation with respect to the image side

$$\frac{l'_{ok}}{l'_k} = - \frac{y_1^2}{2} \sum_{v=1}^{k} \left(\frac{y_v}{y_1}\right)^2 \left(\frac{Q_{vx}^2}{Q_{vx} - Q_{vs}} \Delta_v \frac{1}{nx} + Q_{vx} \frac{1}{r_v} \Delta_v \frac{1}{n}\right).$$

If now we project the aberration l'_{ok}, due to tangential pencils, back into the object, we shall find by the Smith-Helmholtz theorem that

$$n'_k \mathrm{u}'_k l'_{ok} = n_1 \mathrm{u}_1 l_o^{(k)}.$$

Remembering also that the lateral magnification is given by the quotient

$$\frac{l_1}{l'_k} = \frac{n'_k \mathrm{u}'_k}{n_1 \mathrm{u}_1},$$

we obtain finally the following expression for the distortion projected back into the object:

$$\frac{l'_{ok}}{l'_k} = \frac{l_o^{(k)}}{l_1} = - \frac{y_1^2}{2} \sum_{v=1}^{k} \left(\frac{y_v}{y_1}\right)^2 \left(\frac{Q^2_{vx}}{Q_{vx} - Q_{vs}} \Delta_v \frac{1}{nx} + Q_{vx} \frac{1}{r_v} \Delta_v \frac{1}{n}\right). \text{(iii)}$$

The condition that $l_o^{(k)}$ may vanish independently of y_1, whereby the distortion becomes eliminated, is

$$\sum_{v=1}^{k} \left(\frac{y_v}{y_1}\right)^2 \left(\frac{Q^2_{vx}}{Q_{vx} - Q_{vs}} \Delta_v \frac{1}{nx} + Q_{vx} \frac{1}{r_v} \Delta_v \frac{1}{n}\right) = 0. \quad \dots \quad \text{(iv)}$$

This expression may be so transformed that the summation contains $\Delta \frac{1}{ns}$ instead of $\Delta \frac{1}{nx}$. The resulting expression for the distortion will reappear later as the outcome of an entirely different investigation.

We shall now revert to the value of $Q_{vx} - Q_{vs}$ given in § 83 (2). By the introduction of this value in the sum we obtain the equation

$$\frac{l_o^{(k)}}{l_1} = -\frac{y_1^2}{2} \frac{1}{Q_{1x} - Q_{1s}} \sum_{v=1}^{k} \frac{h_v}{h_1} \left(\frac{y_v}{y_1}\right)^3 \left[Q^2_{vx} \Delta_v \frac{1}{nx} + Q_{vx} (Q_{vx} - Q_{vs}) \frac{1}{r_v} \Delta_v \frac{1}{n}\right].$$

Eliminating the common multiple $\frac{Q_{vx}}{Q_{rs}}$, substituting $\underset{v}{\Delta}\frac{1}{nx} = \frac{1}{r_v}\underset{v}{\Delta}\frac{1}{n} - Q_{vx}\underset{v}{\Delta}\frac{1}{n^2}$ and adding and subtracting $\frac{1}{r_v}Q_{vx}^2\underset{v}{\Delta}\frac{1}{n}$, the expression within the brackets becomes

$$\frac{Q_{vx}}{Q_{vs}}\left[Q^2_{vx}\left(\frac{1}{r_v}\underset{v}{\Delta}\frac{1}{n} - Q_{vs}\underset{v}{\Delta}\frac{1}{n^2}\right) - (Q_{vx} - Q_{vs})^2\frac{1}{r_v}\underset{v}{\Delta}\frac{1}{n}\right].$$

Replacing the first part of the expression within the bracket by the equivalent expression given in § 75 (iii), we obtain the following formula for the distortion referred to the object:

$$\frac{l_o^{(k)}}{l_1} = -\frac{y_1^2}{2}\frac{1}{Q_{1x} - Q_{1s}}\sum_{v=1}^{k}\frac{h_v}{h_1}\left(\frac{y_v}{y_1}\right)^3\left[\frac{Q^3_{vx}}{Q_{vs}}\underset{v}{\Delta}\frac{1}{ns} - \frac{Q_{vx}}{Q_{vs}}(Q_{vx} - Q_{vs})^2\frac{1}{r_v}\underset{v}{\Delta}\frac{1}{n}\right]. \text{(v)}$$

Introducing the identities

$$y_1 = \frac{x_1 l_1}{x_1 - s_1} \text{ and } \frac{1}{Q_{1x} - Q_{1s}} = \frac{s_1 x_1}{n_1(x_1 - s_1)} \quad \ldots \quad \text{(vi)}$$

we obtain the final result

$$\frac{n_1\, l_o^{(k)}}{s_1} = -\frac{1}{2}\frac{x_1^3\, l_1^3}{(x_1 - s_1)^3}\sum_{v=1}^{k}\frac{h_v}{h_1}\left(\frac{y_v}{y_1}\right)^3\left[\frac{Q^3_{vx}}{Q_{vs}}\underset{v}{\Delta}\frac{1}{ns} - \frac{Q_{vx}}{Q_{vs}}(Q_{vx} - Q_{vs})^2\frac{1}{r_v}\underset{v}{\Delta}\frac{1}{n}\right]. \text{(vii)}$$

From this it will be seen that the tangential aberration is governed solely by the co-ordinate l_1 with respect to the object and by its third power, as it should be from obvious considerations.

C. The Distortion in Simple Special Cases.

133. To investigate the distortion in simple cases we shall proceed from the expression

$$l_o^{(k)} = -\frac{1}{2}\frac{s_1}{n_1}x_1^3 w_1^3\sum_{v=1}^{k}\frac{h_v}{h_1}\left(\frac{y_v}{y_1}\right)^3\left\{Q^2_{vx}\underset{v}{\Delta}\frac{1}{nx} + Q_{vx}(Q_{vx} - Q_{vs})\frac{1}{r_v}\underset{v}{\Delta}\frac{1}{n}\right\},$$

which is identical with the formula given at the conclusion of the preceding paragraph.

134. Distortion due to a Single Surface.—In this case the equation expressing the defects in the image due to distortion assumes the form

$$l_0^{(k)} = -\frac{1}{2}\frac{1}{n} w^3 x Q_x x^2 s \left\{ Q_x \Delta \frac{1}{nx} + (Q_x - Q_s)\frac{1}{r}\Delta\frac{1}{n} \right\}.$$

It will be seen at once that this expression can vanish only in two cases, namely, when one of the two factors of which it is composed, vanishes, viz.:

(*a*) when the first factor vanishes, $xQ_x = 0$ or $x = r$, that is to say, the position of the diaphragm is identical with that of its image at the centre of the surface, and in this case the position of the object-point is quite arbitrary.

(*b*) when the second factor vanishes,

$$x^2 s \, \mathsf{D} = x^2 s \left(Q_x \Delta \frac{1}{nx} + (Q_x - Q_s)\frac{1}{r}\Delta\frac{1}{n} \right) = 0.$$

This relation implies the mutual dependence of x and s. With the aid of the equation § 31 (ii), *i.e.*,

$$\xi' = \frac{n}{n'}\xi + \frac{n'-n}{n'}\rho,$$

the expression may be reduced to the form

$$\frac{n'^2 \mathsf{D}}{n'-n} = (n'+n)\xi^2 - 2n\xi\rho + n\rho^2 - n'\sigma\rho,$$

from which it will be seen that for any given distance of the stop, there is one and only one value of s which causes D to vanish. This value of s, moreover, is always real, viz.:

$$\frac{1}{s} = \sigma = \frac{1}{\rho}\left[\xi^2 + \frac{n}{n'}(\xi - \rho)^2\right].$$

On the other hand, when the distance of the object s is given, we have the relation

$$\xi = \frac{n\rho \pm \sqrt{n'^2\rho\sigma + nn'\rho\sigma - nn'\rho^2}}{n'+n},$$

that is there are two or one, or no possible positions of the stop for which D vanishes, according as

$$\rho\sigma \gtreqless \frac{n\rho^2}{n'+n}.$$

To obtain a real value for the position of the stop, σ must have the same sign as ρ, and $\frac{\sigma}{\rho}$ must be greater than $\frac{n}{n' + n}$. When the object-point lies at the aplanatic point, in which case

$$\sigma = \frac{n\rho}{n' + n},$$

the distortion cannot be eliminated by making $\mathsf{D} = 0$, since in this case the required value $\xi = \frac{n\rho}{n' + n}$ corresponding to the assumed position of the object-point becomes meaningless.

If none of the cases which we have here specified arise, the expression for the distortion assumes the form

$$\frac{nl_o^{(k)}}{s} = -\frac{1}{2} w^3 x^2 n \frac{n' - n}{n'^2} (x\rho - 1)(n'\xi^2 + n(\xi - \rho)^2 - n'\sigma\rho).$$

The sign of this expression is then governed by the values of σ and ξ. If now, as in the investigation of the spherical aberration, we consider the special case of an infinitely distant object, the resulting equation becomes more easily understood, and we arrive at the conclusion that the displacement of the diaphragm causes the distortion in the principal focal plane to change its sign at the instant that the diaphragm passes through the centre of the surface.

135. Distortion due to a Single Thin Lens.—Applying the conditions of this case to the general expression for the coefficient of distortion, and employing the simple equations given in § 88, we can express r_2, x'_2, s'_2 in terms of $r = r_1$, x_1, s_1, and we then obtain

$$\mathsf{D} = \mathsf{D}_o - \mathsf{D}_1 \rho + \mathsf{D}_2 \rho^2,$$

where

$$\mathsf{D}_o = \frac{n^2}{(n-1)^2}\phi^3 + \frac{1}{n-1}\phi^2\sigma + \frac{3n}{n-1}\phi^2\xi + \frac{1}{n}\phi\sigma\xi + \frac{3n+1}{n}\phi\xi^2$$

$$\mathsf{D}_1 = \frac{2n+1}{n-1}\phi^2 + \frac{n+1}{n}\phi\sigma + \frac{3n+3}{n}\phi\xi$$

$$\mathsf{D}_2 = \frac{n+2}{n}\phi.$$

This method of expressing the distortion due to a single thin lens as a function of the first radius is in keeping with practical

requirements, since in the computation of systems of lenses distortion is generally taken into account by so adjusting the form of the lenses that the distortion assumes a certain prescribed value.

We shall now rearrange the above formula as follows:

By determining D_{min} from $\frac{\partial \mathsf{D}}{\partial \rho} = 0$, so that

$$\rho_{min} = \frac{\mathsf{D}_1}{2\,\mathsf{D}_2} = \frac{3(n+1)}{2(n+2)}\xi + \frac{n+1}{2(n+2)}\sigma + \frac{(2n+1)n}{2(n+2)(n-1)}\phi$$

and further

$$\mathsf{D}_{min} = \mathsf{D}_0 - \frac{\mathsf{D}^2_1}{4\,\mathsf{D}_2},$$

we obtain, quite generally, for

$$\rho = \rho_c + \rho_{min}$$

$$\mathsf{D} = \mathsf{D}_{min} + \mathsf{D}_2\,\rho_c^{\,2} = \mathsf{D}_{min} + \frac{n+2}{n}\phi\,(\rho - \rho_{min})^2.$$

Introducing for simplification

$$\frac{\sigma}{\phi} = \Sigma\,; \quad \frac{\xi}{\phi} = \Xi\,; \quad \frac{\rho_{min}}{\phi} = P_{min}\,; \quad \frac{\mathsf{D}_{min}}{\phi^3} = Y_{min}\,,$$

we obtain the following relation for Y_{min}, viz:

$$\frac{-(n+1)^2}{4n(n+2)}\Sigma^2 - \frac{3n^2+4n-1}{2n(n+2)}\Sigma\,\Xi + \frac{3n^2+10n-1}{4n(n+2)}\Xi^2$$

$$-\frac{2n+3}{2(n+2)}\Sigma + \frac{3}{2(n+2)}\Xi + \frac{(4n-1)n}{4(n-1)^2(n+2)} - Y_{min} = 0,$$

and from the expression for ρ_{min} given above,

$$P_{min} = \frac{n+1}{2(n+2)}\Sigma + \frac{3(n+1)}{2(n+2)}\Xi + \frac{(2n+1)n}{2(n+2)(n-1)}$$

and, as above,

$$Y = Y_{min} + \frac{n+2}{n}(P - P_{min})^2.$$

We have now only to express Y_{min} and P_{min} as functions of Ξ and Σ, to obtain a convenient means of tracing the changes of both functions. We can then calculate for any given value of P the corresponding value of Y. The converse operation of determining the requisite change of form which will result in a prescribed value of Y can only be performed for real values of P, when $Y \geqq Y_{min}$.

The relation of the value of Y_{min} to Σ and Ξ is more clearly seen if we reduce the equation to its normal form by a suitable transformation of the axes.

The displacement of the origin follows from the equations

$$\Sigma = (\Sigma) - \frac{1}{2}; \quad \Xi = (\Xi) - \frac{1}{2},$$

and the angular displacement a is given by

$$\tan 2a = - \frac{3 n^2 + 4 n - 1}{2 n (n + 3)}.$$

As the final result of these substitutions we obtain

$$\lambda_1 s_s^2 + \lambda_2 x_s^2 = Y_{min} - \frac{n^2}{4 (n - 1)^2},$$

whereby the value $\lambda_{1,2}$ is obtained from the equation

$$\lambda_{1\,2} = \frac{n^2 + 4n - 1 \pm \sqrt{(n^2 + 4n - 1)^2 + 4n^2 (3n^2 + 10n + 8)}}{4n (n + 2)}.$$

From this it follows that the values of Σ and Ξ, corresponding to any given values of Y_{min}, represent hyperbolas having a common centre and degenerating into their common pair of asymptotes when $Y_{min} = \dfrac{n^2}{4 (n-1)^2}$.

Values of Σ and Ξ corresponding to any given value of P_{min} represent a straight line in a family of parallel lines.

4. ABERRATIONS OF EXTRA-AXIAL POINTS IN TERMS OF THE FIRST POWERS OF THE ANGULAR APERTURE (u, v).

(Curvature of the Image due to Tangential and Sagittal Rays. Astigmatism.)

136. The derivation of the invariants for oblique refraction given in § 92 (14) and § 93 (16) enable us to formulate the expressions

$$Q = \frac{n \cos j}{r} - \frac{n \cos^2 j}{t}; \quad Q_f = \frac{n \cos j}{r} - \frac{n}{f}.$$

From these expressions we know that infinitely thin pencils of obliquely incident rays undergo astigmatic deformation. In a centred system accordingly, we may conceive an object situated on a surface of revolution about the axis of symmetry to have as its corresponding images two surfaces of revolution, one of which comprises all image-points due to the *tangential* (or *meridional*) pencils, the other all points due to the *sagittal* (or *equatorial*) pencils. The surfaces of revolution are accordingly called **tangential** and **sagittal image surfaces**

A. The Position of Astigmatic Image-Points for Finite Inclinations of the Principal Ray.

137. As in our investigation of the spherical aberration of a point on the axis, we shall again suppose the Gauss image-plane to be situated at the point of intersection O' of the axial rays. Instead of an extra-axial image-point, we shall then find in this plane two lines of confusion, one due to the tangential pencil, and the other to sagittal pencils.

In the two diagrams Figs. 61 and 62, let $P'\bar{O}'_w$ be a principal ray inclined at a finite angle w' to the axis and traversing the middle of the diaphragm at P', and let it correspond to an extra-axial

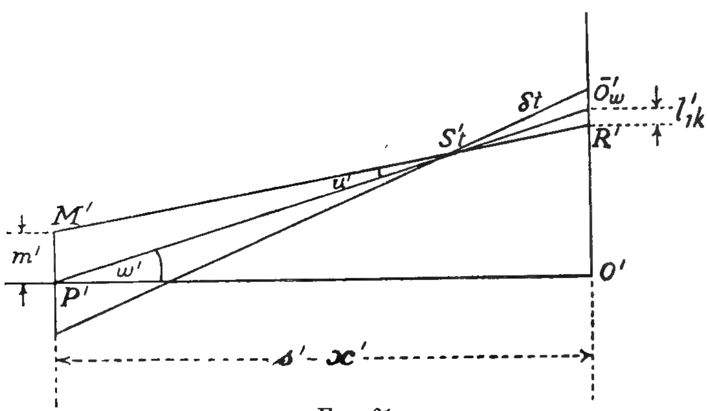

FIG. 61

$P'M' = m'$; $P'O' = s' - x'$; $\bar{O}'_w R' = l'_{1k}$

The line of confusion appearing in the Gauss image-plane due to the tangential curvature of the image.

image-point. This ray contains the two image-points due to pencils of very small apertures u' and v' of the first order of magnitude, viz., S'_t due to the tangential and S'_f to the sagittal pencils. Let

the distances of these points from the point $\bar{O}'_w$ where the principal ray intersects the Gauss plane be denoted by

$$S'_t\,\bar{O}'_w = \delta_t\,; \quad S'_f\,\bar{O}'_w = \delta_f\,;$$

also, let l'_{1k} represent the line of confusion appearing at O'_w due to tangential pencils of aperture u′, and let L'_{1k} be the line of confusion due to sagittal pencils of aperture v′.

On reference to the illustrations we can immediately establish the following relations :

$$\frac{l'_{1k}}{m'} = \frac{-\delta_t \cos w'}{s'-x'-\delta_t \cos w'} \quad \text{and} \quad \frac{L'_{1k}}{M'} = \frac{-\delta_f \cos w'}{s'-x'-\delta_f \cos w'}\,,$$

whence

$$l'_{1k} = -\frac{m'\,\delta_t \cos w'}{s'-x'-\delta_t \cos w'}\,;\; L'_{1k} = -\frac{M'\,\delta_f \cos w'}{s'-x'-\delta_f \cos w'}\,. \quad \text{(i)}$$

It should be noted that the lines of confusion due to the astigmatism of oblique pencils are linear in terms of the aperture co-ordinates m' and M'. These two terms define accordingly all the aberrations specified in § 115 which are linear functions of the aperture.

In relation to w', it is obvious that l'_{1k} and L'_{1k} are even functions of w', beginning with the second power, since they vanish in any case when $w' = 0$.

In the formula for the abscissæ of the image-receiving plane, as given in § 44 (vi) for sagittal pencils, if we multiply the numerator and denominator by $x_{v+1} - r_{v+1}$ and introduce the abbreviation

$$\frac{g_{v+1}}{g_v} = \frac{x_{v+1} - r_{v+1}}{x'_v - r_v}\,,$$

and noting the relation § 44 (vii), we obtain the following recurrence formula :

$$\frac{x_{v+1}-r_{v+1}}{x_{v+1}-\underline{\underline{s}}_{v+1}} \cot w_{v+1} = \frac{g_{v+1}}{g_v} \cdot \frac{x_v - r_v}{x_v - \underline{\underline{s}}_v} \cot w_v - \frac{g_{v+1}}{g_v} \cdot \frac{\sin(j_v - j'_v)}{\sin w_v \sin w'_v}\,.$$

By summation and multiplication of the successive terms by $\frac{g_k}{g_{k-1}}, \frac{g_k}{g_{k-2}} \ldots \frac{g_k}{g_2}$, we obtain the final formula

$$\frac{x'_k - r_k}{x'_k - \underline{\underline{s}}'_k} = \frac{g_k}{g_1} \cdot \frac{x_1 - r_1}{x_1 - \underline{\underline{s}}_1} \cot w_1 - \sum_{v=1}^{k} \frac{g_k}{g_v} \cdot \frac{\sin(j_v - j'_v)}{\sin w_v \sin w'_v}\,. \text{(ii)}$$

This formula holds for systems of finite thickness and having a finite position of the diaphragm. We are unable to establish a corresponding relation for the tangential pencils.

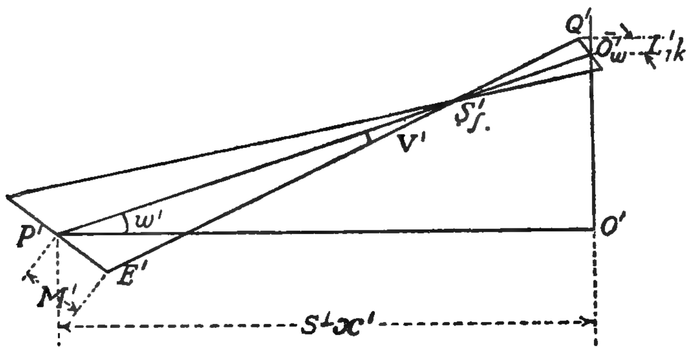

FIG. 62.

$P'E' = M'$; $P'O' = s' - x'$; $\bar{O}'_w Q' = L'_{1,k}$.

The line of confusion in the Gauss image-plane due to the sagittal curvature of the image.

138. Position of the Astigmatic Image Points in a System of Plane Surfaces.—The general investigation of the astigmatism for finite angles of inclination w is practicable in only a very few cases. We shall first consider a system of k parallel planes at right angles to the axis, and find expressions for the distances t'_k and f'_k of the image-points on principal rays inclined at an angle w'_k.

Conjugate tangential intercepts resulting at the v^{th} refraction have been shown in § 100 (ii) to be connected by the relation

$$t'_v = \frac{n'_v}{n_v} \frac{\cos^2 j'_v}{\cos^2 j_v} t_v ,$$

and, since

$$t_v = t'_{v-1} - \frac{d_{v-1}}{\cos j'_{v-1}}$$

and noting that $\cos j'_{v-1} = \cos j_v$,

$$t'_v = \frac{n'_v}{n_v} \frac{\cos^2 j'_v}{\cos^2 j_v} t'_{v-1} - \frac{n'_v}{n'_{v-1}} \frac{\cos^2 j'_v d_{v-1}}{\cos^3 j'_{v-1}} .$$

Similarly,

$$t'_v = \frac{n'_v}{n_{v-2}} \frac{\cos^2 j'_v}{\cos^2 j_{v-2}} t_{v-2} - \frac{n'_v}{n'_{v-2}} \frac{\cos^2 j'_v d_{v-2}}{\cos^3 j'_{v-2}} - \frac{n'_v}{n'_{v-1}} \frac{\cos^2 j'_v d_{v-1}}{\cos^3 j'_{v-1}},$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$t'_k = \frac{n'_k \cos^2 j'_k}{n_1 \cos^2 j_1} t_1 - n'_k \cos^2 j'_k \sum_{v=1}^{k-1} \frac{d_v}{n'_v \cos^3 j'_v}. \quad \ldots \quad \text{(i)}$$

Similarly, we may derive from § 100 (i) the following expression for the sagittal section, viz.:

$$f'_k = \frac{n'_k}{n_1} f_1 - n'_k \sum_{v=1}^{k-1} \frac{d_v}{n'_v \cos j'_v}. \quad \ldots \quad \text{(ii)}$$

There is now no difficulty in applying the formulæ obtained above to express the defects in the quality of the image arising from the astigmatism when the principal rays are inclined at finite angles. In this application the abscissa x' of the image-point which is conjugate to the position of the diaphragm, is furnished by the formulæ for the spherical aberration of a plane system given in § 118.

The quantity t_k' and the other quantities contained in the formulæ relating to the astigmatic image-point are connected by the equation

$$\frac{s_k'}{\cos w'_k} - \delta_t = t_k',$$

that is

$$\delta_t \cos w'_k = s'_k - t'_k \cos w'_k$$

and hence by § 137 (i)

$$l'_{1k} = - m' \frac{s'_k - t'_k \cos w_k'}{t'_k \cos w'_k - x'}. \quad \ldots \quad \ldots \quad \text{(iii)}$$

Similarly, we obtain for the sagittal section

$$L'_{1k} = - M' \frac{s'_k - f'_k \cos w'_k}{f'_k \cos w_k' - x'}. \quad \ldots \quad \ldots \quad \text{(iv)}$$

The difference between the tangential and sagittal intercepts may be expressed very simply in the following manner:

If we suppose the first and last media of the system of planes to be identical it follows that

$$n'_k = n_1; \quad j'_k = j_1,$$

and in this case the first term in the two expressions for t'_k and f'_k becomes t_1 and f_1. Assuming that the object-point is free from aberration so that $f_1 = t_1$, it is clear that the astigmatic difference $f'_k - t'_k$ is independent of the position of the object-point, since by (i) and (ii) we obtain for it the following expression :

$$f'_k - t'_k = n'_k \sum_{v=1}^{k-1} \frac{d_v}{n'_v \cos j'_v} \left(\frac{\cos^2 j'_k}{\cos^2 j'_v} - 1 \right).$$

The factor included in the summation may be transformed in the following manner

$$\frac{\cos^2 j'_k - \cos^2 j'_v}{\cos^2 j'_v} = \frac{\sin^2 j'_v - \sin^2 j'_k}{\cos^2 j'_v}$$

$$= \frac{\sin^2 j'_k}{\cos^2 j'_v} \left(\frac{n'^2_k}{n'^2_v} - 1 \right),$$

and thus we have finally

$$f'_k - t'_k = n_1^2 \sin^2 j_1 \sum_{v=1}^{k-1} \frac{n'_k d_v}{n'_v \cos^3 j'_v} \left(\frac{1}{n'^2_v} - \frac{1}{n'^2_k} \right). \qquad \text{(v)}$$

Since $j_v + w_v = \phi_v \equiv 0$, we can replace j_v by w_v, throughout the formula, as we are only concerned with even angular functions.

139. Position of the Astigmatic Image Points in a System of Thin Lenses.—There is another case which admits of the direct investigation of the astigmatism when the principal rays are inclined at a finite angle w to the axis. This is the case of a system of thin lenses having the diaphragm situated at the common vertex. Harting (**7.**) was the first to investigate this case, and we shall adhere to his procedure.

In this special case, where throughout

$$\phi_v = 0 \ (v = 1 \ldots\ldots k),$$

it follows from the equation

$$j_v + w_v = \phi_v = j'_v + w'_v \ (v = 1 \ldots\ldots k)$$

that the angle of incidence j_v has the same absolute magnitude at every surface where the inclination of the principal ray is w_v. In the formulæ relating to oblique refraction, where the angles of incidence j_v and j_v' occur only as cosine functions, we may accordingly replace j_v by w_v. The various values of w_v are obtained from the equation

$$n'_v \sin w'_v = n_v \sin w_v \ (v = 1 \ldots k).$$

Since n'_v is identical with n_{v+1} and, moreover, in a centred system of lenses

$$w'_v = w_{v+1} \ (v = 1 \ \ldots\ldots \ k-1),$$

it follows that

$$n_{v+1} \sin w_{v+1} = n_v \sin w_v \ (v = 1 \ \ldots\ldots \ k-1).$$

The intercepts of the oblique pencils are obtained accordingly from the general formulæ §§ 92(14) and 93(16), viz.:

$$\frac{n_{k+1}\cos^2 w'_k}{t'_k} = \frac{n_1 \cos^2 w_1}{t_1} + \sum_{v=1}^{k} \frac{n_{v+1}\cos w_{v+1} - n_v \cos w_v}{r_v},$$

$$\frac{n_{k+1}}{f'_k} = \frac{n_1}{f_1} + \sum_{v=1}^{k} \frac{n_{v+1}\cos w_{v+1} - n_v \cos w_v}{r_v}.$$

Further, supposing the system of lenses to be surrounded by air, then

$$n_{k+1} = n_1 = 1,$$

also

$$w'_k = w_1,$$

and, if in the summation we separate the first and last terms, the remainder may be arranged as follows

$$\frac{1}{t'_k} = \frac{1}{t_1} + \frac{1}{\cos^2 w_1}\left\{\cos w_1 \left(\frac{1}{r_k} - \frac{1}{r_1}\right) + \sum_{v=1}^{k-1} \frac{n_{v+1}\cos w_{v+1}}{n_{v+1}-1}\phi_v\right\},$$

$$\frac{1}{f'_k} = \frac{1}{f_1} + \cos w_1 \left(\frac{1}{r_k} - \frac{1}{r_1}\right) + \sum_{v=1}^{k-1} \frac{n_{v+1}\cos w_{v+1}}{n_{v+1}-1}\phi_v,$$

If now we consider the equation which may be derived from § 88(9),

$$\frac{1}{r_k} - \frac{1}{r_1} = -\sum_{v=1}^{k-1} \frac{\phi_v}{n_{v+1}-1},$$

we may write

$$\frac{1}{t'_k} = \frac{1}{t_1} + \frac{1}{U \cos w_1}; \quad \frac{1}{f'_k} = \frac{1}{f_1} + \frac{\cos w_1}{U} \quad \ldots \quad \text{(i)}$$

where

$$\frac{1}{U} = \sum_{v=1}^{k-1} \frac{\frac{n_{v+1}\cos w_{v+1}}{\cos w_1} - 1}{n_{v+1}-1}\phi_v.$$

It will be seen from this equation that U is dependent only upon the power of the lenses and not upon the relation of the radii.

If we are interested to know the two abscissæ $\overline{s}'_k$ and $\overline{\overline{s}}'_k$, as formulated in § 34(ix) and § 44(vi), we may obtain these quantities very simply from t_k' and f_k' since $\phi = 0$, viz.

$$\overline{s}'_k = t'_k \cos w_1 \, ; \quad \overline{\overline{s}}' = f_k' \cos w_1 \, ,$$

whence, by (i),

$$\frac{1}{\overline{s}'_k} = \frac{1}{t_1 \cos w_1} + \frac{1}{U \cos^2 w_1} \, ; \quad \frac{1}{\overline{\overline{s}}'_k} = \frac{1}{f_1 \cos w_1} + \frac{1}{U} \, . \qquad \text{(ii)}$$

In the case of an object free from astigmatism, when accordingly

$$t_1 = f_1 \, ,$$

this formula assumes the simple form

$$\frac{1}{\overline{s}'_k} - \frac{1}{\overline{\overline{s}}'_k} = \frac{\tan^2 w_1}{U} \, . \quad \ldots \quad .. \qquad \text{(iii)}$$

In the case of a system of thin lenses in air, having their diaphragm at the common vertex of the surfaces and transmitting pencils of finite inclination, the reciprocals of the abscissæ on the axis differ by an amount which is independent of the form of the lens components.

In the particular case of an infinitely distant object

$$t_1 = f_1 = \infty$$

and

$$\overline{s}'_k = U \cos^2 w_1 \, ; \quad \overline{\overline{s}}'_k = U \, , \quad \ldots \qquad \text{(iv)}$$

so that finally

$$\overline{\overline{s}}'_k - \overline{s}'_k = U \sin^2 w_1 \, . \quad \ldots \quad \ldots \qquad \text{(v)}$$

B. Formulæ for the Lines of Confusion at Small Inclinations of the Principal Rays.

140. Restricting our investigation to the condition that the inclination w of the principal ray may not contain terms exceeding the second order of magnitude, our next object will be to establish an algebraical relation between the constants of the system, that is the initial and final values of the curvatures of both image surfaces.

We shall proceed in exactly the same manner as in the investigation of the spherical aberration of a point on the axis, and we shall accordingly expand in powers of ϕ the terms occurring in the invariants of the oblique refraction.

Let S be any refracting spherical surface of radius r, P the centre of the diaphragm, and YO the image surface of radius R. Let YP be a principal ray which before refraction passes through the centre P of the diaphragm at an angle w, and intersects the image surface at Y. We must then expand the intercept d_s on this principal ray in terms of ϕ.

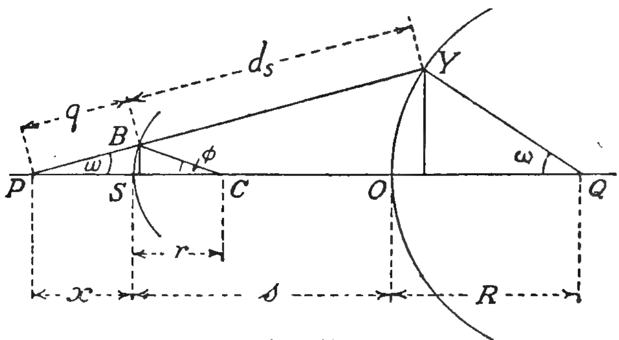

FIG 63.

$SC = r$; $OQ = R$; $BY = d_s$; $SO = s$; $SP = x$.

Curvature of the image. Diagram of notation.

This may be done by separately expanding the segments BP and YP in terms of powers of ϕ and ω respectively. Then

$$BP = q = x\left\{1 - \frac{r^2\phi^2}{2x}\left[\frac{1}{r} - \frac{1}{x}\right]\right\},$$

$$YP = q - d_s = (x - s)\left\{1 - \frac{R^2\omega^2}{2(x-s)}\left[\frac{1}{R} - \frac{1}{x-s}\right]\right\}.$$

Introducing the relation

$$R\omega = r\phi\,\frac{x-s}{x},$$

subtracting one equation from the other, and expressing the result in the form of its reciprocal, we obtain the equation

$$\frac{1}{d_s} = \frac{1}{s} + \frac{A\phi^2}{2},$$

where

$$A = \frac{r^2}{s^2}\left\{\frac{1}{r} - \frac{s}{x^2} - \frac{s^2}{R}\left(\frac{1}{s} - \frac{1}{x}\right)^2\right\}; \quad \dots \quad \text{(i)}$$

and it should be noted that

$$\cos j = 1 - \frac{\phi^2}{2}\left(1 - \frac{r}{x}\right)^2.$$

So far, the curvature which obtains before refraction has been characterized in quite a general way by its radius R. Remembering that there are two distinct image-surfaces, one due to tangential pencils, and the other to sagittal pencils, it is necessary to distinguish two radii, denoting them respectively by R_t and R_f.

To obtain the required relations we shall take the reciprocal of the intercept on the principal ray between the object and the image-surface, which reciprocal $\frac{1}{d_s}$ has just been expressed generally as a function of ϕ, r, R, x, s; and we shall introduce it, first as a tangential quantity and then as a sagittal quantity, in the invariants of the oblique refraction, as defined in § 136.

As before, in § 120, we therefore write

$$Q_t = Q + q_t \frac{\phi^2}{2} = Q' + q'_t \frac{\phi^2}{2};\ q_t = q'_t;$$

$$Q_f = Q + q_f \frac{\phi^2}{2} = Q' + q'_f \frac{\phi^2}{2};\ q_f = q'_f.$$

To deduce q_t and q_f it will be necessary to make use of a few simple invariant relations. Confining the operation to the somewhat simpler case of the sagittal rays, we obtain the following results:

$$q = r^2 (Q - Q_s)^2 \left(\frac{1}{nR_f} - \frac{1}{nr}\right) + \frac{r^2}{ns} Q_x^2 + Q_s - 2\,Q_x$$

$$q'_f = r^2 (Q_x - Q_s)^2 \left(\frac{1}{n'R'_f} - \frac{1}{n'r}\right) + \frac{r^2}{n's'} Q_x^2 + Q_s - 2\,Q_x;$$

hence, by subtraction,

$$0 = (Q_x - Q_s)^2 \left\{\frac{1}{n'R'_f} - \frac{1}{nR_f} - \frac{1}{r}\left(\frac{1}{n'} - \frac{1}{n}\right)\right\} + Q_x^2 \left(\frac{1}{n's'} - \frac{1}{ns}\right),$$

or, using our customary notation,

$$\Delta \frac{1}{nR_f} = \frac{1}{r} \Delta \frac{1}{n} - \frac{Q_x^2}{(Q_x - Q_s)^2} \Delta \frac{1}{ns} \quad \ldots \qquad \text{(ii)}$$

Similarly, by the expansion of $q_t - q'_t$ we may find the curvature in the tangential section. Thus

$$\Delta \frac{1}{nR} = \frac{1}{r} \Delta \frac{1}{n} - \frac{3\,Q_x^2}{(Q_x - Q_s)^2} \Delta \frac{1}{ns} \qquad \ldots \qquad \text{(iii)}$$

The summation of these differences in the curvatures of the images over the whole of the k surfaces, which will cause all but the last and first terms to vanish on the left side, gives the following equations, namely :

$$\left.\begin{aligned}\frac{1}{n'_k R'_{t\,k}} - \frac{1}{n_1 R_{t\,1}} &= \sum_{v=1}^{k} \frac{1}{r_v} \underset{v}{\Delta} \frac{1}{n} - 3 \sum_{v=1}^{k} \frac{Q^2_{v\,x}}{(Q_{v\,x} - Q_{v\,s})^2} \underset{v}{\Delta} \frac{1}{ns}, \\ \frac{1}{n'_k R'_{f\,k}} - \frac{1}{n_1 R_{f\,1}} &= \sum_{v=1}^{k} \frac{1}{r_v} \underset{v}{\Delta} \frac{1}{n} - \sum_{v=1}^{k} \frac{Q^2_{v\,x}}{(Q_{v\,x} - Q_{v\,s})^2} \underset{v}{\Delta} \frac{1}{ns}.\end{aligned}\right\} \text{(iv)}$$

If we assume, as will generally be the case, that the object is plane and free from astigmatism,

$$R_1 = R_{f\,1} = \infty,$$

and therefore the second term disappears from the left side of both equations.

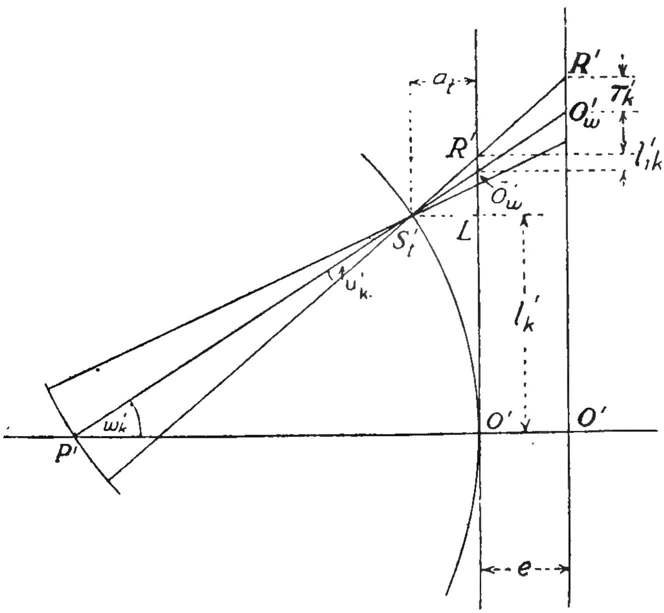

FIG. 64.

$S'_tL = a_t$; $\bar{O}'_wR' = l'_{1k}$; $O'L = l_k'$; $O'\mathbf{O}' = e$; $\mathbf{O}'_w\mathbf{R}' = \tau'_k$.

The tangential line of confusion in the Gauss image-plane and a plane parallel to it.

In this connection we shall proceed to establish a formula for the deterioration of the image within the Gauss plane which is associated with the curvature of the image. To this end we

shall introduce the aperture angles u'_k, v'_k of the oblique pencil. We shall thus obtain the two linear distances l'_{1k} and L'_{1k}, which generally differ in length. Denoting by a_t, a_f the perpendicular distance from the Gauss plane,

$$l'_{1k} = \frac{a_t \, u'_k}{\cos^2 w'_k}\,; \qquad L'_{1k} = \frac{a_f \, v'_k}{\cos^2 w'_k}\,.$$

If now we consider only inclinations of the principal ray of the second order, we can put

$$a_t = \frac{l'^2_k}{2 \, R'_{tk}}\,; \qquad a_f = \frac{l'^2_k}{2 \, R'_{fk}}\,,$$

where l'_k denotes the axial distance of the point where the principal ray meets the Gauss plane.

Since a_t, a_f are themselves quantities of the second order, it follows that, to maintain a degree of approximation of the second order, we need only consider the constant term $\cos^2 w'_k$, which forms the denominator of the expressions l'_{1k} and L'_{1k}. We shall then have finally

$$l'_{1k} = \frac{n'_k \, u'_k \, l'^2_k}{2 \, n'_k \, R'_{tk}}\,; \qquad L'_{1k} = \frac{n'_k \, v'_k \, l'^2_k}{2 \, n'_k \, R'_{fk}}\,. \quad \ldots \quad \text{(v)}$$

In the same way as we projected into the object the quantity representing the deterioration of the image, we may similarly project the line of confusion which arises from the curvature of the image, by introducing the quantities $l_1^{(k)}$ and $L_1^{(k)}$ referred to the object in accordance with the formula of Smith-Helmholtz [§ 84 (4)], viz.

$$n_1 \, u_1 \, l_1^{(k)} = n'_k \, u'_k \, l'_{1k}\,; \qquad n_1 \, v_1 \, L_1^{(k)} = n'_k \, v'_k \, L'_{1k}\,;$$

hence

$$n_1 \, u_1 \, l_1^{(k)} = \frac{n'^2_k \, u'^2_k \, l'^2_k}{2 \, n'_k \, R'_{tk}}\,; \qquad n_1 \, v_1 \, L_1^{(k)} = \frac{n'^2_k \, v'^2_k \, l'^2_k}{2 \, n'_k \, R'_{fk}}\,. \quad \text{(vi)}$$

It is also clear that, within the limits which we are here considering, this law is applicable likewise to the conjugate distances from the axis, l'_k and l_1, so that we shall have

$$n_1 \, u_1 \, l_1^{(k)} = \frac{n_1^2 \, u_1^2 \, l_1^2}{2 \, n'_k \, R'_{tk}}\,; \qquad n_1 \, v_1 \, L_1^{(k)} = \frac{n_1^2 \, v_1^2 \, l_1^2}{2 \, n'_k \, R'_f}\,, \quad \text{(vii)}$$

and finally,

$$l_1^{(k)} = \frac{n_1 \, u_1 \, l_1^2}{2} \left\{ \sum_{v=1}^{k} \frac{1}{r_v} \mathop{\Delta}_{v} \frac{1}{n} - 3 \sum_{v=1}^{k} \frac{Q^2_{v,x}}{(Q_{vx} - Q_{vs})^2} \mathop{\Delta}_{v} \frac{1}{ns} \right\} \quad \text{(viii)}$$

$$L_1^{(k)} = \frac{n_1 \, v_1 \, l^2_1}{2} \left\{ \sum_{v=1}^{k} \frac{1}{r_v} \mathop{\Delta}_{v} \frac{1}{n} - \sum_{v=1}^{k} \frac{Q^2_{vx}}{(Q_{vx} - Q_{vs})^2} \mathop{\Delta}_{v} \frac{1}{ns} \right\}\,.$$

We can give these formulæ a different form by employing the equation given in § 83 (2), viz.

$$Q_{vx} - Q_{vs} = \frac{h_1\, y_1}{h_v\, y_v}(Q_{1x} - Q_{1s}):$$

$$\left.\begin{aligned} l_1^{(k)} &= \frac{n_1\, u_1\, l^2_1}{2\,(Q_{1x} - Q_{1s})^2}\left\{(Q_{1x} - Q_{1s})^2 \sum_{v=1}^{k}\frac{1}{r_v}\underset{v}{\Delta}\frac{1}{n}\right. \\ &\left.\quad - 3\sum_{v=1}^{k}\left(\frac{h_v}{h_1}\right)^2\left(\frac{y_v}{y_1}\right)^2 Q^2_{vx}\underset{v}{\Delta}\frac{1}{ns}\right\} \\ L_1^{(k)} &= \frac{n_1\, v_1\, l_1^2}{2\,(Q_{1x} - Q_{1s})^2}\left\{(Q_{1x} - Q_{1s})^2 \sum_{v=1}^{k}\frac{1}{r_v}\underset{v}{\Delta}\frac{1}{n}\right. \\ &\left.\quad - \sum_{v=1}^{k}\left(\frac{h_v}{h_1}\right)^2\left(\frac{y_v}{y_1}\right)^2 Q^2_{vx}\underset{v}{\Delta}\frac{1}{ns}\right\}. \end{aligned}\right\} \quad \text{(ix)}$$

Noting, as in § 132 (vi), that

$$\frac{n_1}{(Q_{1x} - Q_{1s})^2} = \frac{s_1^2\, x_1^2}{n_1\,(x_1 - s_1)^2}$$

and putting

$$u_1 = \frac{m_1}{s_1 - x_1}; \qquad v_1 = \frac{M_1}{s_1 - x_1},$$

we obtain, after a simple re-arrangement of the terms, another form, which we shall have occasion to refer to later, viz.

$$\left.\begin{aligned} \frac{n_1 l_1^{(k)}}{s_1} &= \frac{m_1 l_1^2 s_1 x_1^2}{2\,(x_1 - s_1)^3}\sum_{v=1}^{k}\left(\frac{h_v}{h_1}\right)^2\left(\frac{y_v}{y_1}\right)^2\left\{3\, Q^2_{vx}\underset{v}{\Delta}\frac{1}{ns}\right. \\ &\left.\quad - (Q_{vx} - Q_{vs})^2\frac{1}{r_v}\underset{v}{\Delta}\frac{1}{n}\right\} \\ \frac{n_1 L_1^{(k)}}{s_1} &= \frac{M_1 l_1^2 s_1 x_1^2}{2\,(x_1 - s_1)^3}\sum_{v=1}^{k}\left(\frac{h_v}{h_1}\right)^2\left(\frac{y_v}{y_1}\right)^2\left\{Q^2_{vx}\underset{v}{\Delta}\frac{1}{ns}\right. \\ &\left.\quad - (Q_{vx} - Q_{vs})^2\frac{1}{r_v}\underset{v}{\Delta}\frac{1}{n}\right\}. \end{aligned}\right\} \quad \text{(x)}$$

C. Astigmatism.

141. We shall now assume that the image-receiving plane, though at right angles to the axis, is not identical with the Gauss image-plane, being, in fact, at a distance e from the latter. On reference to Fig. 64 we shall then have in the new image-receiving plane the lines of distortion

$$\tau'_k = e\, u'_k + \frac{u'_k\, l'^2_k}{2\, R_{tk}}; \qquad \sigma'_k = e\, v'_k + \frac{v'_k\, l'^2_k}{2\, R_{fk}} \quad \ldots \quad \text{(i)}$$

There is accordingly a particular position of the image-receiving plane which causes τ'_k and σ'_k to have equal values of opposite sign for any given aperture angles $u'_k = v'_k$. When this occurs the value of e, as determined by equations § 140 (vii) and (viii), is

$$e = -\frac{n'_k l'^2_k}{2}\left\{\sum_{v=1}^{k}\frac{1}{r_v}\underset{v}{\Delta}\frac{1}{n} - 2\sum_{v=1}^{k}\frac{Q^2_{vx}}{(Q_{vx} - Q_{vs})^2}\underset{v}{\Delta}\frac{1}{ns}\right\}$$

and substituting this value in (i), we obtain

$$-\tau'_k = \sigma'_k = \frac{n'_k u'_k l'^2_k}{2}\sum_{v=1}^{k}\frac{Q^2_{vs}}{(Q_{vx} - Q_{vs})^2}\underset{v}{\Delta}\frac{1}{ns}. \quad \ldots \text{(ii)}$$

It will be seen that the expression for the line of distortion contains only the difference of the two expressions for the curvature of the image, so that it may also be regarded as an expression for the astigmatism. Projecting this line of distortion back into the object we obtain in accordance with the previous expansion § 140 (vi) and (x), two alternative expressions:

$$-\tau_1^{(k)} = \sigma_1^{(k)} = \frac{n_1 u_1 l_1^2}{2}\sum_{v=1}^{k}\frac{Q_{vx}^2}{(Q_{vx} - Q_{vs})^2}\underset{v}{\Delta}\frac{1}{ns} \qquad \text{(iii)}$$

and

$$-\tau_1^{(k)} = \sigma_1^{(k)} = \frac{u_1 l_1^2 x_1^2 s_1^2}{2 n_1 (x_1 - s_1)^2}\sum_{v=1}^{k}\left(\frac{h_v}{h_1}\right)^2\left(\frac{y_v}{y_1}\right)^2 Q^2_{vx}\underset{v}{\Delta}\frac{1}{ns} \qquad \text{(iv)}$$

By introducing the two relations

$$\frac{l_1}{x_1 - s_1} = \frac{y_1}{x_1}; \qquad h_1 = s_1 u_1,$$

(iv) assumes the form

$$c = -\tau_1^{(k)} = \sigma_1^{(k)} = \frac{h_1^2 y_1^2}{2 n_1 u_1}\sum_{=1}^{k}\left(\frac{h_v}{h_1}\right)^2\left(\frac{y_v}{y_1}\right)^2 Q^2_{vx}\underset{v}{\Delta}\frac{1}{ns}. \qquad \text{(v)}$$

The condition for the elimination of astigmatism is that the summation should reduce to zero. In this form the equation shares with that given in § 122 (iii) for the spherical aberration of a point on the axis, the advantage of being suitable for practical computation. It will be seen that, here also, the extent to which each surface contributes to the resulting aberration appears as a separate term.

D. The Curvature of the Image in Anastigmatic (or Stigmatic) Systems.

142. The remaining part common to both formulæ for the curvature of the image, § 140 (viii), is

$$\frac{1}{R} = \sum_{v=1}^{k}\frac{1}{r_v}\underset{v}{\Delta}\frac{1}{n}. \quad \ldots \quad \ldots \quad \ldots \text{(i)}$$

This residual expression indicates the curvature of the two coincident image surfaces when there is no astigmatism, and is known as **Petzval's equation for the curvature of the image.** Seidel (*3.*), in particular, has pointed out that this equation has no significance unless the astigmatism has previously been eliminated in the case of small inclinations of the principal ray. Systems which have been so corrected are described as **anastigmatic** or **stigmatic,** the latter, which has been adopted by British opticians, being the more expressive term.

The above expression is reduced readily to the following form :

$$\left.\begin{aligned}\sum_{v=1}^{k}\frac{1}{r_v}\Delta_v\frac{1}{n} &= \sum_{v=1}^{k}\left\{\frac{1}{r_v}\left(\frac{1}{n_{v+1}}-1\right)-\frac{1}{r_v}\left(\frac{1}{n_v}-1\right)\right\}\\ &= \sum_{v=1}^{k}\left\{\frac{1}{r_v}\left(\frac{1-n_{v+1}}{n_{v+1}}\right)-\frac{1}{r_v}\left(\frac{1-n_v}{n_v}\right)\right\}\\ &= \frac{1-n_{k+1}}{n_{k+1}\,r_k}-\sum_{v=1}^{k-1}\frac{1}{n_{v+1}\,f_v}-\frac{1-n_1}{n_1\,r_1},\end{aligned}\right\}\quad\text{(ii)}$$

where f_v is the focal length of a thin lens of refractive index n_{v+1} and radii r_v and r_{v+1}.

In the case of a system in air $n_{k+1} = n_1 = 1$, and hence the first and last terms of the right side of the equation disappear. The formula thus becomes

$$\frac{1}{R} = -\sum_{\mu=1}^{l}\frac{1}{n_\mu\,f_\mu}, \quad \dots \quad \dots \quad \text{(iii)}$$

where n_μ and f_μ are the refractive index and the focal length respectively of the μ^{th} lens and l is the number of the lenses.

E. The Astigmatism in Simple Special Cases.

143. In discussing the astigmatism arising in special cases we shall proceed from the formula § 141 (v) established for $\sigma_1^{(k)}$.

144. The Astigmatism of a Single Surface.—In this case, making use of the expression obtained for $\Delta\,\dfrac{1}{ns}$ in § 123 (iv) and putting $\dfrac{y}{x} = w$, we obtain

$$\sigma_1^{(k)} = \frac{h^2w^2}{2ur^2}\,\frac{n(n'-n)}{n'^2}\,(x-r)^2\left(\frac{1}{r}-\frac{n'+n}{n}\,\frac{1}{s}\right). \quad \dots \quad \text{(i)}$$

Disregarding the case when $s = 0 = h$, the astigmatism can accordingly be made to vanish only under two alternative conditions, viz.:

(a) when $x = r$, in which case the diaphragm coincides with the centre of curvature of the surface; or

(b) when $s = \frac{n' + n}{n} r$, in which case the object and its image represent a pair of aplanatic points.

When $s = \infty$ the sign of $\frac{n' - n}{r}$ determines the character of the resulting astigmatism, whilst the position of the stop exercises no influence upon it.

145. Astigmatism of a Single Thin Lens.—To ascertain the astigmatism in this case it is necessary to determine the sum occurring in the expression for the astigmatic error, § 141 (v). As in § 135, if we express the unknown quantities x, s, r in terms of ξ, σ, ρ, this sum is obtained from the expression

$$\left.\begin{aligned} L &= L_o - L_1 \rho + L_2 \rho^2, \\ \text{where} \\ L_o &= \frac{n^2}{(n-1)^2} \phi^3 + \frac{n+1}{n-1} \phi^2 \sigma + \frac{2n}{n-1} \phi^2 \xi \\ &\quad + 2 \frac{n+1}{n} \phi \sigma \xi + \phi \xi^2 \\ L_1 &= \frac{2n+1}{n-1} \phi^2 + 2 \frac{n+1}{n} \phi (\xi + \sigma) \\ L_2 &= \frac{n+2}{n} \phi . \end{aligned}\right\} \quad \text{(i)}$$

Proceeding as in the investigation of the distortion in § 135, we may determine the minimum value A_{min}, which occurs when

$$P_{min} = \frac{n+1}{n+2} (\Sigma + \Xi) + \frac{(2n+1) n}{2(n-1)(n+2)} . \qquad \dots \quad \text{(ii)}$$

When the terms are suitably re-arranged, the value of A_{min} is

$$A_{min} = \frac{n^2}{4(n-1)^2} - \frac{1}{n(n+2)} \left((n+1) \Sigma - \Xi + \frac{n}{2} \right)^2 . \quad \text{(iii)}$$

This represents an equation containing the co-ordinates Σ and Ξ as well as the parameter A_{min} of a pair of parallel straight lines, which is reduced to two identical straight lines when $A_{min} = \frac{n^2}{4(n-1)^2}$, whereas when $A_{min} > \frac{n^2}{4(n-1)^2}$ the line is imaginary.

Each value of P_{min} has corresponding to it in the plane of $\Sigma\ \Xi$, as in the case of the distortion, a straight line which is one of a family of parallel straight lines.

The astigmatism A corresponding to any given curvature P can be very simply expressed with the aid of the minimum values, as follows:

$$A = A_{min} + \frac{n+2}{n}(P - P_{min})^2. \quad \dots \quad \text{(iv)}$$

5. THE ABERRATIONS OF EXTRA-AXIAL POINTS DEPENDENT UPON THE SECOND POWERS OF THE ANGULAR APERTURE.

(The Three Defects of Coma in a Wider Sense.)

146. In the investigation of the astigmatism in § 137 and § 138 we ascertained that the lines of confusion for principal rays inclined at finite angles are dependent upon the first powers of the aperture co-ordinates m and M. We have now to investigate the conditions which must be satisfied in order that a sharp image may be formed by pencils of greater aperture, whose principal rays are inclined at finite angles so as to include small quantities of the second order.

To make sure that we may not overlook any of the possible aberrations we shall consider all terms of the second order which can be formed of the two aperture co-ordinates with respect to either plane pencil. We shall then obtain

for the tangential aberrations:—terms involving m^2, mM, M^2,
for the sagittal aberrations:—terms involving M^2, Mm, m^2.

This number of six possible terms decreases, however, very considerably in accordance with a generally applicable principle, according to which the figure of confusion appearing in the Gauss image-plane is necessarily symmetrical with respect to the meridian plane which passes through the object point, when the boundary of the aperture is likewise symmetrical to it.

The point where the meridian plane is intersected by two skew rays which are symmetrical, and which accordingly correspond with two values of M of equal magnitude but opposite sign, is necessarily independent of the sign of M, since both rays intersect in the meridian plane. A direct consequence of this is that the common tangential co-ordinates of the two points where the skew rays intersect the Gauss image-plane are independent of the sign of M. Hence expressions for the tangential aberrations can only contain terms involving even powers of M, and within the limits of

our investigation this includes only terms of the 0^{th} and 2^{nd} degree. From this it is clear that the tangential aberrations involve only terms containing m^2 and M^2.

The sagittal aberrations are subject to precisely the opposite conditions in that they undergo a change of sign whenever M changes its sign. The expressions by which their magnitudes are represented contain accordingly only odd powers of M, and, within the limits of this investigation, these are exclusively terms of the first power of M. In the expressions relating to sagittal aberrations we are accordingly concerned only with the term containing Mm.

We shall now proceed to establish the expressions for these three terms, in which the aperture co-ordinates are of the second order. We shall first consider the term containing m^2.

A. Coma in a Restricted Sense.

147. Dissymmetry in Tangential Pencils of Larger Aperture.—In Fig. 65 let a pencil of rays which are parallel to the axis meet a spherical surface, and let this pencil be bounded by a concentric diaphragm $P_1 P_2$ (which for the sake of simplicity is represented in the diagram as coinciding with the vertex S of the surface). After refraction, which we shall assume to have a converging effect, this pencil exhibits spherical aberration.

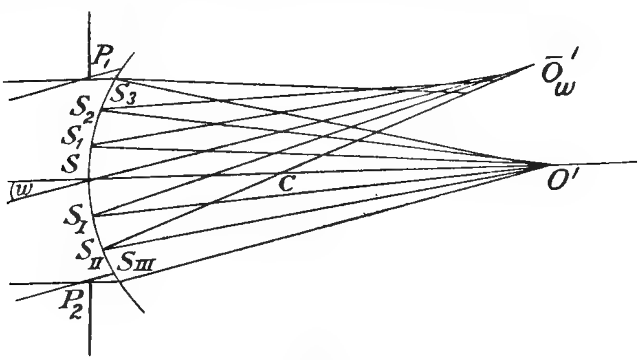

Fig. 65.

Dissymmetry in the path of the rays of a tangential pencil of finite aperture and having its principal ray inclined at an angle w to the axis.

The longitudinal aberration will then increase in proportion to the square of the angle subtended at the centre by the incident pencil, if we confine ourselves to the first two terms of the expansion. This condition is represented in the neighbourhood of O' for six equidistant points of incidence S_3 to S_{III} along the arc of the circle.

An entirely different condition arises when the incident principal ray passes through the centre of the stop S at a finite angle w, in which case we may regard the principal ray as the mean ray of a cylindrical pencil limited by the boundary of the stop. If we exclude the case, which is not one of oblique refraction, in which the stop is situated at the centre of curvature of the surface, we may still find a straight line parallel to the elements of the incident pencil passing through C. We shall refer to this line as the **auxiliary axis.** This auxiliary axis may lie within, or without the boundary of the stop. In the case represented in the diagram the auxiliary axis lies within the boundary of the stop, and for the sake of simplicity we shall assume that it passes exactly through S_{II}. Now, this auxiliary axis behaves as the axis of the oblique pencil exactly in the same way as the principal axis of the system behaves as the axis of the direct pencil of rays. The upper part of Fig. 65 shows for the six selected rays the longitudinal aberrations along this auxiliary axis which result if in the graphic construction we treat the latter as though it were the principal axis of the pencil.

It will be noticed that by far the greater portion of the pencil lies on the upper side of the auxiliary axis. This dissymmetry can be distinctly seen at $\bar{O}'_w$, if we construct the rays in the same way as was done in the case of the principal axis. In the longitudinal aberration along the auxiliary axis, this dissymmetry, which necessarily arises unless the principal axis be made to coincide with the auxiliary axis by means of a specially arranged stop, is not, however, the only difference between direct and oblique refraction.

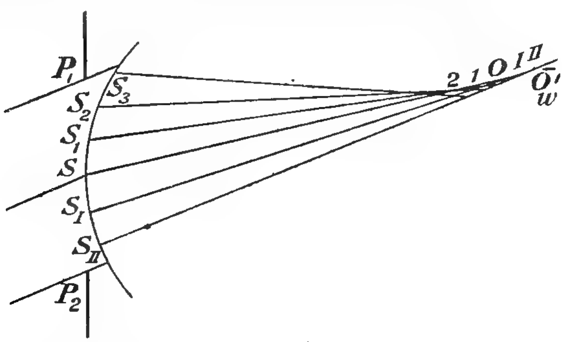

FIG. 66.

Longitudinal aberration of a tangential pencil of finite aperture and having its principal ray inclined at a finite angle.

In Fig. 66, in which the upper part only of the diagram Fig. 65 has been reproduced, the principal ray passing through S appears again still more clearly as the optical centre-line of figure of the pencil. Determining on it the points of intersection of the four representative rays nearest to it, we obtain the range of points

2,1,0,I.II, where 0 denotes the point of intersection of adjacent tangential rays. In other words, the points of intersection follow in the same order of succession as the points of incidence reckoned from their extreme positive position through zero to their extreme negative position. This case invariably arises whenever the selected representative rays are sufficiently near to the principal ray; for, as we have indicated above, the principal ray cannot under any circumstances coincide with the auxiliary axis when we are dealing with oblique refraction.

The longitudinal aberration of oblique pencils may be represented graphically in a similar manner as was suggested in the case of the primary zonal term of the longitudinal spherical aberration. From the preceding considerations it follows, however, that the construction should not be restricted to positive angles.

It does not appear that curves of this kind have yet been published.

The dissymmetry of the path of the rays within a tangential pencil, which has been treated geometrically so as to exhibit it in a striking manner, may also be investigated analytically.

From the investigation of the spherical aberration of points on the axis we know already from § 119 that

$$\boldsymbol{s}' = s' + \mathsf{A}' u'^2 \ldots \qquad \ldots \qquad \ldots \qquad \text{(i)}$$

If now two rays inclined at angles u'_1 and u'_2 are selected, it follows that

$$\boldsymbol{s}'_1 - \boldsymbol{s}'_2 = \mathsf{A}' (u'^2_1 - u'^2_2)$$

and if we adopt the sign convention indicated in Fig. 67 for the portion $\delta t'$ on the ray inclined at an angle u'_1, then

$$\frac{\delta t'}{\boldsymbol{s}'_1 - \boldsymbol{s}'_2} = -\frac{\sin u'_2}{\sin (u'_2 - u'_1)}.$$

This quantity $\delta t'$ is sufficient for our purpose, in that it enables us to investigate the variation of the intercept on a principal ray inclined at an angle u'_1 as a function of the angular aperture $\mathrm{u}' = u'_1 - u'_2$ of the oblique pencil.

If, accordingly, we substitute for $\boldsymbol{s}'_1 - \boldsymbol{s}'_2$ its value and for u'_2 its equivalent

$$u'_2 = u'_1 - (u'_1 - u'_2) = u'_1 - \mathrm{u}',$$

and finally expand the trigonometrical functions as far as the second power, we obtain

$$\delta t' = \mathsf{A}' (2\, u'^2_1 - 3\, u'_1\, \mathrm{u}' + \mathrm{u}'^2). \qquad \ldots \qquad \text{(ii)}$$

From this formula it will be seen that at a sufficiently small angular aperture u' the magnitude of $\delta t'$ varies with the first power of u'. This, as well as our previous investigation, shows the longitudinal aberration to be dependent upon the sign of the angular aperture.

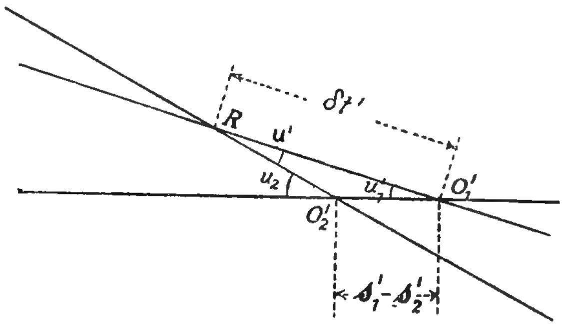

Fig. 67.

$O'_2\,O'_1 = s'_1 - s'_2$; $RO'_1 = \delta t'$

Analytical investigation of the longitudinal aberration of an obliquely incident tangential pencil.

148. Formula for the Line of Confusion.—The continuity of the range of points representing the longitudinal aberration on the principal axis, which was shown above to be a characteristic feature of dissymmetrically incident rays, can now be represented as a function in which the odd terms preponderate, at all events in the neighbourhood of the principal ray. If we confine ourselves to the term of the first order we obtain for the intercept of a tangential pencil having an aperture u of the first order of smallness the expression

$$t = t + \mathfrak{r}\,u \; \ldots \qquad \ldots \qquad \ldots \qquad \text{(i)}$$

The aggregate of rays contracts more on one side of the principal ray (in our case the lower), and on this side there is an aberration which lies within the meridian plane. This aberration arises from the dissymmetry of the paths of the rays and may be called **coma in a restricted sense.** Originally, the term **coma** was applied to the whole of the defects of definition arising from the obliquity of a pencil, including those in a sagittal direction, since in its optical effects the general appearance is comparable to the tail of a comet. When referring to these aberrations collectively we prefer to describe them as **coma in the wider sense.**

For the purpose of mathematical investigation we shall simplify the conditions. In Fig. 68, let BD, RE represent two elements of a plane pencil, BD being the principal ray and RE an adjacent

ray inclined to BD at a small angle du. The symbol du, like u previously, signifies a small quantity of the first order, and has been chosen because in what follows we shall have to resort to differentiation. Let A be the tangential image-point as defined by immediately adjacent rays of the pencil and determinable by the formulæ of §§ 34 and 93. At this point A let the image-receiving plane be described

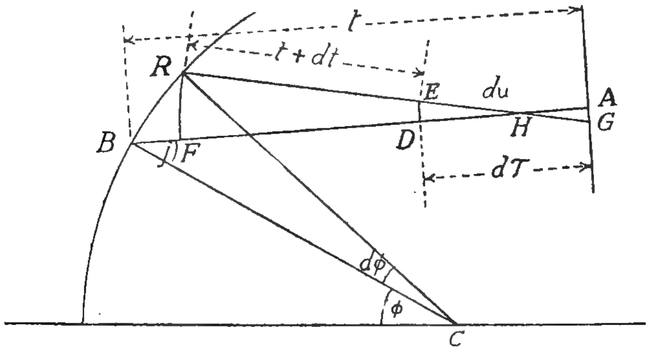

FIG. 68.

$BA = t$; $RE = t + dt$

Investigation of the first tangential line of confusion.

at right angles to the principal ray. Let this plane be intersected at G by the adjacent ray RE at an angle du to BA, so that AG represents the aberration to be determined. Let the adjacent ray RE have its tangential image-point at E. The notation will accordingly be

$$BA = t\ ;\ RE = t + dt\ .$$

The increment dt of the tangential intercept t with respect to du can be resolved into two parts, one of which depends upon the changes in the co-ordinates of the point of incidence R, whilst the other which interests us more particularly, shows the **displacement of the tangential image-point** E dependent on du. We shall denote this displacement by $d\tau$. If now we project upon BA the points R and E by arcs of a circle about the centre H we obtain points F and D, and, if we put $d\tau = AD$, we have the following relations :

$$DA = BA - BF - FD\,,$$

that is

$$-\,d\tau = t - r\,d\,\phi\,\sin j - t - dt$$

or

$$dt = d\tau - r\,d\,\phi\,\sin j\,. \qquad \ldots \qquad \ldots \qquad \text{(ii)}$$

The value of dt is obtained by partial differentiation with respect to ϕ and t of the invariant of the tangential rays, as given in §§ 93 (16) and 136, viz.:

$$Q_t = \frac{n \cos j}{r} - \frac{n \cos^2 j}{t},$$

and eliminating dt with the aid of the following relations given in §§ 93 and 12

$$\frac{dj}{d\phi} = 1 - \frac{du}{d\phi}; \quad \frac{du}{d\phi} = \frac{r \cos j}{t}; \quad J = n \sin j$$

and an equation which follows from (i) and § 93 (ii), viz.:

$$\frac{dt}{d\phi} = \frac{d\tau}{du}\frac{du}{d\phi} - r \sin j = \frac{r \cos j}{t}\frac{d\tau}{du} - r \sin j.$$

The differentiation of Q_t with respect to ϕ, after simplification, furnishes the following equation:

$$\frac{1}{r}\frac{dQ_t}{d\phi} = -\frac{J}{r^2} + \frac{3\,JQ_t}{nt} + \frac{n \cos^3 j}{t^3}\frac{d\tau}{du}$$

and similarly

$$\frac{1}{r}\frac{dQ_t}{d\phi} = -\frac{J}{r^2} + \frac{3\,JQ_t}{n't'} + \frac{n' \cos^3 j'}{t'^3}\frac{d\tau'}{du'}$$

and hence

$$\Delta \frac{n \cos^3 j}{t^3}\frac{d\tau}{du} = -3\,JQ_t\,\Delta\frac{1}{nt}. \quad \ldots \quad \text{(iii)}$$

In this formula it is now permissible to introduce the relation

$$\frac{du'}{du} = \frac{t \cos j'}{t' \cos j},$$

although we are now concerned with an increased aperture, because any correcting terms would introduce aberrations of a higher order. It is evident that

$$\frac{d\tau_{v+1}}{du_{v+1}} = \frac{d\tau'_v}{du'_v}$$

and hence, if there are k surfaces, the above recurrence formula furnishes the following relation:

$$\frac{n'_k \cos^3 j'_k}{t'^3_k}\frac{d\tau'_k}{du'_k} = n_1\left(\frac{h_{1t}}{h_{kt}}\right)^3 \cdot \frac{\cos^3 j_1}{t_1^3}\frac{d\tau_1}{du_1} - 3\left(\frac{h_{1t}}{h_{kt}}\right)^3 \sum_{v=1}^{k}\left(\frac{h_{vt}}{h_{1t}}\right)^3 . J_v Q_{tv} \underset{v}{\Delta} \frac{1}{nt} \text{ (iv)}$$

where in a similar manner as in § 99 (28),

$$\frac{\mathrm{h}_{\nu t}}{\mathrm{h}_{1t}} = \frac{t_\nu}{t'_{\nu-1}} \; \frac{t_{\nu-1}}{t'_{\nu-2}} \; \cdots \; \frac{t_2}{t'_1} \; \frac{\cos j'_{\nu-1}}{\cos j_\nu} \; \frac{\cos j'_{\nu-2}}{\cos j_{\nu-1}} \cdots \frac{\cos j'_1}{\cos j_2} \, . \quad \text{(v)}$$

If we further assume the pencil to be free from aberration at incidence the first term vanishes, and in this case we must write

$$\frac{n'_k \cos^3 j'_k}{t'^3_k} \frac{d\tau'_k}{du'_k} = -3 \left(\frac{\mathrm{h}_{1t}}{\mathrm{h}_{kt}}\right)^3 \sum_{\nu=1}^{k} \left(\frac{\mathrm{h}_{\nu t}}{\mathrm{h}_{1t}}\right)^3 J_\nu Q_{t\nu} \underset{\nu}{\Delta} \frac{1}{nt} \, . \quad \text{(vi)}$$

If now we proceed to determine the lines of confusion we must express AH in terms of known quantities.

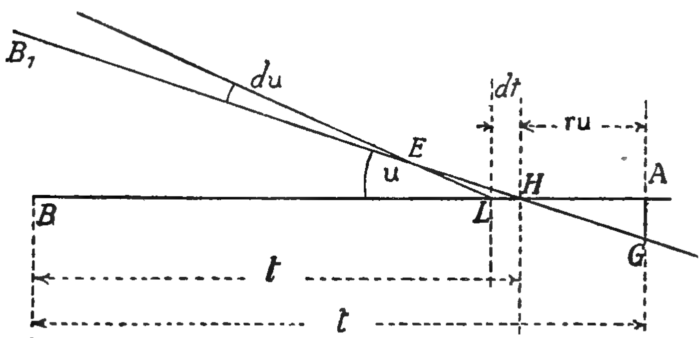

FIG. 69.
Curvature of the caustic curve.

Equation (i) viz. :

$$\mathbf{t} = t + \mathrm{r}\,\mathrm{u}\,,$$

in which we confined ourselves to small quantities u reckoned outwards from the principal ray BA, gives the intercept of a ray inclined at the angle u, viz. :

$$BH = \mathbf{t} = t + \mathrm{r}\,\mathrm{u}\,.$$

Thus in Fig. 69 we can put

$$AH = \mathrm{r}\,\mathrm{u}\,.$$

We will now suppose the secondary ray B_1H, inclined at an angle u to the axis, to carry a tangential pencil of very small aperture du, E being the tangential image-point of this ray B_1H. Let this ray be produced to meet the principal ray BA at L. Then clearly,

$$HL = d\mathbf{t} = \mathrm{r}\,d\mathrm{u}\,,$$

and from the triangle EHL we find by the application of the sine relation

$$\frac{HL}{HE} = \frac{\mathfrak{r}\, d\mathrm{u}}{HE} = \frac{d\mathrm{u}}{\mathrm{u} + d\mathrm{u}} \,.$$

From this, if we neglect $d\mathrm{u}$ in comparison with u, we find

$$HE = \mathfrak{r}\,\mathrm{u}$$

and the increment $d\tau$ of the tangential point of intersection is related to u by

$$d\tau = AH + HE = 2\,\mathfrak{r}\,\mathrm{u}\,, \quad \ldots \qquad \ldots \quad \text{(vii)}$$

so that, if we again substitute the symbol $d\mathrm{u}$ for u to denote its order of magnitude,

$$d\tau = 2\,AH = 2\,HE = 2\,\mathfrak{r}\,d\mathrm{u}\,.$$

Geometrically, the coefficient $\mathfrak{r} = \dfrac{d\tau}{2\,d\mathrm{u}}$ in the expansion denotes accordingly the radius of curvature of the caustic curve.

The length of the tangential line of confusion $AG = l_2$ which appears in an image-receiving plane at A at right angles to BA is

$$l_2 = AH \cdot \mathrm{u} = (\mathfrak{r}\,\mathrm{u})\,\mathrm{u} = \frac{d\tau}{2}\,\mathrm{u}\,,$$

where the small angle is again denoted by u instead of $d\mathrm{u}$. If now, we project this tangential line of confusion back into the object,—a procedure which has already been repeatedly adopted,—and attach the surface index, we know by the formula of Smith-Helmholtz § 84 (4) that

$$n'_k\, l'_{2k}\, \mathrm{u}'_k = n_1\, l_2^{(k)}\, \mathrm{u}_1\,.$$

Reverting to equation (vi) for $d\tau'_k$, we see accordingly that

$$\frac{n'_k \cos^3 j'_k}{t'^3_k}\,\frac{d\tau'_k}{\mathrm{u}'_k} = 2\,\frac{n'_k\, l'_{2k}\, \mathrm{u}'_k \cos^3 j'_k}{t'^3_k\, \mathrm{u}'^3_k} = 2\,\frac{n'_k\, l'_{2k}\, \mathrm{u}'_k}{\left(\dfrac{\mathrm{h}_{kt}}{\mathrm{h}_{1t}}\right)^3 \left(\dfrac{t_1}{\cos j_1}\right)^3 \mathrm{u}_1^3}\,,$$

hence, finally,

$$l_2^{(k)} = -\,\frac{3}{2}\,(n_1\,\mathrm{u}_1)^2 \left(\frac{t_1}{n_1 \cos j_1}\right)^3 \sum_{v=1}^{k} \left(\frac{\mathrm{h}_{vt}}{\mathrm{h}_{1t}}\right)^3 J_v\, Q_{tv}\, \underset{v}{\Delta}\, \frac{1}{nt}\,. \qquad \text{(viii)}$$

149. In the special case of a **system of planes** the summation can be simplified very considerably.

In the first place, it is to be noted that the invariant J_v is reduced in this case to a constant with regard to the index of summation. Moreover, by § 100 (ii),

$$\frac{1}{n't'_v} - \frac{1}{n_v t_v} = \frac{1}{n_v t_v}\left(\frac{n_v^2 \cos^2 j_v - n'^2_v \cos^2 j'_v}{n'^2_v \cos^2 j'_v}\right) = \frac{n_v^2 - n'^2_v}{n_v\, n'^2_v \cos^2 j'_v} \cdot \frac{1}{t_v},$$

so that in the case of a system of planes we obtain by § 136 the following simplified expressions:

$$Q_{vt} \mathop{\Delta}_{v} \frac{1}{nt} = \frac{n'_v \cos^2 j'_v}{t'_v} \mathop{\Delta}_{v} \frac{1}{nt} = \frac{n_v^2 - n'^2_v}{n_v\, n_v'} \frac{1}{t_v\, t_v'} = \frac{n_v\, n_v'}{t_v\, t_v'} \mathop{\Delta}_{v} \frac{1}{n^2}.$$

150. Proceeding finally to principal rays inclined at angles w of the first order of magnitude, we may introduce the following simplifications:

$$t = s\,; \quad \cos j_1 = 1\,; \quad \frac{\mathrm{h}_{vt}}{\mathrm{h}_{1t}} = \frac{h_v}{h_1}\,; \quad J_v = y_v\, Q_{vx}\,; \quad Q_{tv} = Q_{vs}.$$

For the focus-plane through the tangential image-point at right angles to the principal ray, we obtain accordingly by § 148 (vi) the following expression for the tangential line of confusion:

$$l_2^{(k)} = -\frac{3}{2}(n_1\, \mathrm{u}_1)^2 \left(\frac{s_1}{n_1}\right)^3 y_1 \sum_{v=1}^{v} \left(\frac{h_v}{h_1}\right)^3 \frac{y_v}{y_1} Q_{vx}\, Q_{vs} \mathop{\Delta}_{v} \frac{1}{ns}. \quad \text{(i)}$$

In deducing the expression for the line of confusion AG, it was assumed that the image-receiving plane is situated at A at right angles to BA. When this plane is displaced parallel to itself through the distance $A\bar{A}$, the new line of confusion will be $\bar{A}\bar{G}$ (Fig. 70), so that

$$\bar{A}\bar{G} = AG + AG\,\frac{A\bar{A}}{HA}.$$

If now $A\bar{A}$ be an infinitesimal of a higher order than HA, the magnitude of the line of confusion does not change at all, since in this case $\bar{A}\bar{G} = AG$.

This case arises whenever the line of confusion is due to a small inclination w of the principal ray within the Gauss image-plane. The expansion of the expression for the coma is applicable for the first power of the co-ordinate l_1 of the object, whereas the distance of the tangential image-plane (*i.e.*, the versed sine $A\bar{A}$ of the tangential image-plane) is proportional to the second power of l_1. Hence HA, being a quantity of the first order, is of a lower order than $A\bar{A}$.

Finally, the magnitude of the line of confusion within the Gauss image-plane at right angles to the principal axis is $AG \cos w$. In our case $\cos w$ differs from unity only by a small quantity of the second order, so that it need not be considered.

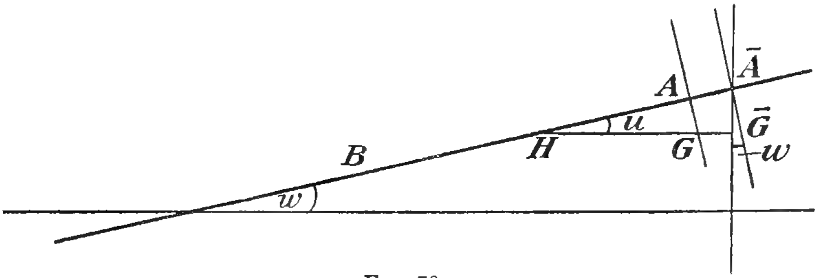

FIG. 70.

The first tangential line of confusion referred to the Gauss image-plane, the principal ray being inclined at a small angle.

Since we are confining ourselves to image defects of the third order, it follows that the formula given for $l_2^{(k)}$ supplies likewise an expression for the tangential aberration in the Gauss plane involving the second power of m. The condition for the elimination of this defect in the definition of the image, *i.e.*, for the **comatic correction in the restricted sense,** is that the terms of the summation should vanish.

By substituting the linear co-ordinates for the angular co-ordinates with the aid of the equations given in § 115, we may write the last expression for $l_2^{(k)}$ as follows:

$$\frac{n_1 l_2^{(k)}}{s_1} = -\frac{3}{2}\frac{m_1^2 l_1}{(x_1 - s_1)^3} s_1^2 x_1 \sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^3 \frac{y_v}{y_1} Q_{vs} Q_{vc} \mathop{\Delta}_{v} \frac{1}{ns} . \qquad \text{(ii)}$$

B. The Trough Defect.

151. The Tangential Aberration in Sagittal Pencils of Considerable Aperture.—In the investigation of the intercepts f of sagittal pencils of small aperture we regarded them as plane pencils. This mode of treatment was applicable to the immediate neighbourhood of the principal ray, inasmuch as the plane at right angles to the principal plane (*i.e.*, the plane of the paper) containing the principal ray at an angle of inclination w to the axis, was actually tangential to the entire cone composed of all rays inclined at an angle w. If now, we increase the aperture of the sagittal pencils, we are no longer justified in supposing that the point of

intersection on the principal ray of adjoining rays will likewise be the point of intersection of these wider pencils. We are not even at liberty to assume, in general, that these wider pencils will necessarily meet the principal ray after refraction, when, as a matter of fact, they will merely cross it. All that we can say with certainty is that two symmetrically disposed sagittal rays will meet at a point situated in the meridian plane.

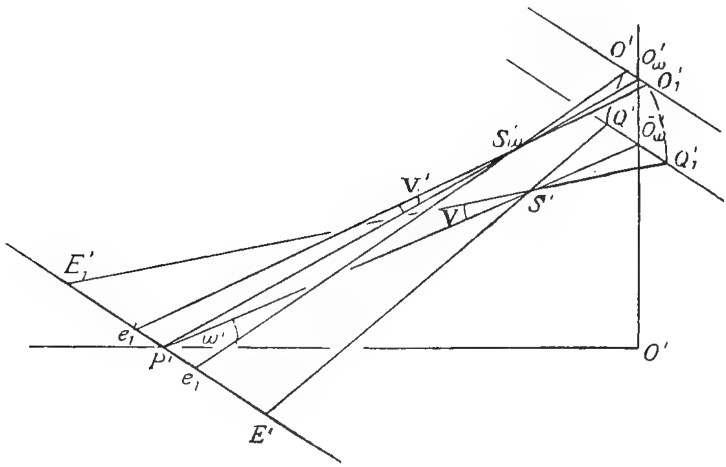

FIG. 71.

Trough-like deformation of a sagittal pencil of considerable aperture.

The above diagram Fig. 71 illustrates the influence which this supposed tangential aberration of wider sagittal pencils has upon the nature of the figure of confusion. In the diagram it is supposed that the ray proceeds from the middle of the aperture plane at a finite inclination w' to the axis. It will then be seen that if we suppose the image-receiving plane to be placed beyond the point of intersection S'_w of the adjacent rays, the intercepted rays will describe on it a figure having, within the assumed limits of accuracy, a parabolic boundary whose successive points correspond to successive elements of the straight line $E'_1P'E'$. The entire pencil is deformed into the shape of a trough, and by reason of this property this species of aberration is appropriately described as the **trough defect.**

In the position of the image-receiving plane here assumed, the depth of the trough $O'_w\ \bar{O}'_w$ represents the tangential aberration of the wider sagittal pencil. As the image-receiving plane is made to approach the plane of the aperture this part of the aberration

will be at first but little affected, whereas on a receiving plane at S' the transverse dimensions corresponding to E'_1E' will vanish entirely, whilst, in the nature of things, the segment corresponding to O_1O_1' cannot at that point have any appreciable breadth. In this case, therefore, the aberration will reduce to a mere tangential line passing through S'.

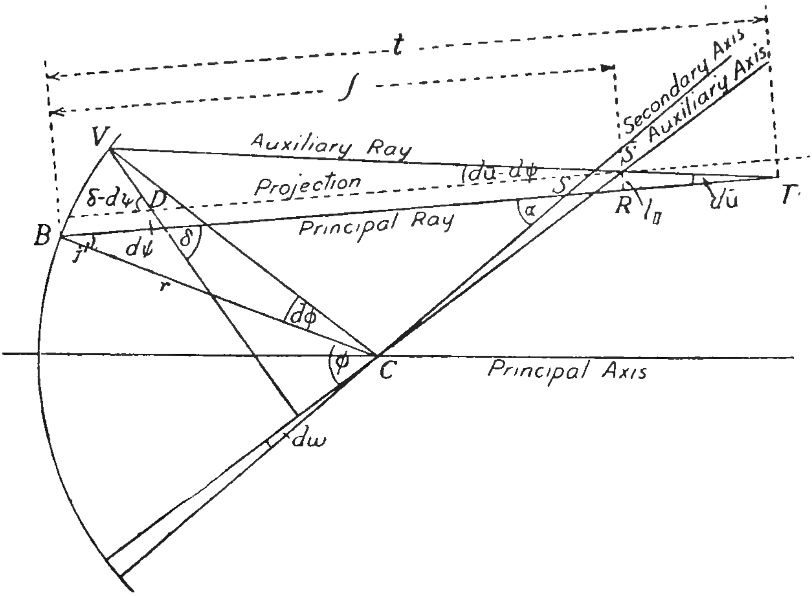

FIG. 72.

$BC = r$; $R\overline{S} = l_{II}$; $DV = D$; $BT = t$; $\delta = 90° - \alpha - d\omega$.

The tangential line of confusion in the sagittal pencil of wider aperture.

152. The Expression for the Line of Confusion.—To obtain a mathematical expression for the aberration which we have just considered, we shall proceed to investigate the relations which prevail in the case of a single surface. With respect to the latter, let the principal ray of the sagittal pencil be defined by the intersection of adjacent sagittal rays at S, so that the straight line through S and the centre C of the spherical surface constitutes the **secondary axis** of the surface. Let a skew ray inclined at an angle dv to the principal plane (the plane of the paper) intersect this plane at the point $\overline{S}$, which naturally is also the point where it meets its projection, and let the latter be inclined at an angle $d\psi$ to the principal ray. The straight line joining $\overline{S}$ and the centre C determines the **auxiliary axis** of the system. If now we suppose the skew ray to be rotated about the auxiliary axis, the resulting cone

will intersect the plane of the paper along the auxiliary ray, which obviously also passes through $\bar{S}$, and we shall suppose that it intersects the principal ray at T at an angle $d\bar{u}$.

In a pencil of infinitely small aperture, S and $\bar{S}$ represent adjacent points, and the distance $S\bar{S}$ is then infinitely small compared with the distance ST, since ST, which is the astigmatic difference of a pencil of finite inclination, has a finite length. Putting $BT = t$ we may thus write

$$l_{II} = (t - f)\, d\bar{u} = t f \left(\frac{1}{f} - \frac{1}{t}\right) d\bar{u} \quad \ldots \quad \ldots \quad \text{(i)}$$

and, by § 94 (24),

$$-\Delta \frac{n l_{II}}{t f d\bar{u}} = \Delta n \left(\frac{1}{t} - \frac{1}{f}\right) = n^2 \sin^2 j \Delta \frac{1}{n t} . \quad \ldots \quad \text{(ii)}$$

The value of the newly introduced term t in the transition from one medium to another, may be calculated by a formula which is analogous to that employed in the calculation of the tangential intercepts (§ 93 (iv)), viz.

$$\Delta \frac{n \cos^2 j}{t} = \frac{1}{r} \Delta (n \cos j) ,$$

whilst below we shall give a formula for the transition from one surface to another.

By reasoning analogous to that applied in § 93 (17) to tangential rays, we have the following relation with respect to the angles $d\bar{u}$ and $d\bar{u}'$, namely

$$d\bar{u}' = \frac{t \cos j'}{t' \cos j}\, d\bar{u} , \quad \ldots \quad \ldots \quad \text{(iii)}$$

and with the aid of this equation the above formula (i) for the difference of the sagittal intercepts becomes

$$-\Delta \frac{n l_{II}}{f \cos j} = J^2 \frac{t\, d\bar{u}}{\cos j} \Delta \frac{1}{n t} . \quad \ldots \quad \text{(iv)}$$

The next step is to express $d\bar{u}$ in terms of dv. The cone described about the auxiliary axis maps out a circle on the spherical surface whose plane is at right angles to the auxiliary axis and to the plane of the paper, and whose periphery contains the point of incidence of the skew ray. The versed sine D in the plane of the paper corresponds with the ordinate H of the point of incidence above the plane of the paper, where this circle is intersected. If we denote the radius of the circle provisionally by R we shall have

$$D = \frac{H^2}{2\mathsf{R}}$$

We may thus write directly $H = f \cdot d\text{v}$, since it has been shown above that the infinitesimal quantity $S\bar{S}$ vanishes with respect to the finite distance $D\bar{S} = BS$, nearly, and for R we clearly have the expression

$$\mathsf{R} = r \sin(\phi + d\phi + d\omega) = r \sin \phi\,;$$

hence D becomes

$$D = \frac{f^2}{2\, r \sin \phi}\, d\,\text{v}^2\,.$$

Further, neglecting infinitesimals in comparison with finite quantities,

$$d\bar{\text{u}} - d\psi = \frac{D}{f} \sin(90^\circ - a - d\omega - d\psi) = \frac{D \cos a}{f}\;;$$

also, since

$$CS = \frac{f - r\cos j}{\cos a} = \frac{f \sin j}{\sin \phi}\;;$$

it follows that

$$d\bar{\text{u}} - d\psi = \frac{D\,(f - r\cos j)}{f^2 \sin j} \sin \phi = \frac{d\text{v}^2}{2r}\,\frac{f - r\cos j}{\sin j} \ldots \qquad \text{(v)}$$

$$d\text{v}' = \frac{f}{f'}\, d\text{v}\,. \quad \ldots \quad \ldots \quad \ldots \qquad \text{(vi)}$$

Introducing the surface index v, the expression becomes after refraction through the v^{th} surface and before refraction at the $(v+1)^{\text{th}}$ surface

$$d\bar{\text{u}}'_v - d\psi'_v = \frac{d\text{v}'^2_v}{2}\,\frac{f'_v - r_v \cos j'_v}{r_v \sin j'_v}$$

$$d\bar{\text{u}}_{v+1} - d\psi_{v+1} = \frac{d\text{v}^2_{v+1}}{2}\,\frac{f_{v+1} - r_{v+1}\cos j_{v+1}}{r_{v+1}\sin j_{v+1}}\,,$$

and, since

$$d\psi'_v = d\psi_{v+1}\,; \qquad d\text{v}'_v = d\text{v}_{v+1}\,,$$

the subtraction of one equation from the other furnishes the following important relation between successive values of $d\bar{\text{u}}_f$:

$$d\bar{\text{u}}_{v+1} = \frac{d\text{v}^2_{v+1}}{2}\left[\frac{f_{v+1} - r_{v+1}\cos j_{v+1}}{r_{v+1}\sin j_{v+1}} - \frac{f'_v - r_v \cos j'_v}{r_v \sin j'_v}\right] + d\bar{\text{u}}'_v\,. \text{(vii)}$$

Moreover, obviously

$$l_{II\,v+1} = l'_{II\,v}\,,$$

and, by (i),

$$l_{II\,v+1} = (\text{t}_{v+1} - f_{v+1})\, d\bar{\text{u}}_{v+1}\,,$$

so that

$$\text{t}_{v+1} = f_{v+1} + \frac{l_{II\,v+1}}{d\bar{\text{u}}_{v+1}}\,. \quad \ldots \quad \ldots \qquad \text{(viii)}$$

It is not advisable to combine all the formulæ into a single expression by introducing the equivalent terms for t and f, so long as the assumption is retained that the principal ray of the pencil is inclined at a finite angle, since to do so would obscure the formulæ.

153. Trough Defect in a System of Plane Surfaces.—In the special case of a system of plane surfaces, it is practicable to condense the successive changes into a single formula. Since in this case all radii are infinite, equations § 152 (vii) for $d\bar{u}$ become

$$d\bar{u}_{v+1} = \frac{dv^2_{v+1}}{2}\left[\cot j'_v - \cot j_{v+1}\right] + d\bar{u}_v \quad \dots \qquad \text{(i)}$$

and, by §§ 152 (iii) and 100 (ii),

$$d\bar{u}'_v = \frac{n_v \cos j_v}{n'_v \cos j'_v}\, d\bar{u}_v . \quad \dots \quad \dots \qquad \text{(ii)}$$

But since

$$j_{v+1} = j'_v$$

the first equation reduces to

$$du_{v+1} = d\bar{u}'_v ,$$

and from this in conjunction with equation (ii), *i.e.*

$$n'_v \cos j_v' d\bar{u}'_v = n_v \cos j_v d\bar{u}_v ,$$

it follows finally that

$$du_v = \frac{n_1 \cos j_1}{n'_{v-1} \cos j'_{v-1}}\, d\bar{u}_1 = \frac{n_1 \cos j_1}{n_v \cos j_v}\, d\bar{u}_1 . \quad \dots \text{(iii)}$$

Further, from § 100 (ii) it follows that

$$t_v \underset{v}{\Delta} \frac{1}{nt} = - \frac{\underset{v}{\Delta}\, n^2}{n_v (n'_v \cos j'_v)^2} ,$$

so that finally, reverting to § 152 (iv), we obtain

$$\underset{v}{\Delta} \frac{nl_{II}}{f \cos j} = - J_v^2 \frac{t_v \underset{v}{\Delta} \frac{1}{nt}}{\cos j_v}\, d\bar{u}_v = J^2 \frac{n_1 \cos j_1}{(n_v \cos j_v\, n'_v \cos j'_v)^2} \underset{v}{\Delta}\, n^2 d\bar{u}_1 . \text{(iv)}$$

Assuming the object to be initially free from aberration, if we form the last term by means of this recurrence formula, we obtain

$$\frac{n_{k+1} l'_{IIk}}{f'_k \cos j'_k} = J^2 n_1 \cos j_1\, d\bar{u}_1 \frac{h_{1f}}{h_{kf}} \sum_{v=1}^{k} \frac{h_{vf}}{h_{1f}} \frac{\underset{v}{\Delta}\, n^2}{(n_v \cos j_v\, n'_v \cos j'_v)^2} \qquad \text{(v)}$$

where, by § 120 (xi),

$$\frac{h_{kf}}{h_{1f}} = \frac{f_k}{f'_{k-1}} \cdot \frac{f_{k-1}}{f'_{k-1}} \cdots\cdots \frac{f_2}{f_1} . \quad \dots \qquad \text{(vi)}$$

Since in the case of an incident plane pencil $d\psi_1$ vanishes, we can now write in accordance with § 152 (v),

$$d\bar{\mathfrak{u}}_1 = -\frac{d\mathrm{v}^2{}_1}{2}\cot j_1 \quad ,$$

and since

$$f'_k\,\mathrm{v}'_k = -\frac{h_{kf}}{h_1} f_1\,\mathrm{v}_1 \quad ,$$

it follows from equation (v), when v_1 is substituted for $d\mathrm{v}_1$, that

$$l'_{II\,k} = -\frac{n^3{}_1}{n_{k+1}}\cos j'_k \cos{}^2 j_1 \sin j_1 \frac{\mathrm{v}^3{}_1}{\mathrm{v}'_k}\sum_{v=1}^{k}\frac{h_{vf}}{h_{1f}}\frac{\underset{v}{\Delta} n^2}{(n_v\cos j_v\, n'_v\cos j'_v)^2}\,. \text{ (vii)}$$

Projecting the line of confusion back into the object and replacing in the coefficient of the sum the values of j in terms of w, we obtain the following expression for the line of confusion in a system of planes

$$l^{(k)}{}_{II} = n^2{}_1\,\mathrm{v}^2{}_1 \sin w_1 \cos w'_k \cos{}^2 w_1 \sum_{v=1}^{k}\frac{h_{vf}}{h_{1f}}\frac{\underset{v}{\Delta} n^2}{(n_v\cos j_v\, n'_v\cos j'_v)^2}\,. \text{ (viii)}$$

154. In the general case of **finite radii,** a self-contained formula can only be obtained when the **principal rays are inclined at infinitely small angles.** In this case we may put

$$f = s = \mathrm{t}\ ;\ \cos \mathrm{j} = 1\ ;\ J = n\sin j = y\,Q_x\ ,$$

also

$$\frac{\mathrm{h}_{vf}}{\mathrm{h}_{1f}} = \frac{h_v}{h_1}\ ,$$

and, putting at the same time $\bar{\mathfrak{u}}_v$ for $d\bar{\mathfrak{u}}_v$, and v_v for $d\mathrm{v}_v$, we obtain

$$-\,l'_{II\,v} = -\frac{n_v\,s'_v}{n'_v\,s_v}\,l_{II\,v} + y^2{}_v\frac{Q^2{}_{vx}}{n'_v}\,s'_v{}^2\,\underset{v}{\Delta}\frac{1}{ns}\,\bar{\mathfrak{u}}'_v\ .\ \dots \qquad \text{(i)}$$

From this equation we have first to eliminate $\bar{\mathfrak{u}}'_v$.

Now, by § 152 (vii), since $s\bar{\mathfrak{u}} = s'\bar{\mathfrak{u}}'$,

$$s'_v\,\bar{\mathfrak{u}}'_v = s_v\,\bar{\mathfrak{u}}_v = \frac{s_v\,\mathrm{v}^2{}_v}{2}\left[\frac{s_v - r_v}{r_v\sin j_v} - \frac{s'_{v-1} - r_{v-1}}{r_{v-1}\sin j'_{v-1}}\right] + s_v\,\bar{\mathfrak{u}}'_{v\text{-}1}$$

$$= \frac{s^2{}_v\,\mathrm{v}_v{}^2}{2}\left[\frac{Q_{vs}}{y_v\,Q_{vx}} - \frac{s'_{v-1}}{s_v}\,\frac{Q_{v-1\,s}}{y_{v-1}\,Q_{v-1\,x}}\right] + \frac{s_v}{s'_{v-1}}\,s_{v-1}\,\bar{\mathfrak{u}}_{v-1}\,.$$

Also, since

$$s_v\,s'_{v-1}\,\mathrm{v}^2{}_v = \frac{s_v\,s'^2{}_{v-1}\,\mathrm{v}'^2{}_{v-1}}{s'_{v-1}} = \frac{s_v}{s'_{v-1}}\,s^2{}_{v-1}\,\mathrm{v}^2{}_{v-1}\ ,$$

it follows from the last equation that

$$s_v\,\bar{u}_v - \frac{s_v^2\,v_v^2}{2}\,\frac{Q_{vs}}{y_v\,Q_{vx}} = \frac{s_v}{s'_{v-1}}\left[s_{v-1}\,\bar{u}_{v-1} - \frac{s^2_{v-1}\,v^2_{v-1}}{2}\,\frac{Q_{v-1s}}{y_{v-1}\,Q_{v-1x}}\right],$$

so that finally

$$s_v\,\bar{u}_v - \frac{s_v^2\,v_v^2}{2}\,\frac{Q_{vs}}{y_v\,Q_{vx}} = \frac{s_v}{s'_{v-1}}\,\frac{s_{v-1}}{s'_{v-2}}\,.\;.\;.\;.\;.\;\frac{s_2}{s'_1}\left[s_1\,u_1 - \frac{s_1^2\,v_1^2\,Q_{1s}}{2\,y_1\,Q_{1x}}\right].$$

Now, in an incident plane pencil, since $\psi_1 = 0$, the general equation § 152 (v), viz.

$$\bar{u}_1 - \psi_1 = \frac{s_1\,v^2_1\,Q_{1s}}{2\,y_1\,Q_{1x}}$$

becomes

$$\bar{u}_1 = \frac{s_1\,v^2_1\,Q_{1s}}{2\,y_1\,Q_{1x}}\,,$$

hence

$$s_v\,\bar{u}_v = \frac{s^2_v\,v^2_v}{2}\,\frac{Q_{vs}}{y_v\,Q_{vx}}\,. \qquad \ldots \qquad \ldots \qquad \text{(ia)}$$

Substituting this value of $s_v\,\bar{u}_v = s'_v\,\bar{u}'_v$ in equation (i), we obtain the expression

$$-\,l'_{IIv} = -\,\frac{n_v\,s'_v}{n'_v\,s_v}\,l_{IIv} + \frac{s'_v}{n'_v}\,\frac{s^2_v\,v^2_v}{2}\,y_v\,Q_{vx}\,Q_{vs}\,\underset{v}{\Delta}\,\frac{1}{ns}\,. \qquad \text{(ii)}$$

This expression supplies a recurrence formula which, when applied to k surfaces, assumes the following resultant form :

$$-\,l'_{IIk} = \frac{s'_k}{2\,n'_k}\,\frac{h_1}{h_k}\,s^2_1\,v^2_1\,y_1\sum_{v=1}^{k}\left(\frac{h_v}{h_1}\right)^3\frac{y_v}{y_1}\,Q_{vx}\,Q_{vs}\,\underset{v}{\Delta}\,\frac{1}{ns}\,, \;\ldots \qquad \text{(iii)}$$

where the term l_{II1} is reckoned as zero in view of the fact that the pencil which meets the first surface was assumed to be plane.

If now, as before, by means of sagittal pencils we project the line of confusion l'_{IIk} back into the object, it follows from the Smith-Helmholtz theorem, of which we have already made frequent use, that

$$n'_k\,l'_{IIk}\,v'_k = n_1\,l^{(k)}_{II}\,v_1\,,$$

hence

$$l'_{IIk} = \frac{n_1}{n'_k}\,\frac{v_1}{v'_k}\,l^{(k)}_{II} = \frac{n_1}{n'_k}\,\frac{s'_k}{s_1}\,\frac{h_1}{h_k}\,l^{(k)}_{II}\,,$$

and, after appropriate substitutions.

$$l^{(k)}_{II} = -\,\frac{1}{2}\,(n_1\,v_1)^2\left(\frac{s_1}{n_1}\right)^3 y_1\sum_{v=1}\left(\frac{h_v}{h_1}\right)^3\frac{y_v}{y_1}\,Q_{vx}\,Q_{vs}\,\underset{v}{\Delta}\,\frac{1}{ns}\,. \qquad \text{(iv)}$$

At the outset the image-receiving plane was supposed to be at the point of intersection of adjacent sagittal rays. If now we imagine, in the first instance, the image-receiving plane to be displaced parallel to itself from the sagittal point of intersection to the position on the axis of the Gauss image-plane, the case will be analogous to that represented in Fig. 64, and, accordingly, $S'_f L = \frac{l^2}{2R_f}$, where l is a small quantity of the first order. Hence the increments occasioned in l'_{IIk} and $l^{(k)}_{II}$ respectively, are of a higher order than the third, to which we have restricted the investigation.

From reasoning similar to that given at the end of § 150, it follows that we need not here consider the effect of the rotation, through a small angle w about the image-point on the axis, of the image-receiving plane after displacement parallel to itself. The investigation which gave the value $l^{(k)}_{II}$ is applicable also to the tangential line of confusion in the Gauss image-plane projected back into the object.

If now we combine into a single expression the two tangential aberrations indicated in §§ 150 (ii) and 154 (iv), which we have shown to be the only two tangential aberrations of this kind that can arise, and then substitute in place of the angular apertures the corresponding expressions in terms of the co-ordinates of the rays, we obtain the following equation :

$$\frac{n_1}{s_1}(l^{(k)}_2 + l^{(k)}_{II}) = -\frac{(3\,m^2{}_1 + M^2{}_1)\,l_1}{2\,(x_1 - s_1)^3} s^2{}_1 x_1 \sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^3 \frac{y_v}{y_1}\, Q_{vs}\, Q_v\, \underset{v}{\Delta} \frac{1}{ns}. \qquad \text{(v)}$$

This discloses the notable fact that the two tangential aberrations of tangential and sagittal pencils of small finite aperture, whose principal rays are inclined at small angles of the first order, have identically similar coefficients, and hence either will vanish if the other vanishes. This condition is expressed by the equation

$$\sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^3 \frac{y_v}{y_1}\, Q_{vs}\, Q_{vx}\, \underset{v}{\Delta} \frac{1}{ns} = 0\,. \qquad \dots \qquad \text{(vi)}$$

C. The Comatic Triangle Defect.

155. The Tangential Difference of the Sagittal Intercepts.—We shall now turn our attention to the third species of comatic aberration. As we know already, this aberration extends in the sagittal direction and its magnitude depends upon the product $M'm'$. If now we give the factor m different values m' and $\overline{m}'$,

the various mean rays in the meridian plane corresponding to different values of u contain different sagittal intercepts, the terminal points of which will be denoted by S' and $\bar{S}'$.

In Fig. 73, where O'_w and $\bar{O}'_w$ are the points in which a plane at O' at right angles to the axis is intersected by the rays through the image-points S' and $\bar{S}'$, the lines of confusion L'_2 and $\bar{L}'_2$ in that plane corresponding to the image-points for a pencil of aperture v are

$$L'_2 = O'_w\, S'.\, \mathrm{v},$$
$$\bar{L}'_2 = \bar{O}'_w\, \bar{S}'.\, \mathrm{v}.$$

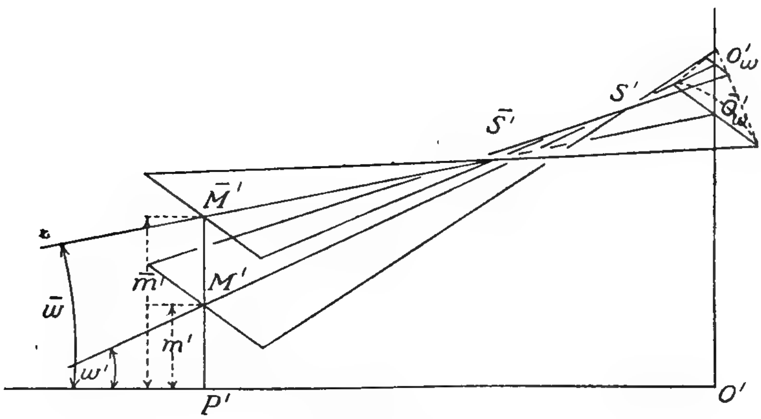

FIG. 73.

$P'M' = m'$; $P'\bar{M}' = \bar{m}'$.

Formation of the comatic triangle defect.

The length of the line of confusion is accordingly proportional to the distance of the image-point from the line of confusion, and within the first degree of approximation, which alone concerns us here, it is a linear function of the aperture co-ordinate m'.

From this consideration we are led to the conclusion that in the case of a rectangular aperture the figure of confusion may be represented by the dotted contours shown in the diagram; and it will be seen that the contour of the figure of aberration is a triangle. Its apex is formed by the intersection with the sagittal pencil upon which the Gauss image-plane happens to be focussed, whilst the side opposite the apex exhibits the comatic trough defect, which, however, does not concern us here. The other two sides are straight lines. In view of the triangular contour of the figure, this defect may conveniently be referred to as the **comatic triangle defect.**

156. Expression for the Line of Confusion due to the Comatic Triangle Defect.—We shall first investigate this aberration in the case of a single surface. In accordance with the general discussion in the preceding article, it is necessary to determine in the first instance the increment of the sagittal intercept with respect to variations of u (also denoted by du for reasons already given).

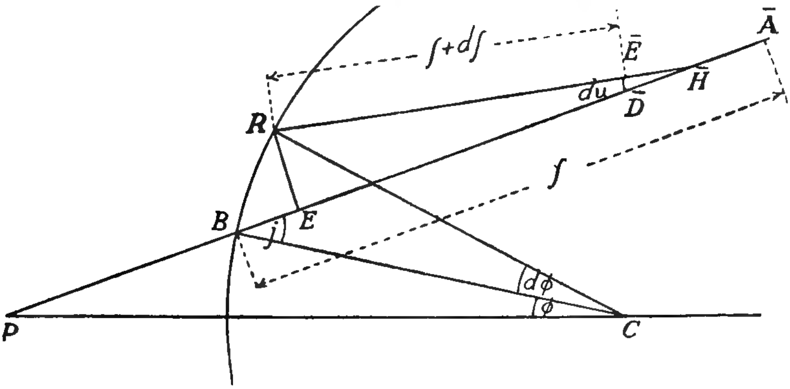

FIG. 74.

$B\bar{A} = f$; $R\bar{E} = f + df$.

Illustrating the expression for the sagittal line of confusion.

In Fig. 74 we have represented two elements of the tangential pencil, viz., the principal ray $P\bar{A}$ and an adjoining ray $R\bar{E}$ inclined to it at an angle du. Let $\bar{A}$ be the position of the sagittal image-point as determined by adjacent sagittal rays. Let the adjoining ray $R\bar{E}$ have its sagittal image-point at $\bar{E}$. As in previous analogous cases, we may introduce the relations

$$B\bar{A} = f;\ R\bar{E} = f + df.$$

The increment df of the sagittal intercept f with respect to du is composed of two parts. The first of these, with which we are already acquainted, is dependent upon variations in the co-ordinates of the point of incidence R, whilst the other indicates the **displacement of the sagittal image-point** $\bar{E}$, which is a function of du. We shall denote this displacement by $d\zeta$. If now we project the points R and $\bar{E}$ upon $B\bar{A}$ by means of arcs of a circle about $\bar{H}$, we shall obtain the points F and $\bar{D}$, and, since $d\zeta = \bar{A}\bar{D}$, we have

$$\bar{D}\bar{A} = B\bar{A} - BF - F\bar{D},$$

that is $$- d\zeta = f - rd\phi \sin j - f - df,$$

or $$df = d\zeta - rd\phi \sin j.$$

We can find the value of $d\mathfrak{f}$ by differentiating the invariant of the sagittal rays (§ 136), viz :

$$Q_{\mathfrak{f}} = \frac{n \cos j}{r} - \frac{n}{\mathfrak{f}}$$

with respect to ϕ. As in § 148,

$$\frac{dj}{d\phi} = 1 - \frac{du}{d\phi} \; ; \quad \frac{du}{d\phi} = \frac{r \cos j}{t} \; ; \quad J = n \sin j \, ,$$

also noting that

$$\frac{d\mathfrak{f}}{d\phi} = \frac{d\zeta}{du} \cdot \frac{du}{d\phi} - r \sin j = \frac{r \cos j}{t} \cdot \frac{d\zeta}{du} - r \sin j \, ,$$

we obtain, by proceeding as in § 148, and after simplification,

$$\frac{1}{r}\frac{dQ_{\mathfrak{f}}}{d\phi} = -\frac{J}{r^2} + \frac{J \cos j}{rt} - \frac{J}{\mathfrak{f}^2} + \frac{n}{\mathfrak{f}^2}\frac{\cos j}{t}\frac{d\zeta}{du}$$

and, similarly,

$$\frac{1}{r}\frac{dQ_{\mathfrak{f}}}{d\phi} = -\frac{J}{r^2} + \frac{J \cos j'}{rt'} - \frac{J}{\mathfrak{f}'^2} + \frac{n'}{\mathfrak{f}'^2}\frac{\cos j'}{t'}\frac{d\zeta'}{du'} \, ;$$

therefore

$$\Delta \, \frac{n \cos j}{\mathfrak{f}^2 t}\frac{d\zeta}{du} = J \, \Delta \left[\frac{1}{\mathfrak{f}^2} - \frac{\cos j}{rt}\right]. \quad \ldots \quad \text{(i)}$$

As in § 148 and previous similar cases, we are justified in introducing in these expressions the ratio

$$\frac{du'}{du} = \frac{t \cos j'}{t' \cos j} \, ,$$

and if the pencil at incidence is assumed to be free from aberration, in which case

$$d\zeta_1 = 0,$$

we obtain the following final expression :

$$\frac{n'_k \cos j'_k \, d\zeta'_k}{\mathfrak{f}'^2_k \, t'_k \, du'_k} = \frac{\mathrm{h}_{1t}}{\mathrm{h}_{kt}}\left(\frac{\mathrm{h}_{1\mathfrak{f}}}{\mathrm{h}_{k\mathfrak{f}}}\right)^2 \sum_{v=1}^{k} \frac{\mathrm{h}_{vt}}{\mathrm{h}_{1t}}\left(\frac{\mathrm{h}_{v\mathfrak{f}}}{\mathrm{h}_{1\mathfrak{f}}}\right)^2 J_v \, \underset{v}{\Delta}\left(\frac{1}{\mathfrak{f}^2} - \frac{\cos j}{rt}\right). \text{(ii)}$$

The terms $\frac{\mathrm{h}_{kt}}{\mathrm{h}_{1t}}$ and $\frac{\mathrm{h}_{k\mathfrak{f}}}{\mathrm{h}_{1\mathfrak{f}}}$ occurring in this expression have already been defined in § 148 (v) and § 153 (vi).

Noting the equations, which follow from § 82 and § 99, viz.:

$$f'_k \, \mathrm{v}'_k = \frac{\mathrm{h}_{kf}}{\mathrm{h}_{1f}} f_1 \, \mathrm{v}_1 \, ; \quad t'_k \, \frac{d\mathrm{u}'_k}{\cos j'_k} = \frac{\mathrm{h}_{kt}}{\mathrm{h}_{1t}} \frac{t_1}{\cos j_1} \, d\mathrm{u}_1 \, ,$$

and multiplying the numerator and denominator of the left-hand side of equation (ii) by v'^2_k, the latter becomes

$$\frac{n'_k \, \mathrm{v}'^2_k \, d\zeta'_k \cos j'_k}{f'^2_k \, t'_k \, \mathrm{v}'^2_k \, d\mathrm{u}'_k} = \frac{n'_k \, \mathrm{v}'^2_k \, d\zeta'_k}{\left(\frac{\mathrm{h}_{kf}}{\mathrm{h}_{1f}}\right)^2 f_1^2 \, \mathrm{v}_1^2 \, \frac{\mathrm{h}_{kt}}{\mathrm{h}_{1t}} \frac{t_1 \, d\mathrm{u}_1}{\cos j_1}} \, . \qquad \text{(iii)}$$

With the aid of this transformed equation an expression for the sagittal aberration projected into the object may readily be formulated.

As before, we shall now write u for du, adhering, however, to the assumption that these angles are infinitesimals of the first order.

Reverting to the preliminary discussion where $d\zeta = \overline{A}\overline{D}$, we may express the length of the line of confusion with respect to an image-receiving plane at A normal to BA, thus,

$$L'_{2k} = d\zeta'_k \, \mathrm{v}'_k \, .$$

This length, projected back into the object by means of sagittal pencils in accordance with the Smith-Helmholtz formula, gives us

$$n'_k \, \mathrm{v}'_k \, L'_{2k} = n_1 \, \mathrm{v}_1 \, L_2^{(k)} \, ,$$

and noting that

$$n_1 \, \mathrm{v}_1 \, L_2^{(k)} = n_k \, \mathrm{v}'^2_k \, d\zeta'_k \, ,$$

we obtain the following expression for the **sagittal coma**:

$$L^{(k)}_2 = \frac{f_1^2 \, t_1}{n^3 \cos j_1} (n_1 \, \mathrm{v}_1) \, (n_1 \, \mathrm{u}_1) \sum_{v=1}^{k} \frac{\mathrm{h}_{vt}}{\mathrm{h}_{1t}} \left(\frac{\mathrm{h}_{vf}}{\mathrm{h}_{1f}}\right)^2 J_v \underset{v}{\Delta} \left(\frac{1}{f^2} - \frac{\cos j}{rt}\right) . \, \text{(iv)}$$

157. Comatic Triangle Defect in a System of Planes. —By a similar reasoning to that pursued in the investigation of the coma in the tangential section, the formula for the comatic triangle defect can be simplified in the case of a system of planes, in that in the general term we have now

$$\underset{v}{\Delta} \frac{1}{f^2} = \left(\frac{n_v^2}{n'^2_v} - 1\right) \frac{1}{f^2_v} = \frac{n_v^2 - n'^2_v}{n_v n'_v} \; \frac{1}{f_v f'_v} = \frac{n_v n'_v}{f_v f'_v} \underset{v}{\Delta} \frac{1}{n^2} \, .$$

Thus we obtain a form which closely resembles the expression for the comatic aberration in the tangential section.

158. If now we consider the case of small inclinations of the principal ray, it will be evident that

$$t = f = s\,; \quad \cos j_1 = 1\,; \quad J_v = y_v Q_v\,;$$

and

$$\frac{\mathrm{h}_{v\,t}}{\mathrm{h}_{1\,t}} = \frac{\mathrm{h}_{v\,f}}{\mathrm{h}_{1\,f}} = \frac{h_v}{h_1}\,; \quad \underset{v}{\Delta}\left(\frac{1}{f^2} - \frac{\cos j}{rt}\right) = -\, Q_{vs} \underset{v}{\Delta} \frac{1}{ns}\,.$$

From the previous remarks in § 150 we know that within the limits of accuracy to which the investigation is confined · no appreciable error will be introduced by assuming that the Gauss plane passes through $\bar{A}$ and that it is at right angles to the principal ray, the inclination of which is small. Equation § 156 (iv) accordingly assumes the form

$$L_2^{(k)} = -\,(n_1\,\mathrm{v}_1)\,(n_1\,\mathrm{u}_1)\left(\frac{s_1}{n_1}\right)^3 y_1 \sum_{v=1}^{k}\left(\frac{h_v}{h_1}\right)^3 \frac{y_v}{y_1}\, Q_{vx}\, Q_{vs} \underset{v}{\Delta} \frac{1}{ns}\,, \quad \text{(i)}$$

which gives the value of the **sagittal line of confusion within the Gauss image-plane.**

If, as in formulæ § 150 (ii) and § 154 (v), we substitute the co-ordinates of the ray in place of the angular apertures u and v, equation (i) becomes

$$\frac{n_1 L_2^{(k)}}{s_1} = -\,\frac{M_1 m_1 l_1}{(x_1 - s_1)^3}\, s_1^2 x_1 \sum_{v=1}^{k}\left(\frac{h_v}{h_1}\right)^3 \frac{y_v}{y_1}\, Q_{vs}\, Q_{vx} \underset{v}{\Delta} \frac{1}{ns}\,. \quad \text{(ii)}$$

We thus arrive at the important conclusion that the three comatic defects of definition in the image, which at finite inclinations of the principal ray involve entirely different functions, whether expressed in an explicit or implicit form, have identical coefficients when the angles of inclination are infinitesimals of the first order, and it will be seen that these identical coefficients express the relation between these aberrations and the principal dimensions of the system.

From this it follows that they cannot be made to vanish separately, but that they either reduce to zero simultaneously or that they all differ from zero, according as the constants of the system satisfy or do not satisfy the condition

$$\sum_{v=1}^{k}\left(\frac{h_v}{h_1}\right)^3 \frac{y_v}{y_1}\, Q_{vs}\, Q_{vx} \underset{v}{\Delta} \frac{1}{ns} = 0\,. \quad \ldots \quad \ldots \quad \text{(iii)}$$

In other words, the value of this sum determines whether the coma (in a wider sense) vanishes or does not vanish with respect to an image area of the first order of magnitude.

159. In the case of a **system of planes** the general term in the sum common to the three expressions for the coma, when simplified, becomes

$$\left(\frac{h_v}{h_1}\right)^3 \frac{y_v}{y_1}\left(\frac{n'_v}{s'^2_v\, x'_v} - \frac{n_v}{s_v^2\, x_v}\right). \qquad \dots \qquad \text{(i)}$$

By a simple transformation this expression can be made to differ from the general term for the spherical aberration on the axis by a constant alone. When, for example, the constant property of the invariants I_v and J_v at all the surfaces of the system of planes is used, the following equations are obtained by division :

$$\frac{y_v s_v' h_1 x_1}{h_v x'_v y_1 s_1} = 1\,; \quad \frac{y_v s_v h_1 x_1}{h_v x_v y_1 s_1} = 1\,,$$

and with the aid of these equations the general term of equation § 158(ii) becomes

$$\frac{s_1}{x_1}\left(\frac{h_v}{h_1}\right)^4 \left(\frac{n_v'}{s'^3_v} - \frac{n_v}{s_v^3}\right). \qquad \dots \qquad \dots \qquad \text{(ii)}$$

It is then only necessary to multiply by $\frac{s_1}{x_1}$ the expression for the aberration given in § 121(iii) in order to obtain its final form in the expression for the summation of the coma, viz. :

$$\left.\begin{aligned}\sum_{v=1}^{k}\left(\frac{h_v}{h_1}\right)^3 \frac{y_v}{y_1}\left(\frac{n'_v}{s'^2_v\, x'_v} - \frac{n_v}{s_v^2\, x_v}\right) &= \frac{n^3_1}{s_1^2 x_1}\left(\frac{1}{n_k'^2} - \frac{1}{n_1^2}\right)\\ &+ \frac{n_1^4}{s_1^3\, x_1}\sum_{v=1}^{k}\frac{d_v}{n_v'}\left(\frac{1}{n'^2_v} - \frac{1}{n_k'^2}\right).\end{aligned}\right\} \qquad \text{(iii)}$$

D. The Coma in Simple Special Cases.

160. Coma of a Single Surface.—At small inclinations of the principal ray the coefficient of the coma was shown in § 158(ii) to have the following form in the case of a single surface (ignoring variations of the numerical factors in these particular cases) :

$$ss\ Q_s\ x\ Q\ \ s\,\Delta\,\frac{1}{ns}.$$

Disregarding the case when $s = 0$, which is of no further interest, there are three principal cases in which this coefficient may vanish; namely

(i) When $s = r$, *i.e.* when the object-point, together with its image, lies at the centre of the surface ;

(ii) when $x = r$, *i.e.* when the centre of the diaphragm, together with its image, lies at the centre of the surface ;

(iii) when $n's' = ns$, *i.e.*, when the object and image points coincide with the pair of aplanatic points.

In all other cases the comatic coefficient differs from zero. As before, the expression may be considerably simplified when the object-point is at an infinite distance. If we imagine the stop to be movable, the comatic coefficient of a single surface changes its sign as the stop crosses the centre of the surface.

161. Coma of a Single Thin Lens.—In the general expression for the coefficient of $n l_2^{(k)}/s_1$, § 158 (ii), if we eliminate r_2, s_2', x_2' by means of ρ, σ, ξ, as in § 124, the coefficient may be written in the form

$$C = C_0 - C_1 \rho + C_2 \rho^2,$$

where

$$\left.\begin{aligned} C_0 &= \frac{n^2}{(n-1)^2}\phi^3 + \frac{2n+1}{n-1}\phi^2\sigma + \frac{n}{n-1}\phi^2\xi \\ &\quad + \frac{n+1}{n}\phi\sigma^2 + \frac{2n+1}{n}\phi\sigma\xi; \quad \ldots \quad \ldots \\ C_1 &= \frac{2n+1}{n-1}\phi^2 + \frac{3n+3}{n}\rho\sigma + \frac{n+1}{n}\phi\xi; \quad \ldots \\ C_2 &= \frac{n+2}{n}\phi. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{aligned}\right\} \quad \text{(i)}$$

As in the investigation in § 135 of the distortion due to a single thin lens, we may determine the value of P_{min} for which C has a minimum value Γ_{min}, namely

$$\mathrm{P}_{min} = \frac{3n+3}{2(n+2)}\Sigma + \frac{n+1}{2(n+2)}\Xi + \frac{n(2n+1)}{2(n-1)(n+2)} \qquad \text{(ii)}$$

and for Γ_{min} we obtain the equation

$$\frac{5n^2+6n+1}{4n(n+2)}\Sigma^2 - \frac{n^2+4n+1}{2n(n+2)}\Sigma\,\Xi + \frac{(n+1)^2}{4n(n+2)}\Xi^2$$
$$+ \frac{2n+1}{2(n+2)}\Sigma - \frac{1}{2(n+2)}\Xi - \frac{(4n-1)n}{4(n-1)^2(n+2)} + \Gamma_{min} = 0. \quad \text{(iii)}$$

From the values of Γ_{min} and P_{min} we may find for every value of P the corresponding value of Γ with the aid of the equation

$$\Gamma = \Gamma_{min} + \frac{n+2}{n}(P - P_{min})^2,$$

but it should be noted that for a given value Γ of the comatic coefficient, if $\Gamma < \Gamma_{min}$, we obtain imaginary values P.

The relation of the value of Γ_{min} to those of Σ and Ξ becomes more evident if we reduce the equation to the standard form by a change of co-ordinates.

It will be seen that the necessary displacement of the origin obtained from the equations

$$\Sigma = (\Sigma) - \frac{1}{2}; \quad \Xi = (\Xi) - \frac{1}{2}$$

is identical with that obtained in the case of distortion, whilst the angle through which the system requires to be rotated follows from the equation

$$\tan 2\,\alpha = \frac{n^2 + 4\,n + 1}{2\,n\,(n+1)}.$$

The final result of the change of co-ordinates is

$$\lambda_1\, s_s^{\,2} + \lambda_2\, x_s^{\,2} = \frac{n^2}{4\,(n-1)^2} - \Gamma_{min},$$

where the values of λ_1, λ_2 are given by the expression

$$\lambda_{1\,2} = \frac{(n+1)(3\,n+1) \pm \sqrt{(n+1)^2(3\,n+1)^2 - 4\,n^3(n+2)}}{4\,n\,(n+2)}.$$

From this it follows that so long as $\Gamma_{min} < \frac{n^2}{4\,(n-1)^2}$ the equation represents an ellipse, since λ_1 and λ_2 are always real positive quantities. When $\Gamma_{min} = \frac{n^2}{4\,(n-1)^2}$ the ellipse reduces to the origin, and when $\Gamma_{min} > \frac{n^2}{4\,(n-1)^2}$ it becomes imaginary.

Values of $\Sigma\,\Xi$ corresponding to any given values of P_{min} determine in the plane of $\Sigma\,\Xi$ a straight line belonging to a family of parallel straight lines.

6. THE CONDITION FOR THE ABSENCE OF ABERRATION IN THE CASE OF A PENCIL OF FINITE APERTURE ABOUT A PRINCIPAL RAY OF SMALL INCLINATION.

A. Proof of the Sine Condition.

162. We shall suppose that a centred system S, spherically corrected for the points O and O' on the axis, is required to produce a distinct image of a plane surface element dq at right angles to the axis, and we shall further suppose that the pencils in question are of finite aperture and that they are confined within the limits of a circular stop which causes the incident rays to be inclined to the axis at angles $u \leqq U$, and the emerging rays at angles $u' \leqq U'$ (Fig. 75).

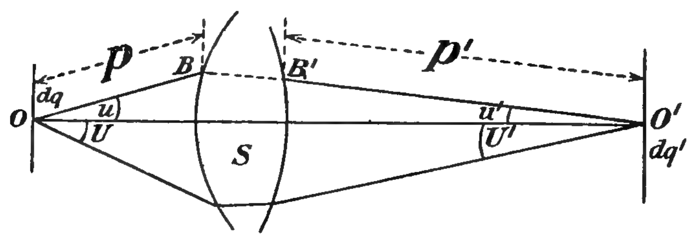

FIG. 75.

Formation of sharply defined images of a surface element normal to the axis by pencils of finite aperture.

$BO = p$; $B'O' = \ '$.

The condition for sharp definition is then identical with the requirement that the image of every linear element dy_s in dq about the point O on the axis formed by partial pencils of any inclination within the prescribed limits, shall be of the same size as the image which is formed by the paraxial rays, that is

$$dy'_s = \beta dy_s .$$

Now, from the previous investigations we know that the formulæ of collinear image-formation hold for any aperture angle u within two plane sections of infinitely small transverse dimensions, which we have distinguished as the tangential and sagittal sections. Therefore the fundamental relation given in § 78 (v), namely

$$\bar{\beta}\gamma = \frac{n}{n'}$$

is applicable to either of the two sections.

163. The Formation by Tangential Pencils of the Images of Radial Linear Elements. In Fig. 76 let $O\ B$ be the principal ray which is inclined at an angle $u \leqq U$, and within its tangential section let $O_1\ B_1$ be an adjacent ray inclined at an infinitely small angle to the principal ray. Then the corresponding emergent ray $B'_1 O'_1$ will be inclined likewise at an infinitely small angle to the conjugate principal ray.

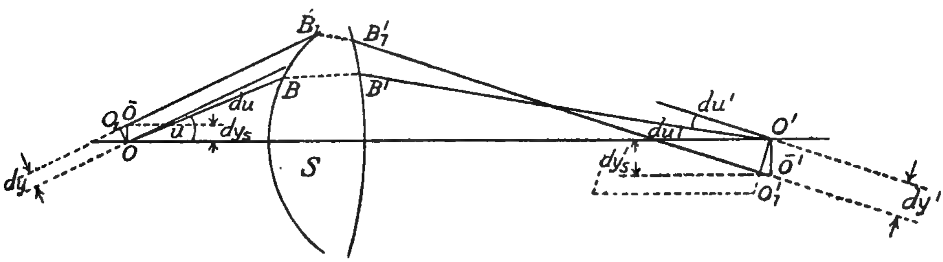

FIG. 76.

$OO_1 = dy$; $O'O'_1 = dy'$; $O\bar{O} = dy_s$; $O'\bar{O}' = dy'_s$.

Formation of an image by tangential partial pencils of radial linear elements.

The investigation contained in Chapter III is applicable to both rays. Accordingly, let perpendiculars be drawn from O and O' to $O\ B$ and $O'\ B'$ respectively, and let them intersect an adjacent ray at O_1 and O'_1. If du and du' denote the infinitely small angular apertures at O and O', then by § 55 (21) and § 78 (v) the ratio of these perpendiculars dy, dy' can be represented by the equation

$$\beta_{tu} = \left(\frac{dy'}{dy}\right)_u = \frac{ndu}{n'du'} \qquad \cdots \qquad \cdots \qquad \text{(i)}$$

Also, at O and O' let two planes be described at right angles to the axis. Then these planes will be intersected by the adjacent rays $O_1\ B_1$ and $B'_1\ O'_1$ at the points $\bar{O}$ and $\bar{O}'$ respectively at distances dy_s and dy'_s from the axis ; and dy_s is a radial line element of dq. Moreover, the triangles $O O_1 \bar{O}$ and $O' O'_1 \bar{O}'$ may be regarded as rectangular, having their right angles at O_1, since the adjacent rays are inclined to the principal rays at angles which are negligibly small in comparison with $\frac{\pi}{2}$.

We obtain, accordingly, the following geometrical relations :

$$dy = dy_s \cos u \ ; \quad dy' = dy'_s \cos u' \, ,$$

and hence

$$\beta_{tu} = \left(\frac{dy'_s}{dy_s}\right)_u = \frac{n \cos u\ du}{n' \cos u'\ du'} = \frac{nd \sin u}{n'd \sin u'} \ . \qquad \text{(ii)}$$

This signifies that the magnitude of the linear elements at right angles to the axis at the points O and O', where the ray traversing the system meets the axis is a function of u, being proportional to the quotient of the differentials of the sines of the aperture angles.

In order that β_{tu} may have the same value at all inclinations, and therefore, in our case, that dy_s may be a radial linear element of a sharply defined conjugate plane surface element dq', the condition must be satisfied that

$$\beta_{tu} = \beta = \frac{n\ d \sin u}{n'\ d \sin u'}, \quad \ldots \quad \ldots \quad \text{(iii)}$$

where β is the value of the lateral magnification of the paraxial rays s. By integrating

$$n'\ \beta\ d \sin u' = n\ d \sin u$$

we obtain

$$n'\ \beta \sin u' = n \sin u,$$

since the constant of integration is zero, owing to the fact that u and u' vanish simultaneously.

The expression

$$\frac{\sin u}{\sin u'} = \frac{n'\ \beta}{n} \ldots \quad \ldots \quad \ldots \quad \text{(iv)}$$

is accordingly the condition that the magnification β_{tu} may be constant for all partial pencils within the tangential section, whatever the inclination u of the corresponding principal rays proceeding from O and meeting at O'.

164. Formation of the Images of Radial Line Elements by Sagittal Pencils.—In Fig. 75, if we suppose the system of rays to be rotated about the axis through an infinitely small angle, the linear elements p and p' will sweep out small portions of conical surfaces, and we will suppose that the two extreme positions of the generating lines comprise the small angles $d\text{v}$ and $d\text{v}'$. These will then be the angular apertures of the partial sagittal pencils which form an image of a linear element of dq lying in the sagittal section, and accordingly normal to the plane of the paper.

In accordance with the formula established for the angular magnification of the sagittal pencils (§§ 78, v and 92, 15) we may now write

$$\gamma_{f1} = \frac{d\text{v}'_1}{d\text{v}} = \frac{f}{f_1'} = \frac{p}{p'_1}.$$

By § 28 (xi) this may be expressed as

$$\frac{p}{p'_1} = \frac{\sin u'_1}{\sin u}$$

and hence

$$\gamma_{f1} = \frac{d\text{v}_1'}{d\text{v}} = \frac{\sin u'_1}{\sin u}.$$

Forming this ratio successively for the other surfaces and noting that for a system of centred surfaces $u_2 = u'_1$; $d\mathrm{v}_2 = d\mathrm{v}'_1$, the multiplication of all the resulting partial ratios furnishes the following equation :

$$\gamma_f = \frac{d\mathrm{v}'}{d\mathrm{v}} = \frac{\sin u'}{\sin u} \quad \ldots \quad \ldots \quad \text{(i)}$$

hence

$$\beta_{fu} = \frac{dy_s'}{dy_s} = \frac{n \sin u}{n' \sin u'}. \quad \ldots \quad \ldots \quad \text{(ii)}$$

From this it follows at once, as in § 163 (ii), that

$$\beta_{fu} = \beta , \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

which is the condition for the formation of a distinct image of a radial element by sagittal pencils of any finite inclination. We have accordingly,

$$\frac{\sin u}{\sin u'} = \frac{n'\beta}{n} , \quad \ldots \quad \ldots \quad \text{(iv)}$$

which is the condition for the magnification due to paraxial rays being identical with the magnification of a linear element at right angles to the axis due to the partial sagittal pencils about principal rays inclined at any angle.

In the case of an object-point near the axis these two proofs apply to all rays of a pencil having an angular aperture U in the meridian plane, but only an infinitely small aperture $d\mathrm{v}$ in the sagittal plane. To investigate the condition for a sharply defined image of such a point near the axis formed by a finite pencil having an aperture U in each direction, we may proceed in the following manner :

In an object-plane at O let $\bar{O}$ be a point near the axis and from this point let a skew ray proceed in any direction. Let the point of incidence on the first lens surface be joined to the object-point O on the axis, and let this ray be regarded as the principal ray. The skew ray just considered will then be adjacent to this principal ray, so that we may apply the result stated at the end of § 105 to its projection upon the principal plane of the principal ray. The condition that the conjugate skew ray must pass through the image-point $\bar{O}$ is then replaced by the other two conditions that its projections on the image-side shall pass through the projections of the image-point, as has been shown above.

This completes the investigation of the conditions under which a sharp image is formed by a solid pencil of finite aperture.

The law which we have just demonstrated is known as **Abbe's sine condition,** which states that in order that a distinct image may be formed of a surface element at right angles to the axis, the

ratio of the sines of conjugate angular apertures must be constant over the whole extent of the aperture for which the system is corrected for spherical aberration, this ratio being equal to the lateral magnification due to paraxial rays multiplied by the quotient of the extreme refractive indices.

If the object-point is at infinity, its distance x_s from the appropriate principal focus becomes in the limit equal to the distance $s = \infty$ from the first surface of the system. Substituting for β its value as given in § 54 (20), viz., $\beta = \frac{f}{s}$, equation (iv) becomes accordingly

$$\left[\frac{\sin u}{\sin u'}\right]_{u=0} = \left[\frac{n'}{n}\,\frac{f}{s}\right]_{s=\infty}.$$

Also, the ordinate of the point of incidence for a distant object-point is

$$h = [s \sin u]_{\substack{s=\infty \\ u=0}},$$

Hence, we may write the sine condition (iv) in the form

$$\frac{h}{\sin u'} = \frac{n'f}{n} = -f', \quad \ldots \quad \ldots \quad \text{(v)}$$

where f and f' are the focal lengths of the paraxial rays in the image space.

Similarly, for an infinitely distant image-point

$$\frac{h'}{\sin u} = -f. \ldots \quad \ldots \quad \ldots \quad \text{(vi)}$$

From this it is easy to determine the values of the focal lengths due to the partial pencils, if we apply to the tangential as well as to the sagittal sections the fundamental formula given in § 54 (18), viz.:

$$f' = -\frac{h}{\tan u'}.$$

Hence in the tangential section

$$f'_{tu} = -\frac{\mathrm{h}_t}{d\,\mathrm{u}'} = -\left[\frac{t\,d\,\mathrm{u}}{d\,\mathrm{u}'}\right]_{\substack{t=\infty \\ d\mathrm{u}=0}}.$$

If now for the tangential rays we take the generally applicable value (§ 163, ii)

$$\frac{d\mathrm{u}}{d\mathrm{u}'} = \beta_{tu}\frac{n' \cos u'}{n \cos u},$$

and note that by § 163 (iii) the sine condition is fulfilled when $\beta_{tu} = \beta$ and that according to the previous assumptions $u = 0$, it follows that

$$t = \infty = s$$

and by § 54 (20),

$$\beta = \frac{f}{s} \, ,$$

and finally

$$f'_{tu} = -\left[s \, \beta \, \frac{n' \cos u'}{n} \right]_{s=\infty} = -f \frac{n'}{n} \cos u' = f' \cos u' \, .$$

Similarly, we obtain for the sagittal rays the equation

$$f'_{fu} = -\frac{\mathrm{h}_f}{d\mathrm{v}'} = -\left[\frac{f\,d\mathrm{v}}{d\mathrm{v}'} \right]_{\substack{f=\infty \\ d\mathrm{v}=0}}$$

and since, by § 164 (iv)

$$\frac{d\mathrm{v}}{d\mathrm{v}'} = \frac{n'\beta}{n} \, ,$$

this becomes

$$f'_{fu} = -\left[s \, \beta \, \frac{n'}{n} \right]_{s=\infty} = -f \frac{n'}{n} = f' \, .$$

Analogous relations hold for the focal lengths in the object-space when the aplanatic image-point is at infinity.

In 1873 Abbe (*2. 420*) expressed the sine law as the condition that the magnification shall be identical for different regions of the aperture. Subsequently, Abbe stated that the points for which a system is spherically corrected cannot properly be described as *conjugate pairs of aplanatic points*, as was customary, unless they also fulfil the sine condition. For, clearly, it is only when the sine condition is fulfilled in a spherically corrected system of lenses that a pencil of rays diverging at a finite angle can form a sharply defined image of a two-dimensional surface element, even of small extent.

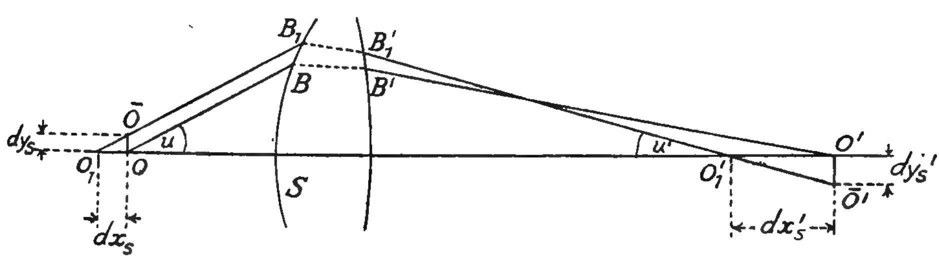

FIG. 77

$O\bar{O} = dy_s$; $O'\bar{O}' = dy_s'$; $OO_1 = dx_s$; $O'O_1' = dx_s'$.

Spherical aberration in the neighbourhood of a pair of aplanatic points.

In this sense, the term is rightly applicable to the pairs of aplanatic points of the sphere, viz. its centre, which is conjugate to itself, and the conjugate points at distances $\frac{n'}{n}r$ and $\frac{n}{n'}r$ from it, whereas it does not rightly apply, for example, to the foci of a reflecting ellipsoid nor to the infinitely distant point and the focus of the reflecting paraboloid.

The great practical importance of the fulfilment of this condition was subsequently demonstrated by Abbe, who showed that in all practicable microscope objectives the sine condition had been fulfilled by empirical methods.

B. The Inconsistency of Herschel's Condition with the Sine Condition.

165. In proof of this, let an adjacent ray, whose inclination differs from u by an infinitely small amount, be drawn from the terminal point O of the radial line element dy_s and let it intersect the axis at O_1. It follows then, from the previous discussion, that the conjugate ray in the image-space will meet the axis at the neighbouring point O'_1 and that its inclination will differ from that of the ray of reference only by an infinitely small amount.

We have accordingly the following purely geometrical relations

$$dy_s = dx_s \tan u\,; \qquad dy_s'\; dx_s' \tan u'$$

$$\frac{dy_s'}{dy_s} = \frac{dx_s' \tan u'}{dx_s \tan u}. \qquad \ldots \qquad \ldots \qquad \text{(i)}$$

In our case, in which it is assumed that O and O' are conjugate aplanatic points, and that accordingly we may rightly describe dy_s' as the distinct image of dy_s, it follows from § 163 (iv) that

$$\frac{\sin u}{\sin u'} = \frac{n'\beta}{n}\,; \qquad \ldots \qquad \ldots \qquad \ldots \qquad \text{(ii)}$$

hence the above equation becomes

$$\beta = \frac{dy_s'}{dy_s} = \frac{dx_s'}{dx_s}\,\frac{n}{n'\beta}\,\frac{\cos u}{\cos u'},$$

and therefore

$$\frac{dx_s'}{dx_s} = \frac{n'\beta^2}{n}\,\frac{\cos u'}{\cos u}. \qquad \ldots \qquad \ldots \qquad \text{(iii)}$$

Now, by § 56 (23) and § 84 (iv) the longitudinal magnification a of the paraxial rays is

$$a = \frac{dx_s'}{dx_s} = \frac{n'\beta^2}{n} \quad \ldots \quad \ldots \quad \text{(iv)}$$

and hence

$$dx_s' = dx_s' \frac{\cos u'}{\cos u} . \quad \ldots \quad \ldots \quad \text{(v)}$$

Expressed in words, this signifies that an infinitely small displacement of either of the two aplanatic points gives rise to spherical aberration at the conjugate aplanatic point, since the displacement dx_s' of the image-point formed by the paraxial pencil differs from the displacement of the axial point of intersection of rays of finite aperture so long as u and u' differ by an absolute amount.

The only cases in which the spherical aberration of points near a pair of aplanatic points can be eliminated are when

$$u = \pm u' ,$$

that is when objects lying in the nodal planes themselves or in the negative nodal planes are in question, or when

$$u = u' = 0 ,$$

that is so long as an infinitely distant image is formed of an infinitely distant object.

In other cases the system gives rise to spherical aberration, the magnitude of which we shall now proceed to investigate. By equation (v)

$$\mathsf{A} = dx'_s - dx'_s = dx'_s \left(\frac{\cos u'}{\cos u} - 1\right) ,$$

from which it will be seen that A can be expanded in the form

$$\mathsf{A} = dx'_s \,(c_2\, u'^2 + c_4\, u'^4 + \;.\;.\;.\;.\;) . \quad \ldots \quad \text{(vi)}$$

The simplest way of determining the coefficient c_2 is probably the following :

$$\frac{\cos u'}{\cos u} - 1 = \frac{\cos^2 u' - \cos^2 u}{\cos u\,(\cos u' + \cos u)} = \frac{\sin^2 u - \sin^2 u'}{\cos u\,(\cos u' + \cos u)}$$

thus by (ii) $$\frac{\cos u'}{\cos u} - 1 = \frac{\sin^2 u'}{\cos u\,(\cos u' + \cos u)} \left(\frac{n'^2\,\beta^2}{n^2} - 1\right) ,$$

hence

$$c_2 = \frac{1}{2}\left(\frac{n'^2\,\beta^2}{n^2} - 1\right)$$

and, finally, by (vi),

$$\mathsf{A} = dx_s \left[\frac{n'\beta^2}{n}\left(\frac{n'^2\beta^2 - 1}{n^2}\right)\frac{u'^2}{2} + c_4\, u'^4 + \ldots\ldots\right] . \quad \text{(vii)}$$

In this form of the equation we recognise at once the general fact that, under similar conditions affecting n', n, β, the spherical aberration varies as the square of the angular aperture u' on the image side. In what follows we shall confine our attention to the more important practical case of real objects, where accordingly s must be supposed to have a negative value. Under this assumption we have

$$\frac{\sin^2 u}{\sin^2 u'} = \frac{n'^2 \beta^2}{n^2} \gtrless 1 ,$$

which indicates whether the greater angular aperture is on the object side or on the image side, showing that an increase of the object distance gives rise to under-correction of the adjacent image-point, whilst a decrease of the object-distance gives rise to over-correction.

Proceeding from the assumption that the system is spherically corrected for two adjacent pairs of points, the value of dx'_s for any finite inclinations u, u' will be the same as that due to paraxial rays, so that, by (iv), for rays at any inclination the ratio will be

$$\frac{dx'_s}{dx_s} = \frac{n' \beta^2}{n} .$$

If we substitute this value in the equation (i), from which the formal relation between dy_s, dy'_s and dx_s, dx'_s was obtained, the equation becomes

$$\frac{dy'_s}{dy_s} = \frac{n' \beta^2}{n} \frac{\tan u'}{\tan u} .$$

Moreover, in § 163 (ii) the magnification in the tangential section was shown to conform to the general expression

$$\beta_{tu} = \frac{dy'_s}{dy_s} = \frac{n \cos u \, du}{n' \cos u' \, du'} .$$

Equating these two expressions, it follows that

$$n^2 \sin u \, du = n'^2 \ \beta^2 \sin u' \, du' ,$$

whence, by integration,

$$- n^2 \cos u = c - n'^2 \ \beta^2 \cos u' .$$

The constant c may be found from the pair of conjugate values

$$u = 0 = u' ,$$

hence $$n^2 (1 - \cos u) = n'^2 \ \beta^2 (1 - \cos u')$$

and if $1 - \cos u$ is expressed in terms of the sines of half the angle, then

$$n^2 \sin^2 \frac{u}{2} = n'^2 \ \beta^2 \sin^2 \frac{u'}{2} . \qquad \ldots \quad \text{(viii)}$$

If, accordingly, two adjacent points are free from aberration it will be seen that the sine condition is fulfilled for the semi-angular apertures; and, clearly, this condition agrees with that affecting the whole angles u, u' in only those three cases which have been stated above.

If the general expressions for β_{tu} and β'_{fu} given in § 163 (ii) and § 164 (ii) are simplified by means of the equation

$$\frac{\sin \frac{u}{2}}{\sin \frac{u'}{2}} = \frac{n'\beta}{n},$$

which follows from (viii), we may obtain the following expressions for a pair of points satisfying the Herschel condition, viz.,

$$\left.\begin{aligned} \beta_{tu} &= \beta \frac{\cos \frac{u'}{2} \cos u}{\cos \frac{u}{2} \cos u'} \\ \beta_{fu} &= \beta \frac{\cos \frac{u}{2}}{\cos \frac{u'}{2}} \end{aligned}\right\} \quad \ldots \quad \ldots \quad \text{(ix)}$$

The conclusions derived from the above investigation may be summarised in the following words of Czapski (*3*. *105*):—"Any optical system, whatever its composition, transmitting finite pencils of any aperture, can only be made to form a sharply defined image of either a surface element normal to the axis, or of an infinitely small portion of the axis itself; and it does so in obedience to the general laws which govern any change in the direction of the rays or in the union of the rays, which can be brought about by dioptrical means. It is impossible to fulfil both conditions conjointly; for the one requirement is inconsistent with the other, and both are opposed to the conditions of the collinear transformation of finite spaces traversed by rays inclined at finite angles."

C. The Relation between the Sine Law and the Comatic Correction.

166. Having established the sine condition for finite angles, it remains to define it particularly for small angular apertures by expansion as a series comprising terms of the third power of u.

In the case of a single surface it was shown in § 28 (xi) that

$$\frac{\sin u'}{\sin u} = \frac{p}{p'}; \text{ and } \frac{n' \sin u'}{n \sin u} = \frac{n'p}{np'}.$$

Substituting the values of p', p derived in § 120 (iv) and (v), namely,

$$p = s\left(1 + \frac{\mathsf{A}\, r^2}{s^3}\,\phi^2\right)\left(1 - \frac{1}{2}\frac{r^2}{ns}\,Q_s\,\phi^2\right),$$

the above expression becomes

$$\frac{n' \sin u'}{n \sin u} = \frac{n's}{ns'}\left\{1 + r^2\phi^2\left(\frac{1}{2}\,Q_s\,\Delta\frac{1}{ns} - \Delta\frac{\mathsf{A}}{s^3}\right)\right\}, \quad \text{(i)}$$

where in the second term we may, of course, replace $r\,\phi$ by h. If now we form the product throughout all k surfaces, and retain only the terms involving ϕ^2, there will remain on the left terms containing the first and the last sine function only, and we obtain accordingly:

$$\frac{n'_k \sin u'_k}{n_1 \sin u_1} = \prod_{v=1}^{k}\frac{n'_v s_v}{n_v s'_v}\left\{1 + \sum_{v=1}^{k}\left(\frac{h_v^2}{2}\,Q_{vs}\,\Delta_v\frac{1}{ns} - h_v^2\,\Delta_v\frac{\mathsf{A}}{s^3}\right)\right\}$$

$$= \prod_{v=1}^{k}\frac{n'_v s_v}{n_v s'_v}\left\{1 + s_1^2 u_1^2\sum_{v=1}^{k}\left[\left(\frac{h_v}{h_1}\right)^2\frac{1}{2}\,Q_{vs}\,\Delta_v\frac{1}{ns} - \left(\frac{h_v}{h_1}\right)^2\Delta_v\frac{\mathsf{A}}{s^3}\right]\right\}. \quad \text{(ii)}$$

In order that this product may be constant for all values of the aperture u_1 the condition which has to be satisfied is obviously

$$\frac{1}{2}\sum_{v=1}^{k}\left(\frac{h_v}{h_1}\right)^2 Q_{vs}\,\Delta_v\frac{1}{ns} - \sum_{v=1}^{k}\left(\frac{h_v}{h_1}\right)^2\Delta_v\frac{\mathsf{A}}{s^3} = 0. \quad \text{(iii)}$$

This sum may be transformed so as to become similar to the expression previously derived in connection with spherical aberration, thus:

$$\left.\begin{aligned}
\sum_{v=1}^{k}\left(\frac{h_v}{h_1}\right)^2\Delta_v\frac{\mathsf{A}}{s^3} &= \sum_{v=1}^{k}\frac{1}{Q_{vx} - Q_{vs}}\left(\frac{h_v}{h_1}\right)^2\Delta_v\left\{\frac{n}{s} - \frac{n}{x}\right\}\frac{\mathsf{A}}{s^3} \\
&= \sum_{v=1}^{k}\frac{1}{Q_{vx} - Q_{vs}}\left(\frac{h_v}{h_1}\right)^2\left\{\Delta_v\frac{n\mathsf{A}}{s^4} - \Delta_v\frac{n\mathsf{A}}{s^3 x}\right\} \\
&= -\frac{1}{2}\sum_{v=1}^{k}\frac{1}{Q_{vx} - Q_{vs}}\left(\frac{h_v}{h_1}\right)^2 Q_{vs}^2\,\Delta_v\frac{1}{ns} \\
&\quad - \sum_{v=1}^{k}\frac{1}{Q_{vx} - Q_{vs}}\left(\frac{h_v}{h_1}\right)^2\Delta_v\frac{n\mathsf{A}}{s^3 x}
\end{aligned}\right\} \quad \text{(iv)}$$

and since

$$Q_x - Q_s = \frac{n}{sx}\left(x - s\right) = \frac{n'}{s'x'}\left(x' - s'\right),$$

$$\Delta \frac{n\,\mathsf{A}}{s^3 x} = \frac{\mathsf{A}'}{s'^2}\frac{n'}{s'x'} - \frac{\mathsf{A}}{s^2}\frac{n}{sx} = (Q_x - Q_s)\,\Delta\,\frac{\mathsf{A}}{s^2\,(x - s)},$$

hence

$$\sum_{v=1}^{k} \frac{1}{Q_{vx} - Q_{vs}}\left(\frac{h_v}{h_1}\right)^2 \underset{v}{\Delta} \frac{n\mathsf{A}}{s^3 x} = \sum_{v=1}^{k}\left(\frac{h_v}{h_1}\right)^2 \underset{v}{\Delta}\, \frac{\mathsf{A}}{s^2\,(x - s)}\,.$$

In the last sum of this equation it will be seen that all terms reduce to zero except the first and last, so that we have

$$\sum_{v=1}^{k} \frac{1}{Q_{vx} - Q_{vs}}\left(\frac{h_v}{h_1}\right)^2 \underset{v}{\Delta} \frac{n\mathsf{A}}{s^3 x} = \left(\frac{h_k}{h_1}\right)^2 \frac{\mathsf{A}'_k}{s'^2_k\,(x'_k - s'_k)} - \frac{\mathsf{A}_1}{s_1^2\,(x - s_1)}\,. \quad \text{(v)}$$

Now, the sine condition has been enunciated for systems which form images of real object-points free from aberration, so that in this case, which alone enters into consideration, $\mathsf{A}'_k = 0 = \mathsf{A}_1$, and we obtain the following equation as an expression satisfying the sine condition:

$$\frac{1}{2}\sum_{v=1}^{k}\left(\frac{h_v}{h_1}\right)^2 Q_{vs}\,\underset{v}{\Delta}\,\frac{1}{ns} + \frac{1}{2}\sum_{v=1}^{k}\frac{1}{Q_{vx} - Q_{vs}}\left(\frac{h_v}{h_1}\right)^2 Q^2_{vs}\,\underset{v}{\Delta}\,\frac{1}{ns} = 0\,. \quad \text{(vi)}$$

Omitting the constant factor $\frac{1}{2}$, this may also be written in the form

$$\sum_{v=1}^{k}\left(\frac{h_v}{h_1}\right)^2 \frac{Q_{vs}\,Q_{vx}}{Q_{vx} - Q_{vs}}\,\underset{v}{\Delta}\,\frac{1}{ns} = 0\,. \quad \ldots \quad \ldots \quad \text{(vii)}$$

In § 83 (2) it was shown that

$$Q_{vx} - Q_{vs} = \frac{h_1\,y_1}{h_v\,y_v}\left(Q_{1x} - Q_{1s}\right).$$

Omitting again the constant factor

$$\frac{1}{Q_{1x} - Q_{1s}},$$

the equation assumes its final form

$$\sum_{v=1}^{k}\left(\frac{h_v}{h_1}\right)^3 \frac{y_v}{y_1}\,Q_{vs}\,Q_{vx}\,\underset{v}{\Delta}\,\frac{1}{ns} = 0\,. \quad \ldots \quad \ldots \quad \text{(viii)}$$

From this it will be seen that the expression which satisfies the sine condition in the special case of small angular apertures is equivalent to the simultaneous elimination of all three species of coma in the special case of small inclinations of the principal ray.

D. Kerber's Summation Formulæ for the Deviations from the Sine Ratio when the Angular Apertures are Finite.

167. In the majority of optical systems the magnification of any zone must be equal to that of the paraxial rays, and since the expression for the magnification is the ratio of the sines of the angular apertures it follows that

$$\frac{\sin u'_k}{\sin u_1} = \frac{h_k}{s'_k} \Big/ \frac{h_1}{s_1} = \frac{h_k}{h_1} \frac{s_1}{s'_k},$$

In spherically corrected systems

$$\mathbf{s}_1 = s_1 ; \quad \mathbf{s}'_k = s'_k ,$$

and if we introduce the quantities e, e' defined in § 46 (i), we obtain the relation :

$$\frac{h_k}{h_1} = \frac{\mathbf{s}'_k \sin u'_k}{\mathbf{s}_1 \sin u_1} = \frac{e'_k}{e_1} .$$

Now, the ratio e'_k / e_1 in a spherically corrected system will generally differ from the prescribed value, and the formula will actually be

$$\frac{e'_k + \delta e'_k}{e_1} = \frac{h_k}{h_1} .$$

If in the identity § 46 (ii)

$$\sin u' + \sin i' = \sin u + \sin i + D ,$$

we express sin i' in terms of sin u' and sin i in terms of sin u (§ 28, v) and then substitute e and e' (§ 46, i), it follows that $e' = e + rD$.

Also, the transition equivalent § 28 (viii) for $\mathbf{s}_{v+1}$, viz.

$$\mathbf{s}_{v+1} = \mathbf{s}'_v - d_v ,$$

when multiplied by sin u'_v, becomes

$$e_{v+1} = e'_v - d_v \sin u'_v = e'_v \left(1 - \frac{d_v}{\mathbf{s}'_v}\right),$$

but since $\delta s = \mathbf{s} - s$,

$$\frac{d}{\mathbf{s}} = \frac{d}{s} - \frac{d\delta s}{s\mathbf{s}} ,$$

and

$$e_{v+1} = e'_v \left(1 - \frac{d_v}{s_v'}\right) + \frac{\delta s' \, d_v \sin u'_v}{s'_v} .$$

Now, by § 46 (v) and § 28 (viii)

$$\frac{h_{v+1}}{h_v} = \frac{s'_v - d_v}{s'_v} = 1 - \frac{d_v}{s'_v} \, ,$$

hence

$$e_{v+1} = e'_v \frac{h_{v+1}}{h_v} + \frac{\delta s'_v \, d_v \sin u'_v}{s'_v} = e'_{v+1} - r_{v+1} D_{v+1} \, ,$$

and we thus obtain the recurrence formula

$$\frac{e'_{v+1}}{h_{v+1}} - \frac{e'_v}{h_v} = \frac{r_{v+1} D_{v+1}}{h_{v+1}} + \frac{\delta s'_v d_v \sin u'_v}{(s'_v - d_v) \, h_v} \, . \tag{i}$$

Summing all these expressions for each of the k surfaces of a centred system from $v = 1$ to $v = k - 1$ and adding the identity

$$\frac{e'_1}{h_1} - \frac{e_1}{h_1} = \frac{r_1 D_1}{h_1} \, ,$$

we obtain

$$-\frac{\delta e'_k}{h_k} = \frac{e'_k}{h_k} - \frac{e_1}{h_1} = \sum_{v=1}^{k} \frac{r_v D_v}{h_v} + \sum_{v=1}^{k-1} \frac{d_v \, \delta s'_v \sin u'_v}{(s'_v - d_v) \, h_v} \, ,$$

hence

$$\delta e'_k = -\frac{h_k}{h_1} \left[\sum_{v=1}^{k} \frac{h_1}{h_v} r_v D_v + \sum_{v=1}^{k-1} \frac{h_1}{h_v} d_v \frac{\delta s'_v \sin u'_v}{s'_v - d_v} \right] . \tag{ii}$$

This determines the **spherical difference of magnification**; for if in accordance with the actual conditions of the case we put

$$\frac{\sin u'_k + \delta \sin u'_k}{\sin u_1} = \frac{h_k}{h_1} \frac{s_1}{s'_k} \, ,$$

we can make use of the identity

$$\frac{\sin u'_k}{\sin u_1} = \frac{s_1}{s'_k} \frac{e'_k}{e_1} = \frac{s_1}{s'_k} \frac{e'_k}{e_1} \left(1 - \frac{\delta s'_k}{s'_k} \right)$$

and hence we may write

$$\frac{h_k}{h_1} \frac{s_1}{s'_k} - \frac{\delta \sin u'_k}{\sin u_1} = \frac{s_1}{s'_k} \left(\frac{h_k}{h_1} - \frac{\delta e'_k}{e_1} \right) \left(1 - \frac{\delta s'_k}{s'_k} \right) \, ,$$

which reduces to

$$\frac{\delta \sin u'_k}{\sin u_1} = \frac{s_1}{s'_k} \left(\frac{\delta e'_k}{e_1} + \frac{h_k \, \delta s'_k}{h_1 \, s'_k} - \frac{\delta e'_k \, \delta s'_k}{e_1 \, s'_k} \right) . \tag{iii}$$

When the **object** is **at an infinitely great distance,** the expression becomes

$$\delta \frac{1}{f} = \frac{\delta \sin u'_k}{s_1 \sin u_1} = \frac{1}{s'_k} \left(\frac{\delta e'_k}{e_1} + \frac{h_k}{h_1} \frac{\delta s'_k}{s'_k} - \frac{\delta e'_k \, \delta s'_k}{e_1 \, s'_k} \right) . \tag{iv}$$

Finally, in a system in which there is no spherical aberration $\delta s'_k$ vanishes, and we obtain the final formula

$$\frac{\delta \sin u'_k}{\sin u_1} = \frac{s_1\,\delta e'_k}{s'_k\,e_1} = -\frac{1}{e_1}\frac{s_1}{s'_k}\frac{h_k}{h_1}\left[\sum_{v=1}^{k}\frac{h_1}{h_v}\,r_v\,D_v + \sum_{v=1}^{k-1}\frac{h_1}{h_v}\,d_v\,\frac{\delta s_v'\sin u'_v}{s'_v - d_v}\right] \quad \text{(v)}$$

where $\delta s'_v$ may be found from the formula given in § 46 (vi).

7. THE ABERRATION OF EXTRA-AXIAL POINTS GOVERNED BY THE THIRD POWER OF THE ANGULAR APERTURE.

(The Four Defects of Spherical Aberration in the Restricted Sense.)

168. We now proceed to consider the next higher power, *i.e.*, the third power, of the co-ordinates of the aperture, so as to establish the requisite condition that pencils of this degree may participate in the formation of a distinct image. It is necessary then to consider the following four terms :

$$m^3\,;\; m^2M\,;\; mM^2\,;\; M^3.$$

By reasoning analogous to that employed in the investigation of the comatic aberration we conclude that of these four terms

m^3 and mM^2 relate to the tangential aberration, and m^2M and M^3 to the sagittal aberration.

We now formulate expressions for these four terms, beginning with m^3.

A. Spherical Aberration of the Tangential Pencil.

169. In Fig. 78 let BD, BD_2 be the axial intercepts of adjacent rays inclined to the principal ray BA at small angles u and $2\,\mathsf{u}$, and let the magnitude of the tangential intercepts be stated, as in § 119, by the following expression, which includes one more term than was necessary in the investigation of the comatic aberrations :

$$\boldsymbol{t} = t + \mathsf{r}\,\mathsf{u} + \mathsf{q}\,\mathsf{u}^2\,. \qquad \dots \qquad \dots \qquad \text{(i)}$$

Then the increase l_3 in the length of the line of confusion due to the aberration which we are considering is represented by the expression

$$l_3 = \mathsf{q}\,\mathsf{u}^2\,.\,\mathsf{u}',$$

and the problem now is to express q in terms of known quantities.

The displacement on the principal ray BA of the point of intersection of a secondary ray inclined at an angle u is

$$AD = \mathsf{r}\,\mathsf{u} + \mathsf{q}\,\mathsf{u}^2\,,$$

and the increment of this displacement due to a pencil of very small aperture du on the first secondary ray which converges to the image-point at A_1 will be

$$DF = dt = \mathrm{r}\, du + 2\ \mathrm{q u}\, du\ .$$

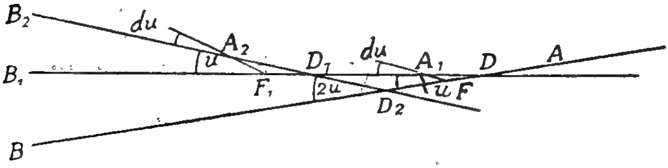

FIG. 78.

Diagram illustrating the tangential line of confusion due to the spherical aberration of extra-axial points.

As in the investigation of the coma, it will be seen from the triangle A_1DF that $DA_1 = \dfrac{DF \sin \mathrm{u}}{\sin du}$, whence it follows from the smallness of the angles that

$$DA_1 = \mathrm{ru} + 2\ \mathrm{qu}^2\quad ,$$

hence

$$d\tau_{o\ u} = AD + DA_1 = 2\ \mathrm{ru} + 3\ \mathrm{qu}^2\ .\ \ldots \qquad \text{(ii)}$$

Since $3\ \mathrm{qu}^2$ is an infinitesimal of a higher order than $2\ \mathrm{ru}$, the magnitude of $d\tau_{o\ u}$, *i.e.*, the distance of the two successive tangential image-points, is equal to that of $d\tau$, the corresponding quantity in the case of the coma (§ 148, vii).

For the other secondary ray B_2D_2 inclined at an angle 2u it follows that

$$AA_2 = d\tau_{o\ 2u} = 4\ \mathrm{r\ u} + 12\ \mathrm{q\ u}^2,$$

$$AA_1 = d\tau_{o\ u} = 2\ \mathrm{r\ u} + 3\ \mathrm{q\ u}^2,$$

and therefore

$$A_1A_2 = d\tau_{1\ u} = 2\ \mathrm{r\ u} + 9\ \mathrm{q\ u}^2.$$

This expression gives the displacement on the first secondary ray of the tangential image-point formed by rays inclined to one another at an angle u. The increment of this displacement is

$$d^2\tau = d\tau_{1\ u} - d\tau_{o\ u} = 6\ \mathrm{q\ u}^2,$$

so that finally

$$\mathrm{q} = \frac{1}{6}\frac{d^2\tau}{d\ \mathrm{u}^2}\ .$$

Forming the third differential quotient of the invariant J in terms of $d\phi$, as in § 156, multiplying by r^{-3} and after simple reductions involving no special difficulties in principle, we obtain the following equation :

$$\frac{1}{r^3}\frac{d^3 J}{d\phi^3} = -\frac{Q_t}{r^3} + 3\,\frac{r^2 Q_t^2 - J^2}{r^2\, nt} + 12\,\frac{J^2 Q_t}{n^2 t^2} + 6\,J\,\frac{\cos j}{n\, t^3}\left(\frac{n\cos^2 j}{t} - Q_t\right)\frac{d\tau}{du}$$

$$-\frac{3\, n\cos^4 j}{t^5}\left(\frac{d\tau}{du}\right)^2 + \frac{n\cos^4 j}{t^4}\,\frac{d^2\tau}{d\mathrm{u}^2}\,. \quad \dots \quad \text{(iii)}$$

If now we form the difference of the values before and after refraction, we should arrive at a recurrence formula for $\frac{d^2\tau}{d\mathrm{u}^2}$ which would be applicable to finite inclinations of the principal ray. If, on the other hand, we confine ourselves at the outset to inclinations to the principal ray which are infinitesimals of the first order, it follows that J and $\frac{d\tau}{d\mathrm{u}}$ are likewise infinitesimals of the first order, so that their products, being quantities of the second order, may be neglected.

Now, under these assumptions, as in § 158,

$$t = s = f\ ; \quad \cos j = 1\,,$$

from which it follows that the expression in equation (iii) reduces to

$$\Delta\frac{n}{s^4}\frac{d^2\tau}{d\mathrm{u}^2} = -\,3\,Q_s^2\,\Delta\frac{1}{n\,s} \quad \dots \quad \dots \quad \text{(iv)}$$

If the object-point is free from aberration, we may obtain from this the following expression :

$$6\,\mathrm{q}'_k = \frac{d^2\tau'_k}{du'^2_k} = -\,\frac{3\,s'^4_k}{n'_k}\sum_{v=1}^{k}\left(\frac{h_v}{h_k}\right)^4 Q^2_{v\,s}\,\underset{v}{\Delta}\frac{1}{ns}\,. \qquad \text{(v)}$$

For the line of confusion in the image we may derive the following expression from the equations established above :

$$l'_{3\,k} = \mathrm{q}'_k\, d\mathrm{u}'^3_k = -\,\frac{1}{2}\,d\mathrm{u}'^3_k\,\frac{s'^4_k}{n'_k}\sum_{v=1}^{k}\left(\frac{h_v}{h_k}\right)^4 Q^2_{v\,s}\,\underset{v}{\Delta}\frac{1}{ns}\,. \quad \text{(vi)}$$

Applying now in the usual way the Smith-Helmholtz formula, and replacing, as before, $d\mathrm{u}'_k$ by u'_k, then, since

$$n'_k\,\mathrm{u}'_k\,l'_{3k} = n_1\,\mathrm{u}_1\,l^{(k)}_3\,,$$

the **line of confusion projected into the object** is

$$l^{(k)}_3 = -\,\frac{1}{2}\,\frac{s_1^4}{n_1^4}\left(n_1\,\mathrm{u}_1\right)^3\sum_{v=1}^{k}\left(\frac{h_v}{h_1}\right)^4 Q^2_{v\,s}\,\underset{v}{\Delta}\frac{1}{ns}\,, \qquad \text{(vii)}$$

and by substituting, in accordance with the alternative notation given in § 115 (ii),

$$\mathfrak{u}_1 = \frac{m_1}{s_1 - x_1} \quad \ldots \quad \ldots \quad \ldots \quad \text{(viii)}$$

we obtain

$$\frac{n_1 l_3^{(k)}}{s_1} = \frac{1}{2}\,\frac{m_1^3 s_1^3}{(x_1 - s_1)^3} \sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^4 Q^2_{v\,s} \underset{v}{\Delta} \frac{1}{ns} \cdot \quad \ldots \quad \text{(ix)}$$

B. The Tangential Difference of the Trough Defect.

170. The trough depth $l'_{II\,k}$ of the sagittal pencil measured in a tangential direction increases by an amount $dl'_{II\,k}$ for a variation $d\mathfrak{u}$.

The magnitude of $dl'_{II\,k}$ can be determined by differentiating the invariant (§ 152, iv)

$$F = \frac{n l_{II}}{\mathfrak{f} \cos j} + \frac{J^2\, \mathfrak{t}\, d\bar{\mathfrak{u}}}{\cos j}\,\frac{1}{n\mathfrak{t}}$$

with respect to ϕ. By multiplying the resulting differential by r^{-1}, and forming the difference of the values before and after refraction, and by noting also that

$$\frac{J^2\, \mathfrak{t}\, d\bar{\mathfrak{u}}}{\cos j}$$

is in itself an invariant quantity of refraction, we thus obtain the following equation :

$$\Delta \frac{n}{\mathfrak{f}t}\frac{dl_{II}}{d\mathfrak{u}} = -\frac{J^2\, \mathfrak{t}\, d\mathfrak{u}}{\cos j}\left[J \Delta \frac{1}{n^2 \mathfrak{t}^2} - \Delta \frac{\cos j}{n\, \mathfrak{t}^2 t}\frac{d\tau}{d\mathfrak{u}}\right]$$

$$-\left[Q_t \frac{\mathfrak{t}\, d\bar{\mathfrak{u}}}{\cos j}\left(2 + \frac{\sin^2 j}{\cos j}\right) - J\, d\bar{\mathfrak{u}}\left(\tan j - \frac{1}{t}\frac{d\tau}{d\mathfrak{u}}\right) + \frac{J\mathfrak{t}}{t}\frac{d^2\bar{\mathfrak{u}}}{d\mathfrak{u}}\right] J \Delta \frac{1}{nt}$$

$$- J Q_t \Delta \frac{l_{II}}{n\mathfrak{f} \cos^3 j} - J \Delta \frac{l_{II}}{\mathfrak{f}^2 \cos j} + \Delta \frac{n}{\mathfrak{f}^2 t}\frac{d\zeta}{d\mathfrak{u}}\, l_{II} \;.$$

In this expression the symbol $\frac{d\tau}{d\mathfrak{u}}$ associated with quantities in terms of $\mathfrak{t}$, corresponds exactly to the quantity $\frac{d\tau}{d\mathfrak{u}}$ associated with the linear element t.

To reduce the expression we shall introduce the symbol $\underset{v}{D}$ which will be readily understood :

$$\underset{v}{D}\left(\frac{\mathfrak{f} - r\cos j}{r \sin j}\right) = \frac{\mathfrak{f}_v - r_v \cos j_v}{r_v \sin j_v} - \frac{\mathfrak{f}_{v-1} - r_{v-1}\cos j'_{v-1}}{r_{v-1}\sin j'_{v-1}} \;.$$

The transition formula § 152 (vii) for $d\bar{u}$, obtained in the investigation of the trough defect, will now assume the form

$$d\bar{u}_v = \frac{dv_v^2}{2} \underset{v}{D} \left(\frac{f - r\cos j}{r \sin j}\right) + du'_{v-1} ,$$

from which it follows by differentiation that

$$\frac{d^2\bar{u}_v}{du_v} = -\frac{dv_v^2}{2} \underset{v}{D} \left(\frac{(f - r\cos j)(t - r\cos j)}{r^2 \sin^2 j}\right) - \frac{dv_v^2}{2}\frac{d\zeta_v}{du_v} \underset{v}{D} \left(\frac{1}{r\sin j}\right)$$

$$+ dv_v \frac{d^2v'_{v-1}}{du'_{v-1}} \underset{v}{D} \left(\frac{f - r\cos j}{r\sin j}\right) + \frac{d^2\bar{u}_{v-1}}{du'_{v-1}} .$$

Differentiating the two invariants

$$K = \frac{t d\bar{u}}{\cos j} \text{ and } L = f . dv ,$$

and introducing the differences in front of and behind the refracting surface, we obtain the expression

$$\underset{v-1}{\Delta} \frac{t d^2\bar{u}}{t\, du} = \frac{t_{v-1}\, d\bar{u}_{v-1}}{\cos j_{v-1}} \left[J_{v-1} \underset{v-1}{\Delta} \frac{1}{nt} - J_{v-1} Q_{v-1,t} \underset{v-1}{\Delta} \frac{1}{n^2 \cos j} - \underset{v-1}{\Delta} \frac{\cos j}{tt}\frac{d\tau}{du}\right]$$

and

$$\underset{v-1}{\Delta} f\frac{d^2v}{du} = r_{v-1} f_{v-1} dv_{v-1} \left[J_{v-1} \underset{v-1}{\Delta} \frac{1}{nf} - \underset{v-1}{\Delta} \frac{\cos j}{tf} \frac{d\zeta}{du}\right] .$$

By a procedure analogous to that adopted in the investigation of coma in the restricted sense, we obtain for $\frac{d\tau}{du}$ the difference

$$\Delta \frac{n \cos^3 j}{t^2 t} \frac{d\tau}{du} = J \left[\Delta \frac{\cos^2 j}{t^2} - \frac{1}{r} \Delta \frac{\cos j}{t} - 2\, Q_t \Delta \frac{1}{nt}\right] .$$

From the differentiation of the transition formulæ we obtain finally the equations

$$t_{v-1} \frac{d^2\bar{u}_{v+1}}{du_{v+1}} + \frac{d\tau_{v+1}}{du_{v+1}} d\bar{u}_{v+1} = f_{v+1} \frac{d^2u_{v+1}}{du_{v+1}} + \frac{d\zeta_{v+1}}{du_{v+1}} d\bar{u}_{v+1} + \frac{dl_{II\,v+1}}{du_{v+1}} \quad \text{(i)}$$

and

$$\frac{dl_{II\,v+1}}{du_{v+1}} = \frac{dl'_{II\,v}}{du'_v} . \quad \ldots \quad \ldots \quad \text{(ii)}$$

This set of equations enables us to determine the successive quantities which arise as the result of the differentiation of $l'_{II,k}$ with respect to $d\mathrm{u}$.

If, now, small inclinations of the principal ray are considered, it is still possible to establish explicit formulæ for this aberration.

Neglecting small quantities of higher orders, we find,

$$\underset{v}{\Delta}\frac{n}{s^2}\frac{dl_{II}}{d\mathrm{u}} = -2\,y_v\,Q_{vx}\,Q_{vs}\,s_v\,d\bar{\mathrm{u}}_v\,\underset{v}{\Delta}\frac{1}{ns} - y_v^2\,Q_v{}^2{}_x\,\frac{d^2\bar{\mathrm{u}}_v}{d\mathrm{u}_v}\,\underset{v}{\Delta}\frac{1}{ns}\,, \quad \text{(iii)}$$

also

$$\frac{d^2\bar{\mathrm{u}}_v}{d\mathrm{u}_v} = -\frac{d\mathrm{v}_v{}^2}{2}\,\underset{v}{D}\left(\frac{s_v{}^2}{y_v{}^2}\,\frac{Q^2{}_{vs}}{Q_v{}^2{}_x}\right) + \frac{d^2\bar{\mathrm{u}}'_{v-1}}{d\mathrm{u}'_{v-1}}\,, \quad \ldots \quad \text{(iv)}$$

and

$$\frac{d^2\bar{\mathrm{u}}'_{v-1}}{d\mathrm{u}'_{v-1}} = \frac{d^2\bar{\mathrm{u}}_{v-1}}{d\mathrm{u}_{v-1}}\,. \quad \ldots \quad \ldots \quad \text{(v)}$$

Introducing this latter relation in the preceding one and performing the operation indicated by the symbol $\underset{v}{D}$, we obtain the very simple recurrence formula

$$\frac{d^2\bar{\mathrm{u}}_v}{d\mathrm{u}_v} + \frac{d\mathrm{v}^2{}_v}{2}\,\frac{s_v{}^2}{y_v{}^2}\,\frac{Q^2{}_{vs}}{Q^2{}_{vx}} = \frac{d^2\bar{\mathrm{u}}_{v-1}}{d\mathrm{u}_{v-1}} + \frac{d\mathrm{v}^2{}_{v-1}}{2}\,\frac{s^2{}_{v-1}}{y^2{}_{v-1}}\,\frac{Q^2{}_{v-1\,s}}{Q^2{}_{v-1\,x}}\,. \quad \text{(vi)}$$

This expression vanishes in front of the first surface, since it is the differential in terms of u of the constant ψ_1, so that we shall have finally

$$\frac{d^2\bar{\mathrm{u}}_v}{d\mathrm{u}_v} = -\frac{d\mathrm{v}_v{}^2}{2}\,\frac{s_v{}^2}{y^2{}_v}\,\frac{Q^2{}_{vs}}{Q^2{}_{vx}}\,. \quad \ldots \quad \ldots \quad \text{(vii)}$$

Substituting in the first expression this quantity as well as the value of $s_v d\bar{\mathrm{u}}_v$ which we obtained in § 154 (i, a) in the course of our investigation of the trough defect, we arrive at the following equation:

$$\underset{v}{\Delta}\frac{n}{s^2}\frac{dl_{II}}{d\mathrm{u}} = -\,s_v{}^2 d\mathrm{v}_v{}^2\,Q^2{}_{vs}\,\underset{v}{\Delta}\frac{1}{ns} + \frac{1}{2}\,s_v{}^2 d\mathrm{v}_v{}^2\,Q^2{}_{vs}\,\underset{v}{\Delta}\frac{1}{ns} \quad \text{(viii)}$$

and hence

$$\underset{v}{\Delta}\frac{n}{s^4}\,\frac{dl_{II}}{d\mathrm{u}\,d\mathrm{v}^2} = -\frac{1}{2}\,Q^2{}_v\,\underset{v}{\Delta}\frac{1}{ns}\,. \quad \ldots \quad \text{(ix)}$$

Assuming that there are k surfaces, we obtain the following expression for the line of confusion after the k^{th} surface:

$$\frac{l'_{III\,k}}{d\mathrm{u}'_k\,d\mathrm{v}'^2_k} = \frac{dl'_{II\,k}}{d\mathrm{u}'_k\,d\mathrm{v}'^2_k} = -\frac{s'^4_k}{n'_k}\sum_{v=1}^{k}\left(\frac{h_v}{h_k}\right)^4 Q^2{}_{vs}\,\underset{v}{\Delta}\frac{1}{ns}\,, \quad \text{(x)}$$

and projecting this in the usual way into the object by means of the equation

$$n_k' \mathrm{u}_k' l'_{III\,k} = n_1 \mathrm{u}_1 l^{(k)}_{III},$$

we obtain the expression

$$l^{(k)}_{III} = -\frac{1}{2}\frac{s_1^4}{n_1^4}(n_1 \mathrm{u}_1)(n_1 \mathrm{v}_1)^2 \sum_{v=1}^{k}\left(\frac{h_v}{h_1}\right)^4 Q^2_{vs} \underset{v}{\Delta}\frac{1}{ns} \quad \ldots \quad \text{(xi)}$$

or, by the alternative notation,

$$\frac{n_1 l^{(k)}_{III}}{s_1} = \frac{1}{2}\frac{m_1 M_1^2 s_1^3}{(x_1 - s_1)^3}\sum_{v=1}^{k}\left(\frac{h_v}{h_1}\right) Q^2_{vs} \underset{v}{\Delta}\frac{1}{ns}. \quad \ldots \quad \text{(xii)}$$

C. The Second Tangential Difference of the Sagittal Intercept.

171. To indicate that the sagittal intercept is a function of u we shall write it in the form of an expanded series:

$$\zeta = f + \mathsf{l}\,\mathrm{u} + \mathsf{k}\,\mathrm{u}^2 .$$

Thus, as in § 169, the increment of the sagittal line of confusion which arises from the second tangential difference of the sagittal intercepts is

$$L_3 = \mathsf{k}\,\mathrm{u}^2\,\mathrm{v};$$

and hence we obtain immediately the following expression for k

$$\mathsf{k} = \frac{1}{2}\frac{d^2\zeta}{d\mathrm{u}^2}.$$

To obtain an expression for $\frac{d^2\zeta}{d\,\mathrm{u}^2}$ we must differentiate the f-invariant for the second time and multiply the result by r^{-2}, which, after reduction, gives us the equation

$$\frac{1}{r^2}\frac{d^2 Q_f}{d\phi^2} = -Q_t\left[\frac{1}{r^2} + \frac{1}{f^2} - \frac{\cos j}{rt}\right] - J^2\left[\frac{Q_t}{n^2\, r\, t\cos j} - \frac{\cos j}{n\, r\, t^2} + \frac{2}{nf^3}\right]$$

$$+ J\left[\frac{4\cos j}{tf^3} + \frac{\cos j}{t^2 f^2} - \frac{Q_t}{n\,t f^2\cos j}\right]\frac{d\zeta}{d\mathrm{u}} - \frac{2\,n^2\cos^2 j}{t^2 f^3}\left(\frac{d\zeta}{d\mathrm{u}}\right)^2$$

$$+ \frac{Q_t}{n\,t^2 f^2}\frac{d\zeta}{d\mathrm{u}}\frac{d\tau}{d\mathrm{u}} + \frac{n\cos^2 j}{t^2 f^2}\frac{d^2\zeta}{d\mathrm{u}^2}, \quad \ldots \quad \text{(i)}$$

whence we might derive a recurrence formula for finite inclinations of the principal ray.

Applying this equation to small inclinations and neglecting infinitesimals of the second order, we deduce at once the simple relation :

$$\Delta \frac{n}{s^4} \frac{d^2 \zeta}{d\,\mathrm{u}^2} = - Q_s^{\,2} \Delta \frac{1}{ns}, \quad \ldots \quad \ldots \quad \text{(ii)}$$

and if we assume that the **object is free from aberration,** we obtain the formula

$$\frac{d^2 \zeta'_k}{d\,\mathrm{u}'^{\,2}_k} = - \frac{s'^{\,4}_k}{n'_k} \sum_{v=1}^{k} \left(\frac{h_v}{h_k}\right)^4 Q^2_{v\,s} \mathop{\Delta}_{v} \frac{1}{ns}. \quad \ldots \quad \text{(iii)}$$

From the preceding investigation the increment of the line of confusion in the image becomes

$$L'_{3\,k} = \mathsf{K}'_k \mathrm{u}'^{\,2}_k \mathrm{v}'_k = \frac{1}{2} \frac{d^2 \zeta'_k}{d\,\mathrm{u}'^{\,2}_k} \mathrm{u}'^{\,2}_k \mathrm{v}'_k \quad \ldots \quad \text{(iv)}$$

and by referring it back to the object in the usual way with the aid of the equation

$$n'_k \mathrm{u}'_k L'_{3\,k} = n_1 \mathrm{u}_1 L_3^{(k)},$$

we obtain the final expression

$$L_3^{(k)} = - \frac{1}{2} \frac{s_1^{\,4}}{n_1^{\,4}} (n_1 \mathrm{u}_1)^2 (n_1 \mathrm{v}_1) \sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^4 Q^2_{vs} \mathop{\Delta}_{v} \frac{1}{ns}, \quad \text{(v)}$$

or, by the alternative notation,

$$\frac{n_1 L_3^{(k)}}{s_1} = \frac{1}{2} \frac{m_1^{\,2} M_1 s_1^{\,3}}{(x_1 - s_1)^3} \sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^4 Q^2_{vs} \mathop{\Delta}_{v} \frac{1}{ns}: \quad \ldots \quad \text{(vi)}$$

D. The Spherical Aberration of the Sagittal Pencil.

172. If the longitudinal aberration $\mathsf{A}'_f \mathrm{v}'^2$ of the sagittal pencil is known, we may find the appropriate line of confusion L'_{III} from the relation

$$L'_{III} = \mathsf{A}'_f \mathrm{v}'^2 \mathrm{v}'. \quad \ldots \quad \ldots \quad \text{(i)}$$

Now, if quantities of the third order are considered, the required longitudinal aberration of the point of intersection of the two sagittal rays inclined at an angle $d\mathrm{v}$ to the principal ray, measured from the point of intersection of adjacent sagittal rays, is equal to the tangential increment $d\,\zeta$ of the sagittal intercepts due to a variation $d\bar{\mathrm{u}}$ in the inclination of the principal ray.

We must accordingly apply the formula of the triangle aberration (§ 156, i) to the pencil which was defined in terms of $d\mathfrak{u}$ and $\mathfrak{t}$ in our investigation of the trough defect, and we thus obtain the equation

$$\underset{v}{\Delta} \frac{n}{j^2} d\zeta = \frac{\mathfrak{t}_v d\mathfrak{u}_v}{\cos j_v} J_v \underset{v}{\Delta} \left[\frac{1}{j^2} - \frac{\cos j}{r\,\mathfrak{t}} \right]. \quad \ldots \quad \text{(ii)}$$

In this equation the values of $\mathfrak{t}_v$ and $d\mathfrak{u}_v$, corresponding to finite inclinations of the principal ray, should be taken successively from the calculation for the trough defect, and it is only necessary that

$$d\,\zeta'_v = d\,\zeta_{v+1}\,.$$

For infinitely small inclinations of the principal ray we then obtain

$$\underset{v}{\Delta} \frac{n}{s^2} d\zeta = -\, s_v\, d\mathfrak{u}_v\, y_v\, Q_{vx}\, Q_{vs} \underset{v}{\Delta} \frac{1}{ns},$$

and, since, by § 154 (i, a),

$$s_v\, d\mathfrak{u}_v = \frac{s_v^2}{2}\, d\mathrm{v}_v \frac{Q_{vs}}{y_v\, Q_{vx}}$$

and

$$s_v'\, d\mathrm{v}_v' = s_v\, d\mathrm{v}_v$$

we have finally

$$\underset{v}{\Delta} \frac{n}{s^4} \frac{d\zeta}{d\mathrm{v}^2} = -\frac{1}{2} Q^2_{vs} \underset{v}{\Delta} \frac{1}{ns}, \quad \ldots \quad \text{(iii)}$$

whence we obtain by the use of the recurrence formula

$$\frac{L'_{III\,k}}{d\,\mathrm{v}'^2_k} = \mathsf{A}_{fk} = \frac{d\,\zeta'_k}{d\mathrm{v}'^2_k} = -\frac{1}{2} \frac{s'^4_k}{n'_k} \sum_{v=1}^{k} \left(\frac{h_v}{h_k}\right)^4 Q^2_{vs} \underset{v}{\Delta} \frac{1}{ns}. \text{(iv)}$$

By applying the equation $n'_k\, \mathrm{v}'_k\, L'_{III\,k} = n_1\, \mathrm{v}_1\, L^{(k)}_{III}$ we then obtain, according to the choice of notation, one or the other of the two equations:

$$L^{(k)}{}_{III} = \frac{1}{2} \frac{s_1^4}{n_1^4} (n_1\, \mathrm{v}_1)^3 \sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^4 Q^2_{vs} \underset{v}{\Delta} \frac{1}{ns} \qquad \text{(v)}$$

or

$$\frac{n_1\, L^{(k)}{}_{III}}{s_1} = \frac{1}{2} \frac{M_1^3\, s_1^3}{(x_1 - s_1)^3} \sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^4 Q^2_{vs} \underset{v}{\Delta} \frac{1}{ns}. \qquad \text{(vi)}$$

It will thus be seen that so long as the principal ray is inclined at small angles w these four entirely different expressions for defects which may be attributed to spherical aberration in the restricted sense are related by the fact that they have the same coefficient. This is identical with the coefficient obtained in the investigation

of the radius of the circle of confusion of a point on the axis (*see* § 122, v and vi). All four expressions of the spherical aberration accordingly vanish simultaneously.

The methods given in the preceding articles for determining the lines of confusion for angular apertures up to and including quantities of the third order, when the principal ray is inclined at finite angles, enable us to form the following conclusions regarding the correction appropriate to extra-axial points.

In the case of a centred spherical system the union of the rays forming the image of an object-point situated on the axis can be extended to terms up to the third order. On the other hand, to obtain an equally sharp image of an extra-axial point at a finite distance from the axis it is necessary to satisfy four conditions. Of these four conditions one is astigmatic and the other three comatic.

When the spherical correction of a point on the axis has been obtained by satisfying a single condition, the sharp definition of the image extends to the terms of the fifth order. To ensure a similar degree of optical sharpness in the case of an extra-axial point at a finite distance from the axis it is necessary to satisfy, in addition to the four conditions named, four other conditions of spherical aberration in the restricted sense, and five additional conditions of coma in the wider sense. These latter correspond to the five aperture-terms

$$m^4\,;\quad m^3 M\,;\quad m^2 M^2\,;\quad m M^3\,;\quad M^4\,;$$

which, with the first power of l, form products of the fifth degree.

In the case of points close to the axis the correction of the astigmatism follows naturally. The correction of the spherical aberration is identical with the removal of the spherical aberration of paraxial rays, and the eight conditions are satisfied by establishing a constant sine ratio. It will thus be seen that only two conditions remain to be satisfied.

8.—SEIDEL'S THEORY OF ABERRATIONS OF THE THIRD ORDER.

A. Kerber's Investigation of Seidel's Five Image Defects.

173. In the most general case of refraction, namely, that of a skew ray, as investigated in Chapter II, § 41, if we expand the trigonometrical functions into a series, and neglect all terms beyond the third order, we shall ultimately arrive at a general expression for the departures from the Gauss values which may arise if we take account of the spherical aberration in the sense defined.

Of these expressions there are three, as illustrated in Fig. 79, namely, one for the **longitudinal aberration** δs, such as was used in the investigation of the longitudinal spherical aberration of points on the axis; the second for the **tangential aberration** δg, and the third for the **sagittal aberration** δG, the magnitudes of which

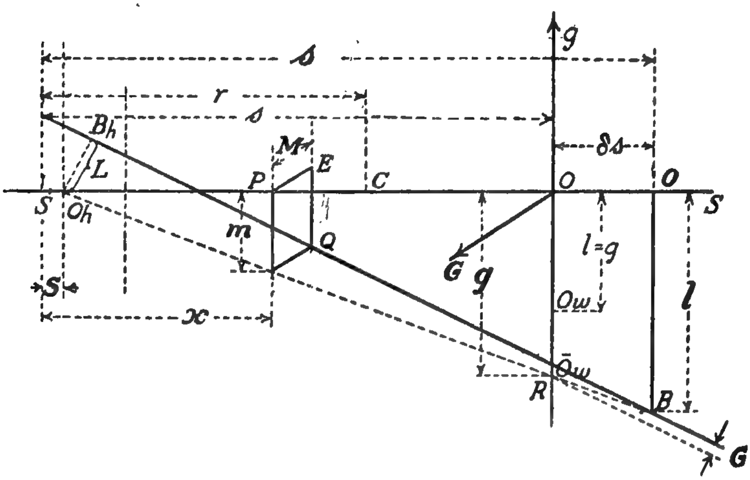

FIG. 79.

$SC = r$; $SO = \mathfrak{s}$; $\mathbf{SO = s}$; $SO_h = S$; $SP = x$; $O_hB_h = \mathbf{L}$; $OB = \mathbf{l}$; $PE = \mathbf{M}$; $EQ = \mathbf{m}$; $OO_w = l = g$; $OR = \mathfrak{g}$; $R\bar{O}_w = \mathbf{G}$; $OO = \delta s$.

Kerber's investigation of Seidel's image defects.

have been determined in the individual cases previously investigated. We shall now study these three aberrations in succession, and we shall show, in particular, that the two last aberrations, which interest us more especially, are related to the longitudinal aberration by very simple expressions.

174. The Longitudinal Aberration.—It will be recollected that in § 41, the determination of the path of a skew ray was reduced by the introduction of the plane of incidence to the determination of the path of a ray in this auxiliary plane of incidence and proceeding from a point on the axis. Clearly, our previous investigation of the longitudinal aberration A of points on the axis is applicable here.

By means of the equation derived in § 120, viz.:

$$\mathfrak{s} - s = \delta s = \mathsf{A}u^2 = \mathsf{A}\,\frac{r^2}{s^2}\,\phi^2,$$

we may write the invariant given in § 120 (vii) in the form

$$Q = Q_s + \left[\frac{1}{2}\,\frac{Q_s^2}{n}\,su^2 + \frac{n\,\delta s}{s^2}\right]. \quad \ldots \quad \text{(i)}$$

In this simple case the angle u corresponds to the angle v in the plane of incidence of the skew ray, which was defined in § 41 as follows :

$$\cos \mathrm{v} = \cos(\varepsilon - \gamma)\cos\delta,$$

and, since we are confining the investigation to powers of the third and lower orders, the angle may be expressed thus

$$d\mathrm{v}^2 = (d\varepsilon - d\gamma)^2 + d\delta^2\,. \quad \ldots \quad \ldots \quad \text{(ii)}$$

As in the preceding investigation, remembering that we are concerned with small angles only, we shall again denote them by the symbols v, ε, γ, δ. As the angles v, ε, γ, δ occur only in terms of their second powers, we may in the case of their first powers confine ourselves to the Gauss values, so that in the values particularised in § 41, viz.

$$\varepsilon = \frac{\boldsymbol{l}}{\boldsymbol{S} - \boldsymbol{s}} \quad \gamma = \frac{\boldsymbol{l}}{r - \boldsymbol{s}}\,; \quad \delta = \frac{\boldsymbol{L}}{\boldsymbol{s} - \boldsymbol{S}}\,, \quad \ldots \quad \text{(iii)}$$

we may replace letters in heavy type by the corresponding letters in light type, since any higher degree of exactness would disappear in the squares of these quantities. By an easy transition we obtain accordingly

$$\varepsilon - \gamma = \frac{l\,(r - S)}{(S - s)\,(r - s)} \text{ and } \delta = \frac{L}{s - S},$$

hence

$$\mathrm{v}^2 = \left(\frac{r - S}{r - s}\right)^2\left(\frac{l}{S - s}\right)^2 + \left(\frac{L}{S - s}\right)^2. \quad \ldots \quad \text{(iv)}$$

We shall next eliminate the quantities relating to B_h by means of the following set of equations derived from § 38 (i), (ii), viz.

$$\frac{\boldsymbol{x} - \boldsymbol{S}}{\boldsymbol{s} - \boldsymbol{S}} = \frac{\boldsymbol{m}}{\boldsymbol{l}}\,; \quad \frac{\boldsymbol{L}}{\boldsymbol{M}} = \frac{\boldsymbol{s} - \boldsymbol{S}}{\boldsymbol{s} - \boldsymbol{x}}\,; \quad \frac{\boldsymbol{l}}{\boldsymbol{l} - \boldsymbol{m}} = \frac{\boldsymbol{s} - \boldsymbol{S}}{\boldsymbol{s} - \boldsymbol{x}}\,, \quad \ldots \quad \text{(v)}$$

in which we may likewise confine ourselves to a consideration of the paraxial quantities.

With the aid of the identity

$$r - S \equiv r - x + x - S$$

we obtain from (iv)

$$\mathrm{v}^2 = \left(\frac{r-x}{r-s} \cdot \frac{l-m}{x-s} - \frac{m}{r-s}\right)^2 + \left(\frac{M}{x-s}\right)^2$$

$$= \frac{m^2 + M^2}{(x-s)^2} - \frac{2\,m\,l}{(x-s)^2}\,\frac{x\,Q_x}{s\,Q_s} + \frac{l^2}{(x-s)^2}\,\frac{x^2\,Q^2_x}{s^2\,Q^2_s}\,.$$

Since, after refraction, the paraxial quantities l, L, l', L' remain in perspective relation to the centre of the sphere, it will be seen from Fig. 79 that they are connected by the following ratios :

$$\frac{l}{s-r} = \frac{l'}{s'-r}\,; \quad \frac{m}{x-r} = \frac{m'}{x'-r}\,; \quad \frac{M}{x-r} = \frac{M'}{x'-r}\,,$$

from which it follows, as may be noted here, that the quantities

$$\frac{l\,x}{x-s}\,; \quad \frac{m\,s}{x-s}\,; \quad \frac{Ms}{x-s}$$

have the invariant property for the refraction at a surface. By means of these ratios we find that $\mathrm{v}^2 s^2 = \mathrm{v}'^2 s'^2$, and thus we are now able to apply the expansion to the intercept indicated in Fig. 20 (§ 41), viz.

$$\mathbf{s}_s = s_s + \delta s_s = s + \delta s_s\,,$$

and obtain the expressions

$$Q_{\mathbf{s}_s} = Q_s + \frac{1}{2}\,\frac{Q_s^{\,2}}{n}\,s\,\mathrm{v}^2 + \frac{n\,\delta\,s_s}{s^2}$$

$$Q_{\mathbf{s}_s} = Q_s + \frac{1}{2}\,\frac{Q_s^{\,2}}{n'}\,s'\,\mathrm{v}'^2 + \frac{n'\,\delta s_s'}{s'^2}\,,$$

whence

$$\Delta\,\frac{n\,\delta s_s}{s^2} = -\,\frac{1}{2}\,Q_s^{\,2}\,s^2\,\mathrm{v}^2\,\Delta\,\frac{1}{ns}$$

$$= -\,\frac{1}{2}\,\frac{m^2 + M^2}{(x-s)^2}\,s^2\,Q_s^{\,2}\,\Delta\,\frac{1}{ns} + \frac{ml}{(x-s)^2}\,sx\,Q_s\,Q_x\,\Delta\,\frac{1}{ns}$$

$$-\,\frac{1}{2}\,\frac{l^2}{(x-s)^2}\,x^2\,Q_x^{\,2}\,\Delta\,\frac{1}{ns}. \quad \ldots \quad \ldots \quad \text{(vi)}$$

The longitudinal aberration thus referred to the secondary axis now requires to be determined with reference to the principal axis. This may be accomplished with the aid of equation (v) established in § 41, viz.

$$\mathfrak{s} - r = (\mathfrak{s}_s - r)\cos\gamma = (\mathfrak{s}_s - r)\left(1 - \frac{\gamma^2}{2}\right).$$

As we have seen (iii),

$$\gamma = -\frac{l}{s - r},$$

hence

$$s + \delta s - r = s + \delta s_s - r - (\mathfrak{s}_s - r)\frac{l^2}{2(s-r)^2},$$

$$= s + \delta s_s - r - (s - r)\frac{l^2}{2(s-r)^2},$$

or

$$\frac{n\,\delta s}{s^2} = \frac{n\,\delta s_s}{s^2} - \frac{1}{2}\frac{l^2}{(x-s)^2}\frac{x^2}{rs}\frac{(Q_x - Q_s)^2}{Q_s};$$

and, similarly,

$$\frac{n'\,\delta s'}{s'^2} = \frac{n'\delta s'_s}{s'^2} - \frac{1}{2}\frac{l^2}{(x-s)^2}\frac{x^2}{rs'}\frac{(Q_x - Q_s)^2}{Q_s},$$

whence

$$\Delta\frac{n\delta s}{s^2} = \Delta\frac{n\,\delta s_s}{s^2} + \frac{1}{2}\frac{l^2x^2}{(x-s)^2}(Q_x - Q_s)^2\frac{1}{r}\Delta\frac{1}{n}$$

$$= -\frac{1}{2}\frac{m^2 + M^2}{(x-s)^2}s^2\,Q_s^2\,\Delta\frac{1}{n\,s} + \frac{ml}{(x-s)^2}sx\,Q_s\,Q_x\,\Delta\frac{1}{ns}$$

$$-\frac{1}{2}\frac{l^2\,x^2}{(x-s)^2}\left[Q^2_x\,\Delta\frac{1}{ns} - (Q_x - Q_s)^2\frac{1}{r}\Delta\frac{1}{n}\right]. \qquad \text{(vii)}$$

175. The Tangential Aberration. From Fig. 79 it will be seen that the tangential aberration $\mathfrak{g} - g$ is $OR - OO_w$, and since $g = l$, its magnitude is

$$\delta g = \mathfrak{g} - l.$$

Also, let

$$\delta l = \mathfrak{l} - l = \mathfrak{l} - \mathfrak{g} + \delta g.$$

It now remains to express the difference $\mathfrak{l} - \mathfrak{g}$ in terms of the longitudinal aberration, as determined above. This may be accomplished with the aid of the ratios

$$\mathfrak{m} : \mathfrak{g} : \mathfrak{l} = (\mathfrak{x} - \mathfrak{S}) : (s - \mathfrak{S}) : (\mathfrak{s} - \mathfrak{S}),$$

thus

$$\delta g = \delta l - \frac{\mathfrak{m} - \mathfrak{g}}{\mathfrak{x} - s}\delta s.$$

For δl we may obtain another expression from the equation

$$\boldsymbol{l} = (r - \boldsymbol{s}) \tan \bar{\gamma}\,; \quad l = (r - s) \tan \gamma,$$

where $\bar{\gamma}$ corresponds to values of $\boldsymbol{s}$, and therefore

$$\delta l = (r - s)\, \delta \tan \gamma - \delta s \tan \bar{\gamma}\,,$$

hence

$$\delta g = (r - s)\, \delta \tan \gamma - \delta s \left(\tan \bar{\gamma} + \frac{\boldsymbol{m} - \boldsymbol{g}}{\boldsymbol{x} - s} \right),$$

and a corresponding relation may be obtained in terms of the accented quantities. In the coefficients of δs we may, without adversely affecting the desired degree of accuracy, replace $\bar{\gamma}$ by γ, and the quantities in heavy type by those represented in light type. We shall then have

$$\tan \bar{\gamma} + \frac{\boldsymbol{m} - \boldsymbol{g}}{\boldsymbol{x} - s} = \frac{1}{s} \left(\frac{ms}{x - s} - \frac{lx}{x - s} \frac{Q_x}{Q_s} \right) \text{ nearly.}$$

In order to re-introduce in our expression the term $\frac{n\delta s}{s^2}$, if we form the equivalent of $\frac{n\delta g}{s}$, then since $\bar{\gamma}$, γ are not changed by the refraction, we can obtain the following equations :

$$\frac{n\delta g}{s} = - \frac{Q_s}{r}\, \delta \tan \gamma - \frac{n\delta s}{s^2} \left(\frac{ms}{x - s} - \frac{lx}{x - s} \frac{Q_x}{Q_s} \right)$$

and

$$\frac{n'\delta g'}{s'} = - \frac{Q_s}{r}\, \delta \tan \gamma - \frac{n'\delta s'}{s'^2} \left(\frac{ms}{x - s} - \frac{lx}{x - s} \frac{Q_x}{Q_s} \right),$$

so that

$$\Delta \frac{n\delta g}{s} = - \left(\frac{ms}{x - s} - \frac{lx}{x - s} \frac{Q_x}{Q_s} \right) \Delta \frac{n\delta s}{s^2}$$

$$= \frac{1}{2} \frac{(m^2 + M^2)\, m}{(x - s)^3} s^3 Q^2_s \Delta \frac{1}{ns} - \frac{1}{2} \frac{(3\, m^2 + M^2) l}{(x - s)^3} s^2 x Q_s Q_x \Delta \frac{1}{ns}$$

$$+ \frac{1}{2} \frac{ml^2}{(x - s)^3} s x^2 \left[3\, Q_x{}^2 \Delta \frac{1}{ns} - (Q_x - Q_s)^2\, \frac{1}{r} \Delta \frac{1}{n} \right]$$

$$- \frac{1}{2} \frac{l^3}{(x - s)^3} x^3 \left[\frac{Q_x^3}{Q_s} \Delta \frac{1}{ns} - \frac{Q_x}{Q_s} (Q_x - Q_s)^2 \frac{1}{r} \Delta \frac{1}{n} \right].$$

176. The Sagittal Aberration. The sagittal aberration is represented in Fig. 79 by $R\bar{O}_w$, and its magnitude is

$$\delta G = G - 0 = G = \frac{\mathbf{M}}{\mathbf{s} - x}\,\delta s\,.$$

For the same reason, as before, we may replace the quantities in heavy type in the coefficients of δs by the corresponding quantities in light type, and we shall again give the expression in a form in which $\frac{n\delta s}{s^2}$ appears on the right side, viz.:

$$\frac{n\delta G}{s} = -\frac{Ms}{x - s}\frac{n\delta s}{s^2}\,,$$

thus

$$\Delta\frac{n\delta G}{s} = -\frac{Ms}{x - s}\Delta\frac{n\delta s}{s^2}$$

$$= \frac{1}{2}\frac{(m^2 + M^2)\,M}{(x - s)^3}s^3 Q_s^2\,\Delta\frac{1}{ns} - \frac{mMl}{(x-s)^3}s^2 x\,Q_s\,Q_x\,\Delta\frac{1}{ns}$$

$$+\frac{1}{2}\frac{Ml^2}{(x - s)^3}sx^2\left[Q_x^2\,\Delta\frac{1}{ns} - (Q_x - Q_s)^2\frac{1}{r}\Delta\frac{1}{n}\right].$$

These expressions for the differences can be made to supply, as before, a set of recurrence formulæ in the following manner. Appending the surface index, we have

$$\underset{v}{\Delta}\frac{n\,\delta g}{s} = A_v\,; \quad \underset{v}{\Delta}\frac{n\delta G}{s} = B_v\,.$$

where A_v is a sum composed of four terms and B_v of three terms. In the case of a system of k surfaces in which the object is originally free from aberration,

$$\delta g_1 = 0\,; \;\; \delta G_1 = 0\,,$$

and the general equations

$$n'_k\frac{\delta g'_k}{s'_k} = n_k\frac{\delta g_k}{s_k} + A_k\,; \qquad n'_k\frac{\delta G'_k}{s'_k} = n_k\frac{\delta G_k}{s_k} + B_k$$

assume the final forms

$$n'_k\frac{\delta g'_k}{s'_k} = \frac{h_1}{h_k}\sum_{v=1}^{k}\frac{h_v}{h_1}A_v\,; \qquad n'_k\frac{\delta G'_k}{s'_k} = \frac{h_1}{h_k}\sum_{v=1}^{k}\frac{h_v}{h_1}B_v\,.$$

If now we project the tangential aberration $\delta g'_k$ of tangential pencils, and the sagittal aberration $\delta G'_k$ of sagittal pencils, in the usual way back into the object, we obtain finally the following expressions for the lines of aberration :

For the **tangential line of aberration :**

$$\frac{n_1\,\delta g_1^{(k)}}{s_1} = \frac{1}{2}\frac{(m_1^2 + M_1^2)\,m_1}{(x_1 - s_1)^3}\,s_1^3 \sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^4 Q^2_{vs}\,\underset{v}{\Delta}\frac{1}{ns}$$

$$- \frac{1}{2}\frac{(3\,m_1^2 + M_1^2)\,l_1}{(x_1 - s_1)^3}\,s_1^2\,x_1 \sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^3 \frac{y_v}{y_1}\,Q_{vs}\;Q_{vx}\,\underset{v}{\Delta}\frac{1}{ns}$$

$$+ \frac{1}{2}\frac{m_1\,l_1^2}{(x_1 - s_1)^3}\,s_1\,x_1^2 \sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^2 \left(\frac{y_v}{y_1}\right)^2 \left[\,3\,Q^2_{vx}\,\underset{v}{\Delta}\frac{1}{ns} - (Q_{vx} - Q_{vs})^2\frac{1}{r_v}\,\underset{v}{\Delta}\frac{1}{n}\right]$$

$$- \frac{1}{2}\frac{l_1^3}{(x_1 - s_1)^3}\,x_1^3 \sum_{v=1}^{k} \frac{h_v}{h_1}\left(\frac{y_v}{y_1}\right)^3 \left[\frac{Q^3_{vx}}{Q_{vs}}\,\underset{v}{\Delta}\frac{1}{ns} - \frac{Q_{vx}}{Q_{vs}}(Q_{vx} - Q_{vs})^2\frac{1}{r_v}\,\underset{v}{\Delta}\frac{1}{n}\right]; \text{(i)}$$

and for the **sagittal line of aberration :**

$$\frac{n_1\,\delta G_1^{(k)}}{s_1} = \frac{1}{2}\frac{(m_1^2 + M_1^2)\,M_1}{(x_1 - s_1)^3}\,s_1^3 \sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^4 Q^2_{vs}\,\underset{v}{\Delta}\frac{1}{ns}$$

$$- \frac{m_1\,M_1\,l_1}{(x_1 - s_1)^3}\,s_1^2\,x_1 \sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^3 \frac{y_v}{y_1}\,Q_{vs}\;Q_{vx}\,\underset{v}{\Delta}\frac{1}{ns}$$

$$+ \frac{1}{2}\frac{M_1\,l_1^2}{(x_1 - s_1)^3}\,s_1\,x_1^2 \sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^2 \left(\frac{y_v}{y_1}\right)^2 \left[\,Q^2_{vx}\,\underset{v}{\Delta}\frac{1}{ns} - (Q_{vx} - Q_{vs})^2\frac{1}{r_v}\,\underset{v}{\Delta}\frac{1}{n}\right]. \quad \text{(ii)}$$

If we compare these results with those previously obtained we can establish the following relations :

$$\delta g_1^{(k)} = l_3^{(k)} + l_{III}^{(k)} + l_2^{(k)} + l_{II}^{(k)} + l_1^{(k)} + l_0^{(k)}$$

and

$$\delta G_1^{(k)} = L_3^{(k)} + L_{III}^{(k)} + L_2^{(k)} + L_1^{(k)}.$$

The agreement of the results of these entirely distinct investigations gives further confirmation of their completeness.

B. Seidel's Image Defects due to Non-spherical (Deformed) Surfaces.

177. We shall define the nature and amount of the deviation from the spherical form by adding to the radius, at every point B of the arc b of the tangential section, a certain increment Σ, as indicated generally in § 24. The radial increment is given by the expression

$$\Sigma = \frac{1}{4}\,\kappa b^4 = \frac{1}{4}\,\kappa r^4\,\phi^4.$$

It will readily be seen that if at B in Fig. 80, we describe a curve parallel to the new meridian curve, the angle θ between the new normal and the radius of the circle for small inclinations is given by

$$\theta = \frac{d\Sigma}{db} = \kappa b^3 = \kappa r^3\,\phi^3.$$

From Fig. 81 it follows immediately, that the angle θ, which represents the change of the inclination of the normals, corresponds to a change of direction η' of the refracted ray of the magnitude

$$\eta' = i' - \mathrm{i}' - \theta$$

or

$$\eta' = \frac{n - n'}{n'}\,\theta = \frac{n - n'}{n'}\,\kappa r^3\,\phi^3\ ,$$

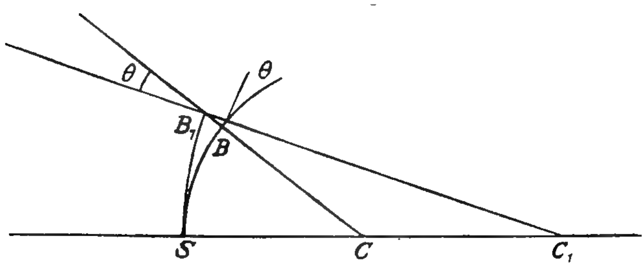

FIG. 80.
$\widehat{SB} = b\ ;\ BB_1 = \Sigma.$
Illustrating Seidel's image-defects of non-spherical surfaces.

which is equivalent to a displacement of the image-point in the Gauss plane by an amount

$$s'\eta' = \frac{n - n'}{n'}\,\kappa r^3\phi^3 s'\ .$$

This displacement can be resolved into a tangential component $\delta\mathrm{g}'$ and a sagittal component $\delta\mathrm{G}'$.

Thus

$$s'\eta' = \sqrt{\delta\mathrm{g}'^2 + \delta\mathrm{G}'^2}\,.$$

Similarly, we may resolve the arc b into components $\mathbf{h}$ and $\mathbf{H}$, thus

$$r\phi = \sqrt{\mathbf{h}^2 + \mathbf{H}^2},$$

and accordingly

$$\frac{n'\delta g'}{s'} = (n - n')\,\kappa\,(\mathbf{h}^2 + \mathbf{H}^2)\,\mathbf{h} \ldots \quad \ldots \quad \text{(i)}$$

$$\frac{n'\delta G'}{s'} = (n - n')\,\kappa\,(\mathbf{h}^2 + \mathbf{H}^2)\,\mathbf{H}\,. \quad \ldots \quad \text{(ii)}$$

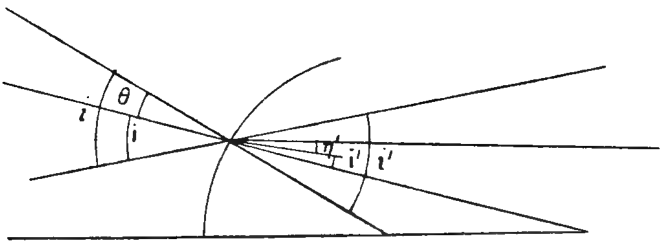

FIG. 81.
Change of direction η' of a ray refracted at a non-spherical surface.

Now from Fig. 79 it will be seen that

$$\frac{\mathbf{h}}{l} = \frac{SO_h}{OO_h} = \frac{S}{S - s},$$

and by eliminating the quantity S with the aid of the formula

$$\frac{S - x}{S - s} = \frac{m}{l},$$

we find that

$$\mathbf{h} = \frac{lx - ms}{x - s};$$

also,

$$\mathbf{H} = -\frac{Ms}{x - s},$$

whence

$$\mathbf{h}^2 + \mathbf{H}^2 = \frac{(m^2 + M^2)\,s^2 - 2mlsx + l^2x^2}{(x - s)^2} \quad \ldots \quad \text{(iii)}$$

and therefore

$$\frac{n\delta g'}{s'} =$$

$$\frac{(m^2 + M^2)\,ms^3\kappa\Delta n - (3m^2 + M^2)\,lxs^2\kappa\Delta n + 3ml^2sx^2\kappa\Delta n - l^3x^3\kappa\Delta n}{(x - s)^3} \quad \text{(iv)}$$

and

$$\frac{n\delta G'}{s'} = \frac{(m^2 + M^2)\,Ms^3\kappa\Delta n - 2mMls^2x\kappa\Delta n + Ml^2sx^2\kappa\Delta n}{(x - s)^3}\,. \ldots \quad \text{(v)}$$

It now only remains to establish with the aid of the recurrence formulæ embodied in these equations a summation for a system in which all the surfaces are deformed. Applying the equations connecting the accented and unaccented quantities, as given at the beginning of this section, we arrive in the usual way at the following expressions for the tangential and sagittal aberrations projected into the object, viz.

$$\left.\begin{aligned}\frac{n_1\delta\mathfrak{g}_1^{(k)}}{s_1} &= \frac{(m_1^2+M_1^2)\,m_1}{(x_1-s_1)^3}\,s_1^3\sum_{v=1}^{k}\left(\frac{h_v}{h_1}\right)^4\kappa_v\underset{v}{\Delta n}\\ &- \frac{(3m_1^2+M_1^2)l_1}{(x_1-s_1)^3}\,s_1^2x_1\sum_{v=1}^{k}\left(\frac{h_v}{h_1}\right)^3\frac{y_v}{y_1}\kappa_v\underset{v}{\Delta n}\\ &+ \frac{3m_1\,l_1^2}{(x_1-s_1)^3}\,s_1x_1^2\sum_{v=1}^{k}\left(\frac{h_v}{h_1}\right)^2\left(\frac{y_v}{y_1}\right)^2\kappa_v\underset{v}{\Delta n}\\ &- \frac{l_1^3}{(x_1-s_1)^3}\,x_1^3\sum_{v=1}^{k}\frac{h_v}{h_1}\left(\frac{y_v}{y_1}\right)^3\kappa_v\underset{v}{\Delta n},\end{aligned}\right\}\quad \text{(vi)}$$

$$\left.\begin{aligned}\frac{n_1\delta\mathfrak{G}_1^{(k)}}{s} &= \frac{(m_1^2+M_1^2)\,M_1}{(x_1-s_1)^3}\,s_1^3\sum_{v=1}^{k}\left(\frac{h_v}{h_1}\right)^4\kappa_v\underset{v}{\Delta n}\\ &- \frac{2m_1\,M_1\,l_1}{(x_1-s_1)^3}\,s_1^2x_1\sum_{v=1}^{k}\left(\frac{h_v}{h_1}\right)^3\frac{y_v}{y_1}\kappa_v\underset{v}{\Delta n}\\ &+ \frac{M_1\,l_1^2}{(x_1-s_1)^3}\,s_1x_1^2\sum_{v=1}^{k}\left(\frac{h_v}{h_1}\right)^2\left(\frac{y_v}{y_1}\right)\kappa_v\underset{v}{\Delta n}.\end{aligned}\right\}\quad \text{(vii)}$$

These expressions require to be added to the similarly constituted formulæ which we have established for the aberration of spherical surfaces. The total tangential aberration, for example, is represented by the quantity

$$\frac{n_1}{s_1}\left[\delta g_1^{(k)}+\delta\mathfrak{g}_1^{(k)}\right].$$

It will be seen at once from these expressions that the deformation of the spherical surfaces influences the image defects due to spherical aberration, coma, astigmatism, and distortion. The only expression which remains entirely unaffected is that giving a measure of the curvature of the image after the astigmatism has been corrected.

When, as is usually the case, the system contains both spherical and deformed surfaces, it is necessary to put $\kappa_v = 0$ when the v^{th} surface happens to be spherical.

C. The Seidel Elimination Formulæ.

178. The formulæ of Seidel-Kerber for the sagittal and tangential line of confusion, as given above, contain two sets of quantities, which are not independent of each other. These are the quantities s_v, h_v in one case and x_v, y_v in the other, which correspond respectively to the two object-points at distances s_1 and x_1. As a rule, though not necessarily, we shall denote by x_μ the distance from the μ^{th} surface of the system of the diaphragm in front of the surface, in which case, since s_1 always represents the distance of an object-point proper, it will be seen that the quantities occurring in the formula relate to the positions of the object, the diaphragm and their images.

Seidel (*2.*) was the first to show that in the general formulæ one set of these quantities is capable of elimination. The requisite equations have been given in § 83.

179. Defects in Terms of *s* and *h*.—Turning our attention, in the first place, to the elimination of the quantities in terms of x and y, it may be remarked that they are entirely absent in the expression for the spherical aberration. They make their first appearance in the formulæ established in the case of coma. Since by reduction it will be seen that

$$S_v = D_{xs}\left[\left(\frac{h_1}{h_v}\right)^2 \frac{1}{Q_{vs}} + h_1^2 \sum_{\lambda=2}^{v} \frac{d_{\lambda-1}}{n'_{\lambda-1}\, h_\lambda\, h_{\lambda-1}}\right],$$

and S_v is a multiple of D_{xs},
where

$$D_{xs} = \frac{1}{s_1} - \frac{1}{x_1}$$

is solely a function of the constants before and after the first refraction. The coefficients of the expression for coma may now be written thus—

$$\left.\begin{aligned}\frac{\text{Coma}}{}{}_s &= \sum_{v=1}^{k}\left(\frac{h_v}{h_1}\right)^3 \frac{y_v}{y_1}\, Q_{vs}\, Q_{vx} \underset{v}{\Delta}\left(\frac{1}{ns}\right)\\ &= \sum_{v=1}^{k}\left(\frac{h_v}{h_1}\right)^4 Q^2_{vs} \underset{v}{\Delta}\frac{1}{ns}\left(1 + S_v\right).\end{aligned}\right\} \quad \text{(i)}$$

In this form of the expression it will be seen that the first term on the right side represents the spherical aberration of the object-point s_1.

In the case of a spherically corrected system the expression for the coma has been shown above to merge into the sine condition, so that

$$\underline{\text{Sine condition}}_s = \left[\underline{\text{Coma}}_s - \underline{\text{Spherical correction}}_s\right]_{Sph_s = 0}$$

$$= \sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^4 Q^2_{vs} \underset{v}{\Delta} \frac{1}{ns} S_v \ldots \qquad \ldots \qquad \text{(ii)}$$

The coefficients of other image defects may be transformed in a similar manner. For the curvature of the image due to the sagittal pencils, we obtain accordingly the expression

$$f\text{- }\underline{\text{Curvature}}_s = \sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^2 \left(\frac{y_v}{y_1}\right)^2 \left[Q^2_{vx} \underset{v}{\Delta} \frac{1}{ns} - (Q_{vx} - Q_{vs})^2 \frac{1}{r_v} \underset{v}{\Delta} \frac{1}{n} \right]$$

$$= \sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^4 Q^2_{vs} \underset{v}{\Delta} \frac{1}{ns} \left(1 + S_v\right)^2 - D^2_{xs} \sum_{v=1}^{k} \frac{1}{r_v} \underset{v}{\Delta} \frac{1}{n} \; ; \quad \text{(iii)}$$

and for the distortion

$$\underline{\text{Distortion}}_s = \sum_{v=1}^{k} \frac{h_v}{h_1} \left(\frac{y_v}{y_1}\right)^3 \left[\frac{Q^3_{vx}}{Q_{vs}} \underset{v}{\Delta} \frac{1}{ns} - \frac{Q_{vx}}{Q_{vs}} (Q_{vx} - Q_{vs})^2 \frac{1}{r_v} \underset{v}{\Delta} \frac{1}{n} \right]$$

$$= \sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^4 Q^2_{vs} \underset{v}{\Delta} \frac{1}{ns} \left(1 + S_v\right)^3 - D^2_{xs} \sum_{v=1}^{k} \frac{1}{r_v} \underset{v}{\Delta} \frac{1}{n} \left(1 + S_v\right) . \quad \text{(iv)}$$

These expressions enable us to recognise the influence which the displacement of the stop exercises upon the correction of the image defects at points out of the axis. They show at once that the coma is a function of the first order of D_{xs}, whilst the astigmatism is a function of the second order, and the distortion a function of the third order.

Fuller investigation shows that the differential quotients in terms of x_1 of the tangential image defects producing distortion, curvature of the image in the meridian section, and coma, which in the expression for $\frac{n\delta g_1^{(k)}}{s_1}$ occur as the fourth, third and second terms, contain respectively, the expressions for the defect of the next lower order in terms of l.

The relationship here indicated may be expressed in the form of the following algorithms, in which

$$V = -\frac{x_1^3}{2(x_1-s_1)^3}\sum_{v=1}^{k}\frac{h_v}{h_1}\left(\frac{y_v}{y_1}\right)^3\left[\frac{Q^3_{vx}}{Q_{vs}}\underset{v}{\Delta}\frac{1}{ns}-\frac{Q_{vx}}{Q_{vs}}(Q_{vx}-Q_{vs})^2\frac{1}{r_v}\underset{v}{\Delta}\frac{1}{n}\right]:$$

$$\text{4th term of } \frac{n\,\delta g_1^{(k)}}{s_1} = (s_1-x_1)^2\, l_1^3\, \frac{V}{(s_1-x_1)^2},$$

$$\text{3rd term of } \frac{n\,\delta g_1^{(k)}}{s_1} = (s_1-x_1)\, m_1\, l_1^2\, \frac{\partial V}{\partial x_1},$$

$$\text{2nd term of } \frac{n\,\delta g_1^{(k)}}{s_1} = \frac{(3m_1^2+M_1^2)\, l_1}{6}\frac{\partial}{\partial x_1}\left[(s_1-x_1)^2\frac{\partial V}{\partial x_1}\right],$$

$$\text{1st term of } \frac{n\,\delta g_1^{(k)}}{s_1} = \frac{(m_1^2+M_1^2)\, m_1}{s_1-x_1}\frac{\partial}{\partial x_1}\left\{(s_1-x_1)^2\frac{\partial}{\partial x_1}\left[(s_1-x_1)^2\frac{\partial V}{\partial x_1}\right]\right\}.$$

180. Defects expressed in Terms of x and y.—In the alternative, we may eliminate the quantities s_v and h_v in the expressions for the image defects. For this purpose we shall introduce the quantity

$$X_v = D_{xs}\left[\left(\frac{y_1}{y_v}\right)^2\frac{1}{Q_{vs}} + y_1^2\sum_{\lambda=2}^{v}\frac{d_{\lambda-1}}{n'_{\lambda-1}\, y_\lambda\, y_{\lambda-1}}\right]. \quad \ldots \quad \text{(i)}$$

The coefficient of the distortion in the plane of the object at a distance s_1 will then be

$$\underline{\text{Dist.}}_s = \sum_{v=1}^{k}\left(\frac{y_v}{y_1}\right)^4 Q^2_{vx}\underset{v}{\Delta}\frac{1}{nx}(1-X_v) + 2\,D_{xs}\sum_{v=1}^{k}\left(\frac{y_v}{y_1}\right)^2 Q_{vx}\underset{v}{\Delta}\frac{1}{nx}$$
$$+\, D_{xs}\sum_{v=1}^{k}\left(\frac{y_v}{y_1}\right)^2 Q^2_{vx}\underset{v}{\Delta}\frac{1}{n^2}.$$

The last two terms of the sum may be simplified with the aid of the formulæ in § 75, thus:

$$\underline{\text{Dist.}}_s = \sum_{v=1}^{k}\left(\frac{y_v}{y_1}\right)^4 Q^2_{vx}\underset{v}{\Delta}\frac{1}{nx}[1-X_v] - D_{xs}\sum_{v=1}^{k}\left(\frac{y_v}{y_1}\right)^2\underset{v}{\Delta}\frac{1}{x^2}.$$

This latter sum is so constituted that in two successive binomials the adjacent terms cancel one another, so that only the first and last terms of the entire sum remain.

Putting

$$\mathbf{D}_{k1} = \frac{y_k^2}{y_1^2}\left(\frac{1}{x'^2_k} - \frac{1}{x_1^2}\right),$$

we obtain finally the expression

$$\underline{\text{Dist.}}_s = \sum_{v=1}^{k}\left(\frac{y_v}{y_1}\right)^4 Q^2_{vx} \underset{v}{\Delta} \frac{1}{nx}[1 - X_v] - D_{xs}\,\mathbf{D}_{k1}. \quad \ldots \quad \text{(ii)}$$

By similar simplification the following results are obtained, viz.:

$$\int\text{-}\underline{\text{Curv.}}_s = \sum_{v=1}^{k}\left(\frac{y_v}{y_1}\right)^4 Q^2_{vx} \underset{x}{\Delta} \frac{1}{nx}[1-X_v]^2 - D_{xs}\sum_{v=1}^{k}\left(\frac{y_v}{y_1}\right)^2 \underset{v}{\Delta}\frac{1}{x^2}[1-X_v]$$

$$+ D^2_{xs}\sum_{v=1}^{k}\frac{1}{r_v}\underset{v}{\Delta}\frac{1}{n}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

$$\underline{\text{Coma}}_s = \sum_{v=1}^{k}\left(\frac{y_v}{y_1}\right)^4 Q^2_{vx} \underset{v}{\Delta} \frac{1}{nx}[1-X_v]^3 - D_{xs}\sum_{v=1}^{k}\left(\frac{y_v}{y_1}\right)^2 \underset{v}{\Delta}\frac{1}{x^2}[1-X_v]^2$$

$$+ D^2_{xs}\sum_{v=1}^{k}\frac{1}{r_v}\underset{v}{\Delta}\frac{1}{n}[1 - X_v], \quad \ldots \quad \ldots \quad \ldots \quad \text{(iv)}$$

$$\underline{\text{Sph.}}_s = \sum_{v=1}^{k}\left(\frac{y_v}{y_1}\right)^4 Q^2_{vx} \underset{v}{\Delta} \frac{1}{nx}[1-X_v]^4 - D_{xs}\sum_{v=1}^{k}\left(\frac{y_v}{y_1}\right)^2 \underset{v}{\Delta}\frac{1}{x^2}[1-X_v]^3$$

$$+ D^2_{xs}\sum_{v=1}^{k}\frac{1}{r_v}\underset{v}{\Delta}\frac{1}{n}[1-X_v]^2. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(v)}$$

If we regard x_v as the distance of the diaphragm, these expressions enable us to trace the manner in which the correction of the system is affected by a change in the distance s_1 of the object, since the latter occurs in the quantity D_{xs} only. It will be seen that the expressions for the distortion, the curvature of the image field due to the sagittal rays, the coma, and the spherical aberration are functions in which the degree of D_{xs} increases from the first by unit steps. On the other hand, if we regard x_1 as the distance of a second object-point, the different image defects corresponding with a given distance s_1 are expressed in terms of quantities relating to a second object distance, viz. x_v, y_v, and the difference D_{xs} of the reciprocals of the two distances whereby the position of the object is defined.

181. Seidel's Proof of the Inconsistence of the Constant Sine Ratio and the Fulfilment of Herschel's Condition.—In the last set of equations, if we regard x_1 as a second object distance, assuming that both object-points are in close proximity, and then expand the expression for the spherical correction with respect to s_1 in terms of powers of D_{xs}, which, in accordance with the previous assumption, need not be carried beyond the first power, then since

$$D_{xs} = - D_{xs}$$

it follows that

$$\underline{\text{Sph.}}_s = \sum_{v=1}^{k} \left(\frac{y_v}{y_1}\right)^4 Q^2_{vx} \underset{v}{\Delta} \frac{1}{nx} + 4\, D_{sx} \sum_{v=1}^{k} \left(\frac{y_v}{y_1}\right)^4 Q^2_{vx} \underset{v}{\Delta} \frac{1}{nx} \left[\left(\frac{y_1}{y_k}\right)^2 \frac{1}{Q_{vx}} \right.$$

$$\left. + y_1^2 \sum_{\lambda=2}^{v} \frac{d_{\lambda-1}}{n'_{\lambda-1}\, y_\lambda\, y_{\lambda-1}} \right] - D_{xs}\, \mathbf{D}_{k1}\,. \quad \ldots \quad \text{(i)}$$

From the first set of formulæ we now find that

$$\underline{\text{Sph.}}_s = \underline{\text{Sph.}}_s + 4 \left[\underline{\text{Coma}}_s - \underline{\text{Sph.}}_s \right] - D_{xs}\, \mathbf{D}_{k1}\,. \quad \text{(ii)}$$

If the system is spherically corrected for the point at a distance x_1, the second term on the right side represents the sine condition, and if we assume that the system is also corrected aplanatically, we obtain

$$\underline{\text{Sph.}}_s = - D_{xs}\, \mathbf{D}_{k1}\,. \quad \ldots \quad \ldots \quad \text{(iii)}$$

The condition that the spherical aberration may be zero is

$$\mathbf{D}_{k1} = \frac{1}{y_1^2} \left(\frac{y_k^2}{x'^2_k} - \frac{y_1^2}{x_1^2} \right) = 0\,. \quad \ldots \quad \text{(iv)}$$

This expression, however, can only vanish in the case of a telescope used for star observations and accordingly focussed for infinity, when

$$x_1 = x'_k = \infty$$

or when

$$\frac{y_k}{x'_k} = \pm \frac{y_1}{x_1}\,,$$

that is when

$$u'_k = \pm u_1\,.$$

This case arises only when the object-point is situated at the front nodal point or at a distance from the principal focus equal to the focal length—*i.e.*, when the system forms a virtual or real image of unit magnification.

Summarising these investigations, but disregarding the three special cases to which we have just referred, it may be said in general, that the object-point s_1 near an aplanatic point x_1 is subject to aberration, so that the fulfilment of the sine condition is not reconcilable with that of Herschel. This conclusion is in perfect agreement with the result of the investigation in § 165.

Similar proofs apply to changes in the other image defects occasioned by changes in the position of the object.

It follows from this that no aplanatic point may be chosen if it is required that the variations of the coma, the curvature of the image field, and the distortion shall vanish in the neighbourhood of a spherically corrected point.

D. The Image Defects of Hemisymmetrical and Holosymmetrical Systems.

182. A **hemisymmetrical system** may be defined as a combination of two parts in which all the constants (*i.e.*, the radii, thicknesses, distances and stops) of one part are m times the corresponding quantities of the other part. This condition introduces at the outset various consequences respecting the zero invariants. It may be noted here that the hemisymmetrical system becomes a holosymmetrical one when $m = 1$.

We shall commence by considering either half separately, and assume that the rays are parallel, in which case the first system must be reversed. We then have the following data for each of the components **I** and **II** :—

I.	**II** (m times the dimensions of **I**).
$s_1 = \infty$	$s_1 = \infty$
s_1'	ms'_1
s_2	ms_2
$\vdots$	$\vdots$
s_l'.	ms'_l

and, of course, also

$$Q_{rs} \qquad\qquad \frac{1}{m} Q_{vs}$$

If now, we reverse the first half of the system, placing it before the second, so that the region of the diaphragm is traversed by parallel pencils, all the elements bearing algebraical signs, including the axial intercepts, will have their signs reversed; moreover, the accented quantities of the first half of the system will become the unaccented quantities, and *vice versâ*. The previous index v will now be replaced by $_{l+1-v}$ in the first and $_{l+v}$ in the second half of the system. According to the previous assumption that the space containing the stop is traversed by parallel rays, we shall obtain for the two halves of the system the following quantities:

I.	**II.**
s_1	$s_{l+1} = \infty$
s'_1	$s'_{l+1} = -ms_l$
$\vdots$	$s_{l+2} = -ms'_{l-1}$
$\vdots$	$\vdots$
$s'_l = \infty$	$s'_k = s'_{l2} = -ms_1$

and

$$Q_{l+vs} = -\frac{1}{m} Q_{l+1-vs}$$

Precisely similar considerations are applicable to the axial intercepts with respect to the position of the stop, thus

$$Q_{l+vx} = -\frac{1}{m} Q_{l+1-vx};$$

and since

$$\underset{l+v}{\Delta} \frac{1}{nx} = \frac{1}{n'_{l+v}\, x'_{l+v}} - \frac{1}{n_{l+v}\, x_{l+v}}$$

we obtain

$$\underset{l+v}{\Delta} \frac{1}{nx} = -\frac{1}{m} \frac{1}{n_{l+1-v}\, x_{l+1-v}} + \frac{1}{m} \frac{1}{n'_{l+1-v}\, x'_{l+1-v}}$$

$$= \frac{1}{m} \underset{l+1-v}{\Delta} \frac{1}{nx}.$$

For the values in terms of s, if in the region of the stop the rays are parallel, the corresponding relation is

$$\underset{l+v}{\Delta} \frac{1}{ns} = \frac{1}{m} \underset{l+1-v}{\Delta} \frac{1}{ns}.$$

If now we turn our attention to the ordinates h_v, y_v of the points where the surfaces are intersected, since the rays are parallel in the region of the stop, then in the first case

$$h_l = h_{l+1},$$

and thus we find

$$h_{l-1} = h_l \frac{s'_{l-1}}{s_l},$$

also

$$h_{l+2} = h_{l+1} \frac{s_{l+2}}{s'_{l+1}} = h_l \frac{-ms'_{l-1}}{-ms_l} = h_{l-1},$$

or, generally,

$$\frac{h_{l+v}}{h_1} = \frac{h_{l+1-v}}{h_1}.$$

Incidentally it may be noted that according to this equation

$$h_k = h_1,$$

hence, putting

$$u_1 = \frac{h_1}{s_1},$$

$$u'_k = \frac{h_k}{s'_k} = -\frac{1}{m}\frac{h_1}{s_1} = -\frac{1}{m} u_1.$$

From this it follows that

$$\gamma = -\frac{1}{m}$$

and again from § 84 (v)

$$\beta = -m.$$

From this it will be seen that in hemisymmetrical systems, instead of the condition that the region of the stop shall be traversed by parallel pencils, we may specify the condition that the image and object shall be similar and their dimensions in the ratio of 1 to m.

With regard to the ordinates y_v, they are obviously in the ratio 1 to m but they have opposite signs in the two systems, thus

$$\frac{y_{l+v}}{y_1} = -m\frac{y_{l+1-v}}{y_1}.$$

Having in this way simplified the elements of the hemisymmetrical systems which occur in the formulæ of aberration, we shall now consider the latter themselves.

$$\underline{\text{Sph.}}_s = \sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^4 Q^2_{v\,s} \underset{v}{\Delta} \frac{1}{ns}$$

$$= \sum_{v=1}^{l} \left(\frac{h_v}{h_1}\right)^4 Q^2_{v\,s} \underset{v}{\Delta} \frac{1}{ns} + \sum_{v=1} \left(\frac{h_{l+v}}{h_1}\right)^4 Q^2_{l+v\,s} \underset{l+v}{\Delta} \frac{1}{ns}.$$

From the above it follows that if the sequence of the sum is reversed, the second term on the right side becomes finally

$$\frac{1}{m^3}\sum_{v=1}^{l}\left(\frac{h_{l+1-v}}{h_1}\right)^4 Q^2_{l+1-v\,s}\ \underset{l+1-v}{\Delta}\frac{1}{ns} = \frac{1}{m^3}\sum_{v=1}^{l}\left(\frac{h_v}{h_1}\right)^4 . Q^2_{v\,s}\ \underset{v}{\Delta}\frac{1}{ns},$$

and thus

$$\underline{\mathrm{Sph.}}_s = \left(1+\frac{1}{m^3}\right)\sum_{v=1}^{l}\left(\frac{h_v}{h_1}\right)^4 Q^2_{v\,s}\ \underset{v}{\Delta}\frac{1}{ns}.$$

It will thus be seen that the spherical correction of a symmetrical system which reproduces the object in true scale proportions, depends upon the degree of the correction for an infinitely distant object of the spherical aberration attained in the component member.

$$\underline{\mathrm{Coma}}_s = \sum_{v=1}^{k}\left(\frac{h_v}{h_1}\right)^3 \frac{y_v}{y_1} Q_{v\,s}\ Q_{v\,x}\ \underset{v}{\Delta}\frac{1}{ns} = \sum_{v=1}^{l}\left(\frac{h_v}{h_1}\right)^3 \frac{y_v}{y_1} Q_{v\,s}\ Q_{v\,x}\ \underset{v}{\Delta}\frac{1}{ns}$$
$$+\sum_{v=1}^{l}\left(\frac{h_{l+v}}{h_1}\right)^3 \frac{y_{l+v}}{y_1} Q_{l+v\,s}\ Q_{l+v\,x}\ \underset{l+v}{\Delta}\frac{1}{ns}.$$

As before, the last term should be written thus:

$$-\frac{1}{m^2}\sum_{v=1}^{l}\left(\frac{h_{l+1-v}}{h_1}\right)^3 \frac{y_{l+1-v}}{y_1} Q_{l+1-v\,s}\ Q_{l+1-v\,x}\ \underset{l+1-v}{\Delta}\frac{1}{ns}$$
$$= -\frac{1}{m^2}\sum_{v=1}^{l}\left(\frac{h_v}{h_1}\right)^3 \frac{y_v}{y_1} Q_{v\,s}\ Q_{v\,x}\ \underset{v}{\Delta}\frac{1}{ns},$$

so that we get

$$\underline{\mathrm{Coma}}_s = \left(1-\frac{1}{m^2}\right)\sum_{v=1}^{l}\left(\frac{h_v}{h_1}\right)^3 \frac{y_v}{y_1} Q_{v\,s}\ Q_{v\,x}\ \underset{v}{\Delta}\frac{1}{ns}.$$

When $m = +1$, as in the case of a holosymmetrical system designed for copying in natural size, the term $\underline{\mathrm{Coma}}_s$ vanishes, independently of the correction of the elements for parallel rays, and it does so even when the spherical aberration is not corrected. On the other hand, in the case of a hemisymmetrical system designed for copying on a reduced scale m, the coma depends upon the degree of its correction in the component elements for an infinitely distant object.

In a similar manner we may obtain the following expression for the astigmatism

$$\underline{\mathrm{Astigmatism}}_s = \left(1+\frac{1}{m}\right)\sum_{v=1}^{l}\left(\frac{h_v}{h_1}\right)^2\left(\frac{y_v}{y_1}\right)^2 Q^2_{v\,x}\ \underset{v}{\Delta}\frac{1}{ns},$$

from which it will be seen that in holosymmetrical and hemisymmetrical objectives the removal of the astigmatism for the particular object-point depends generally upon the degree of correction of the astigmatism in the component elements for distant objects.

In conclusion, to determine the distortion in a plane situated at the object distance s_1, we may express this image defect in terms of x_v, y_v and D_{xs}:

$$\underline{\text{Dist}}_s = \sum_{v=1}^{k} \left(\frac{y_v}{y_1}\right)^4 Q^2_{v\,x} \underset{v}{\Delta} \frac{1}{nx} (1 - X_v) - D_{xs}\, \mathbf{D}_{k,1}\,.$$

In a hemisymmetrical system,

$$y_k = -\,my_1\,; \quad x'_k \doteq -\,mx_1\,,$$

and thus

$$\mathbf{D}_{k1} = 0\,.$$

As regards the expression remaining on the right-hand side, we may accordingly write

$$\underline{\text{Distortion}}_s = \underline{\text{Coma}}_x$$

so that

$$\underline{\text{Coma}}_x = \sum_{v=1}^{k} \left(\frac{y_v}{y_1}\right)^3 \frac{h_v}{h_1}\, Q_{vx}\, Q_{vs} \underset{v}{\Delta} \frac{1}{nx}$$

vanishes, since the two parts into which the expression may be divided in the manner already described, are equal and of opposite sign.

From this it will be seen that the distortion vanishes quite generally in hemisymmetrical systems designed for producing accurate scale reductions.

In the special case when the component system is spherically corrected with respect to the stop, the coma condition merges into the sine condition, and in this case we have, irrespective of the distance s_1 of the object,

$$\underline{\text{Dist.}}_s = \underline{\text{Sinc}}_x\,.$$

This condition is always fulfilled in hemisymmetrical systems, since the inclinations of the principal ray before and after traversing the system are equal and of opposite sign.

It follows accordingly, that hemisymmetrical systems which are spherically corrected for the stop form undistorted images of objects irrespective of their distance. This is a special case of the Bow-Sutton condition referred to in § 131 limited to small inclinations of the principal ray.

9. HISTORICAL NOTES.

183. *On Section 2.*

A few references to older writers on the spherical longitudinal aberration have been given in Czapski's work (*3*. 118). First among these was Roger Bacon, who in the 13th century pointed out the existence of spherical aberration in spherical mirrors. Next, he quotes Kepler (*1*.), Barrow (*1*. *2*.), Gregory (*1*.), Newton (*1*. *2*.), Huygens (*1*.) Smith (*1*. *2*. *3*.), and Lagrange (*1*. *2*.). Of the earliest general treatises he mentions Euler (*2*.) and Kluegel (*1*.), as well as Priestley (*1*. *2*.), and Wilde (*1*.) In more recent times the longitudinal aberration has been investigated by J. Herschel (*1*. to *6*.), Santini (*1*.), Seidel (*1*.), and Mossotti (*1*.).

Investigations of the longitudinal aberration involving the next higher order of terms, the so-called zonal terms, were published by Keller (*1*.), Bauer (*1*.), Kerber (*2*. *3*.) and Schupmann (*1*.).

184. *On Section 3.*

In 1827 Airy (*3*.) formulated in the case of astronomical telescopes the tangent condition which bears his name. The distortion which occurs in the image-plane of projection systems at finite inclinations of the principal ray, does not appear to have received closer attention until the use of photographic lenses led to a notable increase in the inclinations of the principal rays. A more precise history of the efforts made to improve the correction of distortion has been published by von Rohr (*2*.). For our present purpose it will be sufficient to note from this account that the removal of distortion was first practically attained in the construction of symmetrical lenses. This type was suggested in 1858 by Rothwell (*1*.) in connection with Sutton's investigations. A complete investigation of the behaviour of the rays in a symmetrical system, was presented by Bow (*1*.), and in connection with this investigation, Sutton (*1*.) enunciated the rule known as the Bow-Sutton condition for the absence of distortion for objects at any distance. These results were soon forgotten, and it is only comparatively recently that Lummer (*2*. and *3*.) again took up the investigation of the condition for the correction of distortion. An analytical treatment of this problem was published by von Rohr (*1*.) at the instigation of Abbe.

185. *On Section 4.*

The investigation of anastigmatic image surfaces was first undertaken by Airy (*3*.) in 1827. Pursuing this investigation, Coddington (*1*.) established formulæ giving the curvature of the image-field at the vertex, both for tangential and sagittal pencils at small inclinations of the principal ray. These formulæ became firmly established in England and are not likely to be again lost sight of.

In France Breton de Champ (*1* . *2* . *3* .), probably independently of Coddington, investigated the curvature of the image and arrived at similar results, but he omitted to apply them in detail to practically important cases, as had already been done by Airy. Petzval, as early as 1840, was led by the practical application of his calculations to the invention of the portrait lens which bears his name and which was corrected for astigmatism, so that the resulting curvature of the image was capable of being strictly investigated as such. The formula for the curvature of the image in this sense was given by Petzval (*1* .) in 1843 without a complete account of its derivation, and it was enunciated in such a form that it was liable to be interpreted as the necessary and sufficient condition for the realisation of a flat and stigmatic image-field; as a matter of fact, his formula has frequently been thus misinterpreted. The claim made on Petzval's behalf has been emphatically contested from two directions. Seidel (*3* .), in our opinion without sufficient grounds, has expressed his doubts as to whether Petzval was ever himself in possession of the correct derivation of the formula for the curvature of the image, and showed incidentally that it was only after applying the correction for spherical aberration, coma, and astigmatism that Petzval's equation acquired the general quality claimed for it of being a means of stating the curvature of the image of the 5th order. Somewhat later, Zinken, writing under the name of Sommer (*1* . *2*), published another derivation of the formulæ of the curvature of the image for tangential and sagittal pencils, and on the strength of this derivation showed that, to obtain a flat anastigmatic image, it was not sufficient to satisfy Petzval's equation. The curvature of the image formed by tangential pencils was investigated by Kerber (*5* .). In conclusion we may mention the investigation by Harting (*7* .), which has already been referred to in §139. In his paper the author determines the position of the astigmatic image-points and extends his investigation to finite inclinations of the principal ray, in the case of infinitely thin systems of lenses having the stop at the common vertex.

186. *On Sections 5 and 6.*

A separate paper on coma has been published by Dennis Taylor (*2* .).

The sine condition was enunciated in 1873 by Abbe (*2* .), who emphasised its significance in regard to the correction of systems transmitting wide pencils. At the beginning of 1874 a proof of the condition from photometric considerations was published by Helmholtz (*4* .). Likewise, Hockin (*1* .) contributed in 1881 a proof of the sine condition. In his investigation, he proceeded from a pair of points on the axis, free from aberration, and for infinitely small quantities established the condition for the constancy of the optical paths between two laterally adjacent points conjugate to those

points on the axis, including all conceivable paths within a finite solid angle. As a matter of fact, the sine condition is implied in Kirchhoff's law of radiation enunciated in 1860, as pointed out by Drude (*3*.462). Amongst the investigators of problems in geometrical optics, Seidel (*7*.) has shown that the fulfilment of the sine condition for small angular apertures is identical with the annulment of the coefficient of the coma in a spherically corrected system, or to use Seidel's terms, that it is identical with the fulfilment of Fraunhofer's condition. Kerber (*11*.) has likewise furnished a proof of his proposition. Elaborations of his sine condition have been discussed by Eppenstein (126) in the revised edition of Czapski's treatise (*3*.)

187. *On Section 7.*

On the subject of spherical aberration at points away from the axis we have not been able to find any observations in the literature consulted by us.

188. *On Sections 1 to 8.*

The general scheme of an elegant solution of these problems is due to Petzval (*1*.) and notably to Seidel (*3*. and *7*.). Kerber proceeded on similar lines, and his derivation, as used by us, embodies what is probably the shortest and most direct course. A general investigation of the problems connected with aberration was undertaken by Schleiermacher (*1*. *2*. *3*.), but his presentation of the subject is perplexing and lacking in elegance, as has already been pointed out by Czapski. In addition, we may mention C. Moser (*2*. and *4*.) and Charlier (*1*.), and, in conclusion, the investigations of Thiesen (*1*.) and Classen (*1*.), who proceeded on similar lines. In these investigations the expressions for the five defects are deduced from the principle of least time, but the authors give no expressions indicating in direct explicit terms the significance of the constituents of the system. The effect of non-spherical surfaces (more correctly of surfaces of revolution of the second order) upon thin pencils has been investigated by Detels (*1*.). Attention should also be drawn to a paper by C. Moser (*1*.). Problems affecting the magnitude of the circle of confusion and the distribution of light therein, with which we have not concerned ourselves in this book, have been investigated by Gauss (*2*.), J. C. E. Schmidt (*1*. 514), Seidel (*4*. *5*.), Kerber (*3*.), von Hoegh (*2*.), A. Steinheil (*2*.), Finsterwalder (*1*.), Charlier (*2*.) and Everett (*1*.). The most comprehensive of these is the paper of Finsterwalder, which is based upon Seidel's results. He investigates first the caustic surface corresponding to an extra-axial object-point, and he next establishes the connection between the points where a ray cuts the plane of the aperture and the image receiving plane, and finally is thereby enabled to determine the distribution of the intensity of light within the figure of confusion conforming to various positions of these two planes.

Chapter VI.

THEORY OF CHROMATIC ABERRATION.

(A. Koenig.)

189. In the previous investigations each medium taking part in the formation of an optical image was supposed to have a definite refractive index. As a matter of fact, the refractive index is a function of the wave-length, or the colour of the light, and the relation between these two quantities differs in the various media. When, accordingly, a pencil of white light, which is composed of rays of different wave-lengths, falls upon a refracting surface it splits up into pencils of coloured rays, unless it happens to meet the surface at right angles, and these rays proceed in different directions from the point of incidence. It will thus be seen that it is only in purely catoptrical systems that the object has corresponding to it an image in the sense in which we have studied it so far; whilst in dioptrical and cata-dioptrical systems the object has corresponding to it a series of differently coloured superimposed images ranged one behind the other. Such a series of images can be regarded as an **image exhibiting chromatic aberrations.** These **chromatic aberrations,** which may be defined as the **deviations arising from colour dispersion,** affect the whole of the factors incidental to the formation of images, in that they are all governed by the refractive index and consequently vary with the colour. We shall make a distinction between the chromatic aberrations of the primary factors which determine the position and magnitude of the images and the chromatic variations of the various image-defects.

In what follows we shall establish the most important formulæ for the computation of these chromatic defects. We shall explain the influence of the defects upon the quality of the image and we shall discuss methods of devising systems which are free from chromatic aberration. With regard to the first proposition, it will not be necessary to consider the method of evaluating the constants connecting the object and image by introducing the values of n for different colours and then ascertaining the chromatic aberration by forming the appropriate differences. The better course is to establish expressions for finite and infinitesimal differences in which we may be able to recognise the connection between

the aberrations and the changes in the refractive indices, as well as between the aberrations and the radii, thicknesses and distances; or again, such formulæ as may assist us in calculating the magnitudes of the defects, in so far as these defects can be calculated with approximately the same degree of accuracy as obtains in every other respect, whereas under the conditions of the other procedure the accuracy diminishes in the ratio of the variation to the varied quantity itself. These formulæ give us, moreover, appropriate information as to the manner in which the chromatic defects can be removed by the compensation of the partial defects due to the component parts (*i.e.* the surfaces or lenses) of a dioptric system. This process of **correction with respect to a pair of wave-lengths** is known as **achromatisation** of a system **with reference to a given constant of the image-formation,** or as the **achromatisation of the constant of image-formation.** The result of this procedure is called **achromatism,** and a system in which this has been attained is described as **achromatic.** To distinguish aberrations which are of opposite sign, a system in which the sense of the aberration is the same as in a simple thin converging lens adjusted for infinity, will be regarded as **chromatically under-corrected,** whilst we shall call a system **chromatically over-corrected** if its aberration is of the opposite kind. To particularise the colours and to define the constituent parts of the spectrum it is common usage to state the lines of the spectrum, notably those of Fraunhofer, instead of referring to the numerical values of their wave-lengths.

1. THE CHROMATIC VARIATION OF THE POSITION AND MAGNITUDE OF THE IMAGE.

A. Chromatic Aberrations of the First Order.

190. The Chromatic Variation of the Position of the Image.—The quantities with which we are here concerned are those occurring in the Gaussian theory of image-formation. We shall begin with the chromatic variation of the position of the image, which constitutes the **longitudinal part of the chromatic aberration** or the **chromatic elongation.** We shall distinguish quantities corresponding to another colour of shorter wave-length by Roman letters. Let the variation of the value in passing from the first to the second colour be indicated by the symbol V. The relation between the intercepts, before and after refraction at a surface, is given by the zero invariant introduced in § 31 (i), viz.

$$Q_s = n\left(\frac{1}{r} - \frac{1}{s}\right) = n'\left(\frac{1}{r} - \frac{1}{s'}\right).$$

In the absence of any remark to the contrary we shall for convenience write Q in the place of Q_s. We shall then have

$$VQ = n\left(\frac{1}{r} - \frac{1}{s}\right) - \mathrm{n}\left(\frac{1}{r} - \frac{1}{\mathrm{s}}\right) \text{ or } VQ = Q\,\frac{Vn}{n} + \frac{\mathrm{n}}{s\mathrm{s}}\,Vs\,.$$

Since Q is invariant so also is VQ, and hence

$$Q\Delta\,\frac{Vn}{n} = -\,\Delta\,\frac{\mathrm{n}}{s\mathrm{s}}\,Vs\,. \quad \ldots \quad \ldots \quad \text{(i)}$$

In a system of k surfaces, if we remember that

$$Vs_{v+1} = Vs'_v$$

and if we assume that the object is free from aberration, the chromatic variation of the position of the k^{th} image formed by the system will be

$$Vs'_k = -\,\frac{s'_k\,\mathrm{s}'_k}{\mathrm{n}'_k}\sum_{v=1}^{k}\left(\frac{h_v}{h_k}\right)\left(\frac{\mathrm{h}_v}{\mathrm{h}_k}\right)Q_v\,\underset{v}{\Delta}\frac{Vn}{n}\,. \quad \ldots \quad \text{(ii)}$$

This aberration vanishes in a centred system of surfaces having the object at their common centre of curvature, since $Q=0$. In a system of parallel planes the magnitude of the aberration is best derived by forming the differences direct from the expression for s'_k/n'_k, as given in § 121 (ii), viz.

$$\frac{s'_k}{n'_k} = \frac{s_1}{n_1} - \sum_{v=1}^{k-1}\frac{d_v}{n'_v}\,,$$

and thus

$$\frac{Vs'_k}{n'_k} = \frac{s'_k\,Vn'_k}{\mathrm{n}'_k\,n'_k} - \frac{s_1\,Vn_1}{\mathrm{n}_1\,n_1} + \sum_{v=1}^{k-1}\frac{d_v\,Vn'_v}{\mathrm{n}'_v\,n'_v}\,. \quad \ldots \quad \text{(iii)}$$

When the first and last media are air, all but the last term on the right side vanish. In this case the system is therefore always chromatically over-corrected.

In a system having longitudinal chromatic aberration the point upon which the image should be focussed and the magnitude of the circle of confusion, which serves as the true measure of the defect of the image, cannot be determined by the rules of geometrical optics. It is accordingly only as an arbitrary expedient that we may define the **lateral chromatic aberration** (Ch) of a point on the

axis as the radius of the circle of confusion formed at the image of the first colour by a pencil consisting of rays of the second colour. Accordingly

$$Ch'_k = \mathrm{u}'_k \, Vs'_k \, . \qquad \ldots \qquad \ldots \qquad \text{(iv)}$$

To determine the radius $Ch^{(k)}$ of the circle of confusion projected back into the object by means of rays of the first colour we make use of the Smith-Helmholtz equation [§ 84(4)]:

$$n'_k \, u'_k \; Ch'_k = n_1 \, u_1 \; Ch^{(k)}$$

and of the relations

$$u'_k = \frac{h_k}{s'_k} \; ; \quad \mathrm{u}'_k = \frac{\mathrm{h}_k}{\mathrm{s}'_k} \; ; \quad u_1 = \frac{h_1}{s_1} ,$$

whence we find

$$Ch^{(k} = - \frac{n'_k \, u_1}{\mathrm{u}'_k \, n_1} s_1^2 \sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right) \left(\frac{\mathrm{h}_v}{\mathrm{h}_1}\right) Q_v \underset{v}{\Delta} \frac{Vn}{n} . \qquad \ldots \qquad \text{(v)}$$

The radius of the chromatic circle of confusion is accordingly simply proportional to the angular aperture. When the object is at an infinitely great distance the angular circle of confusion is proportional to the linear aperture, as explained in § 122.

191. The Chromatic Variation of the Magnification. Next, we must determine the chromatic variation of the axial magnification α, the lateral magnification β, and the angular magnification or the ratio of convergence γ. The expressions for these quantities are by § (82)

$$\left. \begin{aligned} \alpha &= \frac{n_1}{n'_k} \left(\frac{s'_k}{s_1} \frac{h_1}{h_k}\right)^2 \\ \beta &= \frac{n_1}{n'_k} \frac{s'_k}{s_1} \frac{h_1}{h_k} \\ \gamma &= \frac{s_1}{s'_k} \frac{h_k}{h_1} , \end{aligned} \right\} \qquad \ldots \qquad \text{(i)}$$

which clearly exhibit their close relationship. It follows also immediately from them, that in a hemisymmetrical system traversed by hemisymmetrical pencils of rays the magnification is independent of the colour when Vs_1 and Vs'_k are related in a ratio conforming to the hemisymmetrical principle; in particular, therefore, when $Vs_1 = Vs'_k = 0$, that is to say when the chromatic aberration in

the individual elements has been corrected for parallel rays. To establish the respective formulæ we shall again assume that the object is free from aberration. Noting that

$$s_{v+1} = s'_v - d_v\,,$$

we obtain the following two expressions for the auxiliary quantity H which contains the chromatic variation of $\frac{h_k}{h_1}$:

$$\left.\begin{aligned} H = 1 + \frac{V\frac{h_k}{h_1}}{\frac{h_k}{h_1}} &= \prod_{v=1}^{k-1}\left(1 + \frac{d_v}{s_v'}\frac{Vs'_v}{s_{v+1}}\right) = \prod_{v=1}^{k-1}\left(1 - d_v\frac{h_v}{h_{v+1}}V\sigma'_v\right) \\ \text{and,}\qquad & \\ \frac{1}{H} = 1 + \frac{V\frac{h_1}{h_k}}{\frac{h_1}{h_k}} &= \prod_{v=1}^{k-1}\left(1 - \frac{d_v}{s_v'}\frac{Vs'_v}{s_{v+1}}\right). \end{aligned}\right\}\quad \text{(ii)}$$

Next, introducing by way of abbreviation,

$$\left.\begin{aligned} S &= 1 + \frac{Vs'_k}{s'_k}, \\ N &= \frac{1 + \frac{Vn_1}{n_1}}{1 + \frac{Vn'_k}{n'_k}}, \end{aligned}\right\}\quad \ldots \quad \ldots \quad \text{(iii)}$$

we obtain the following expressions for the ratio of the magnifications for the two colours :

$$\left.\begin{aligned} 1 + \frac{V\alpha}{\alpha} &= NS^2 / H^2 \\ 1 + \frac{V\beta}{\beta} &= NS / H \\ 1 + \frac{V\gamma}{\gamma} &= H / S. \end{aligned}\right\}\quad \ldots \quad \ldots \quad \text{(iv)}$$

The formulæ for H and $1/H$ are particularly well adapted, in conjunction with an approximate calculation of the path of an axial ray (say with the aid of four-figure logarithms), for computing the above auxiliary quantities to a high degree of accuracy.

By assuming the change of n to be infinitely small, the formulæ thus established for finite differences may be reduced to simple differential formulæ which enable us to recognize much more clearly

all the conditions obtaining in any given case. Moreover, the results obtained by means of these formulæ (involving only the calculation of an axial ray of the first colour) are applicable in many cases with sufficient nearness to finite colour intervals. These formulæ are:

$$\left.\begin{aligned}
ds'_k &= -\frac{s'^2_k}{n'_k}\sum_{v=1}^{k}\left(\frac{h_v}{h_k}\right)^2 Q_v \underset{v}{\Delta}\frac{dn}{n}\\
\frac{da}{a} &= \frac{dn_1}{n_1} - \frac{dn'_k}{n'_k} + 2\frac{ds'_k}{s'_k} - 2\sum_{v=1}^{k-1}\frac{d_v\, ds'_v}{s'_v\, s_{v+1}}\\
\frac{d\beta}{\beta} &= \frac{dn_1}{n_1} - \frac{dn'_k}{n'_k} + \frac{ds'_k}{s'_k} - \sum_{v=1}^{k-1}\frac{d_v\, ds'_v}{s'_v\, s_{v+1}}\\
\frac{d\gamma}{\gamma} &= -\frac{ds'_k}{s'_k} + \sum_{v=1}^{k-1}\frac{d_v\, ds'_v}{s'_v\, s_{v+1}}\,.
\end{aligned}\right\} \qquad \text{(v)}$$

These formulæ may be further simplified by grouping the expressions in accordance with the appropriate surfaces. Thus

$$\frac{ds'_k}{s'_k} - \sum_{v=1}^{k-1}\frac{d_v\, ds'_v}{s'_v\, s_{v+1}} = \sum_{v=1}^{k}\underset{v}{\Delta}\frac{ds}{s}\ . \ \ldots \qquad \ldots \qquad \text{(vi)}$$

These simplified formulæ, however, have no longer the advantage that for the achromatism of the image position (*i.e.*, $ds'_k = 0$) the conditions for the achromatism of a, β, γ are expressed in a particularly convenient form.

When the first and last media are identical the conditions that ensure zero values of Va, $V\beta$, $V\gamma$ merge into a single condition. Geometrically, this condition signifies that the emerging rays of the first and second colours corresponding to an incident ray are parallel. Moreover, when all thicknesses and distances are equal to zero this condition is identical with that of the longitudinal achromatisation of the image.

Instead of seeking to achromatise the image with respect to the quantity β, so as to ensure that each coloured image measured in its own position may be identical in size, it will be more expedient in the computation of systems having longitudinal chromatic aberration, to bring about conditions whereby two images of similar size are projected from the point of crossing of the principal image rays of either colour into the plane in which the image is focussed. Denoting the distance of this plane from the image planes of the two colours respectively by c and c, where $\mathrm{c} - c = -Vs'_k$, the new requirement can be satisfied by the condition

$$\beta\left(1 + \frac{c}{s'_k - x'_k}\right) = (\beta + V\beta)\left(1 + \frac{\mathrm{c}}{\mathrm{s}'_k - \mathrm{x}'_k}\right). \quad \text{(vii)}$$

If we regard c, c, $V\beta$ and $V(s'_k - x'_k)$ as infinitesimals of the first order, the above relation, when infinitesimals of the second order are neglected, can be replaced by the following condition, which is satisfied in the case of a system of thin lenses in air when $x'_k = 0$, namely :

$$\frac{Vs'_k}{s'_k - x'_k} = \frac{V\beta}{\beta} \quad \ldots \quad \ldots \quad \ldots \quad \text{(viii)}$$

It is identical with the condition for the compensation of the chromatic difference of the lateral magnification when $s'_k - x'_k = \infty$, that is to say when the image or the crossing point of the principal rays on the image side is at infinity. In the case of hemisymmetrical systems it can be shown, without neglecting infinitesimals of the second order and even when the principal rays are inclined at finite angles, that the projections of the differently coloured images into a focussing plane which is hemisymmetrical with respect to the object-plane are of equal size. To realise this it is only necessary to trace the path of a ray of either colour from the extra-axial object-point through the system in such a way that either passes through the middle of the aperture stop (*see* Chapter IX).

192. The Chromatic Variations of Any Chosen Plane Expressed in Terms of the Chromatic Variations of the Focal Plane.—In any system for which the chromatic variations of the principal foci and of the two focal lengths are known, the Gaussian theory will enable us to derive from them the chromatic variation for the position of any image as well as of the corresponding values α, β, γ. We shall here limit ourselves to the formation of the difference formulæ corresponding with the equations

$$x_s x'_s = ff' \text{ and } \beta = \frac{f}{x_s},$$

where x_s and x'_s are the axial intercepts in the object and image-spaces referred to their respective principal focal planes. These formulæ are :

$$\frac{Vx_s - VS}{x_s} + \frac{Vx_s' - VS'}{ff'}(x_s + Vx_s - VS) = \frac{V(ff')}{ff'},$$

and

$$V\beta = \frac{f}{x_s + Vx_s - VS}\left(\frac{Vf}{f} - \frac{Vx_s - VS}{x_s}\right),$$

where Vx_s and Vx'_s are the chromatic variations of the positions of the object and image, and VS and VS' denote those of the principal foci in the object-space and the image-space. According to the

type of the system, Vx'_s does not vanish for any distance of the object, or it does so for two, or for all distances. The last case arises when VS, VS' and $V(ff')$ vanish simultaneously, and the achromatisation system is then said to be **stable.** When, moreover, the system is enclosed on both sides by the same medium, $V\beta$ reduces throughout to zero. If we consider any system, $V\beta$ vanishes only for a certain distance of the object. In order that it may vanish for all object distances it is necessary for Vf and $Vx_s - VS$ to vanish. In the event of $Vx_s - VS$ only reducing to zero we have the relation:

$$\frac{V\beta}{\beta} = \frac{Vf}{f}.$$

As an example of **stable achromatism** we may instance a symmetrical concentric system achromatised for an infinitely distant object-point, since such a system is likewise achromatic in relation to the principal front focus and the middle of the system.

B. The Chromatic Variations in Simple Special Cases.

193. We shall now investigate the conditions for the achromatisation of a few simple forms of dioptric systems. In all these cases we shall suppose the system to be enclosed on both sides by air, and we shall therefore also speak, without qualification, of the compensation of the chromatic difference of magnification.

194. Single Lens of Finite Thickness. We shall first investigate the case of a single lens of appreciable thickness. The condition for the compensation of the longitudinal chromatic aberration assumes in this case the form

$$Vs'_2 = -s'_2\,\mathrm{s}'_2\left[\left(\frac{h_1}{h_2}\right)\left(\frac{\mathrm{h}_1}{\mathrm{h}_2}\right) Q_1\frac{Vn'_1}{n'_1} - Q_2\frac{Vn_2}{n_2}\right] = 0\,. \;\dots\quad \text{(i)}$$

Since

$$n'_1 = n_2 = n\,;\quad Vn'_1 = Vn_2 = Vn\,,$$

this condition, if we exclude the case where $s'_2 = 0$, becomes

$$h_1\,\mathrm{h}_1\,Q_1 = h_2\,\mathrm{h}_2\,Q_2 \;\dots\qquad\dots\qquad\dots\quad \text{(ii)}$$

that is to say, the exterior or, which comes to the same thing, the interior angles of incidence of a ray of the first colour near the axis, are related inversely as the ordinates of the point of incidence for the second colour. For the ratio of the angles of incidence we

may substitute the negative value of the ratio of the deviations. When this condition is satisfied the chromatic difference of the magnification vanishes in that

$$\frac{V\gamma}{\gamma} = -\frac{d}{n} \cdot \frac{h_1}{h_2} Q_1 \frac{Vn}{n} = 0 \quad , \ldots \qquad \ldots \quad \text{(iii)}$$

where d is the thickness of the lens.

It may, on the other hand, be required to reduce $V\gamma$ to zero independently of other quantities. Proceeding from the equation

$$\gamma\sigma_1 = (n-1)(\rho_1 - \rho_2) + \frac{(n-1)^2}{n} d \rho_1 \rho_2 + \sigma_1 \left(1 + \frac{n-1}{n} d \rho_2\right) \quad \text{(iv)}$$

and establishing the differences, we obtain the following condition

$$\frac{\sigma_1 V\gamma}{Vn} = \rho_1 - \rho_2 + \left(1 - \frac{1}{n\mathrm{n}}\right) d \rho_1 \rho_2 + \frac{\sigma_1 d \rho_2}{n\mathrm{n}} = 0 \, . \quad \text{(v)}$$

When $\sigma_1 = 0$, it follows that

$$r_1 - r_2 = \left(1 - \frac{1}{n\mathrm{n}}\right) d \, . \quad . \ . \qquad \ldots \quad \text{(vi)}$$

195. System of Thin Lenses.—The longitudinal chromatic aberration of a system of thin lenses reduces to the form

$$V\sigma'_k = \sum_{v=1}^{k} \frac{\phi_v}{\nu_v} \left(\frac{h_v}{h_k}\right) \left(\frac{\mathrm{h}_v}{\mathrm{h}_k}\right) , \ldots \qquad \ldots \quad \text{(i)}$$

where σ'_k is the reciprocal of the intercept after refraction through the k^{th} lens ; ϕ_v the power to the v^{th} lens ; h_v the incidence height of a ray near the axis, and $\nu_v = \dfrac{n_v - 1}{Vn_v}$ the **reciprocal relative dispersion** or the value of ν for the v^{th} lens. We shall call $\dfrac{1}{\nu}$ the **disperser** and ϕ_v/ν_v the **dispersive power** of the v^{th} lens. It should be noted that the longitudinal aberration and likewise the chromatic difference of magnification are independent of the coflexure of the component lenses. A single thin converging lens is chromatically under-corrected, whilst a single diverging lens is chromatically over-corrected ; and the chromatic aberration of the principal focus is expressed by the product of the focal length and the disperser, when Vn, and consequently also Vs, can be regarded as small quantities.

We shall in the first place investigate the case of two thin lenses separated by an infinitely small distance. Let F denote the total focal length of the combination. By § 88 the power of the combination of the two thin lenses is

$$\Phi = \phi_1 + \phi_2 = \sigma'_2 - \sigma_1 = (\gamma - 1)\,\sigma_1\,, \quad \ldots \quad \text{(ii)}$$

which represents the fundamental equation. The condition of achromatism for any position of the object on the axis is:

$$\frac{\phi_1}{\nu_1} + \frac{\phi_2}{\nu_2} = 0\,, \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

or, if we denote by k the difference of the curvatures of the two surfaces of the individual lenses, which in the expression for the power of a lens is the factor governing its form and which we shall briefly refer to as the k-value, we obtain the alternative expression

$$k_1 V n_1 + k_2 V n_2 = 0\,. \quad \ldots \quad \ldots \quad \text{(iv)}$$

whence we can calculate

$$\left.\begin{aligned} \phi_1 &= \frac{\nu_1}{\nu_1 - \nu_2}\,\Phi = \frac{\nu_1}{\nu_1 - \nu_2}\,(\gamma - 1)\,\sigma_1 \\ \text{and} \qquad & \\ \phi_2 &= -\frac{\nu_2}{\nu_1 - \nu_2}\,\Phi = -\frac{\nu_2}{\nu_1 - \nu_2}\,(\gamma - 1)\,\sigma_1 \end{aligned}\right\} \quad \ldots \quad \text{(v)}$$

or

$$\left.\begin{aligned} k_1 &= \frac{\Phi}{(\nu_1 - \nu_2)\,V n_1} \\ \text{and} \qquad & \\ k_2 &= \frac{\Phi}{(\nu_1 - \nu_2)\,V n_2} \end{aligned}\right\}. \quad \ldots \quad \ldots \quad \text{(vi)}$$

Achromatisation is possible only so long as $\nu_1 \gtrless \nu_2$, from which it follows that it is necessary to use two different types of glass in the two lenses. The lens having the higher ν has the same sign as the total focal length, whilst that having the lower ν is of the opposite sign. The powers of the component lenses become smaller the greater the difference of the values of ν, and smaller as the values of ν themselves diminish. If, for example, it is required to achromatise a pair of lenses for visual rays, a combination of fluorspar and quartz having high values of ν necessitates the use of component lenses of similar powers to those of a combination of the same focal length and made up of lenses of Jena flint glasses of the types O.726 (n = 1·54) and O.335 (n = 1·637),

in which the difference of the values of ν is only about one half that of the fluorspar and quartz combination. In the case of the ordinary crown-glass of $\nu = 60$ and the ordinary flint glass of $\nu = 36$ (assuming the colour interval to correspond to that represented by the Fraunhofer lines C to F), we find $\phi_1 = 2{\cdot}5\,\Phi$ and $\phi_2 = -1{\cdot}5\Phi$. In two combinations having equal values of ν the k-values are inversely related as the refractive indices diminished by unity; for, to give a concrete case, since

$$\frac{K}{k} = \frac{n_F - n_C}{N_F - N_C} = \frac{\dfrac{n_F - n_C}{n_D - 1}(n_D - 1)}{\dfrac{N_F - N_C}{N_D - 1}(N_D - 1)} = \frac{\dfrac{1}{\nu}(n_D - 1)}{\dfrac{1}{\nu}(N_D - 1)},$$

it will be seen that

$$\frac{K}{k} = \frac{n_D - 1}{N_D - 1}.$$

In a system of k **thin lenses in contact** the chromatic aberration for any distance of the object is expressed according to (i) by the formula

$$V\sigma'_k = \sum_{v=1}^{k} \frac{\phi_v}{\nu_v}. \quad \ldots \quad \ldots \quad \ldots \quad \text{(vii)}$$

When the distance of the object is zero, since the dispersive powers of the component lenses, ϕ_1/ν_1 and ϕ_2/ν_2, just counteract one another, besides satisfying the fundamental equation,

$$\Phi = \sum_{v=1}^{k} \phi_v, \quad \ldots \quad \ldots \quad \ldots \quad \text{(viii)}$$

it follows that the powers of $k-2$ lenses can be given any value we choose. The chromatic aberration does not depend upon the order of succession of the component lenses. It remains likewise unaffected when one of the lenses is resolved into several others having the same value of ν and an aggregate power equal to that of the resolved lens.

In its refracting and colour-dispersing effects in the formation of the Gaussian images, a combination of k thin lenses in close proximity is equivalent to a single lens of strength Φ and having a value of ν given by

$$N = \frac{\sum_{v=1}^{k} \phi_v}{\sum_{v=1}^{k} \dfrac{\phi_v}{\nu_v}}. \quad \ldots \quad \ldots \quad \ldots \quad \text{(ix)}$$

For any given value of ν, the value of N can be varied within the limits $-\infty$ and $+\infty$ by a suitable choice of the value of ϕ_ν/Φ. This may be accomplished by means of two lenses. The practical optician is thus enabled to introduce any values of ν into his calculations, although the range of the optical glasses which have been rendered available up to the present extends only from $\nu = 20$ to $\nu = 70$. In the case of a value of ν arising in the computation which is not contained in the list of available glasses, all that is necessary is to replace this lens by a suitable combination of two or more lenses.

In the special case of a combination of cemented lenses having the same refractive index for a given colour as the single lens, the combination affords a means of varying the chromatic aberration without disturbing the spherical aberrations for this particular colour; on the other hand, this change affects the chromatic variations of the spherical aberrations, with which we shall have to concern ourselves later. It may arise that a combination of lenses has a smaller value of ν than a single lens composed of a glass having the smallest value of ν among the available glasses of the required refractive index, so that in this case the chromatic effect of the combination exceeds that attainable in a single lens of glass of similar refractive index. A lens combination of this kind is said to be **hyperchromatic**. Rudolph (*1.*) applied this principle to a dispersing member in photographic lenses.

Conversely, it is theoretically advantageous to be able to replace a system of k thin lenses in contact by a single lens having the values of Φ and N, defined above. Thus the case of thin lenses separated by finite distances discussed in § 196 includes that of systems of thin lenses separated by finite distances. In an achromatic system of lenses $N = \infty$. The value $N = 0$ represents a special limiting case which is not realisable in practice, since one lens at least would necessarily have an infinitely great value of k. In the case of all the individual lenses having the same sign the value of N lies between ν_{max} and ν_{min}.

196. System of Two Separated Thin Lenses.—We will now consider the achromatisation of two thin lenses separated by a finite distance d_1. In establishing the fundamental equation we may either fix the position of the image at the back of the last lens, or we may choose a value for the magnification:

$$\left.\begin{aligned} (\sigma_1 + \phi_1)\frac{h_1}{h_2} + \phi_2 &= \sigma_2' \\ (\sigma_1 + \phi_1) + \phi_2\frac{h_2}{h_1} &= \gamma\,\sigma_1 \\ \frac{h_2}{h_1} &= 1 - d_1(\phi_1 + \sigma_1)\,. \end{aligned}\right\} \quad \ldots \quad \ldots \quad \text{(i)}$$

In the case of incident parallel light we may regard the position of the principal focus or the focal length as our specified datum, in which case :

$$\left.\begin{aligned} \phi_1 \frac{h_1}{h_2} + \phi_2 &= \Sigma \\ \phi_1 + \phi_2 \frac{h_2}{h_1} &= \Phi \\ \frac{h_2}{h_1} &= 1 - d_1\phi_1 . \end{aligned}\right\} \qquad \ldots \qquad \ldots \qquad \text{(ii)}$$

The corresponding chromatic conditions are :

$$V\sigma'_2 = \frac{\phi_1}{\nu_1}\left(\frac{h_1}{h_2}\right)\left(\frac{\mathrm{h}_1}{\mathrm{h}_2}\right) + \frac{\phi_2}{\nu_2}$$

$$= \frac{\phi_1}{\nu_1\left\{1-d_1(\sigma_1+\phi_1)\right\}^2\left\{1-\frac{d_1\phi_1}{\nu_1}\,\frac{1}{1-d_1(\phi_1+\sigma_1)}\right\}} + \frac{\phi_2}{\nu_2} = 0 , \text{ (iii)}$$

since

$$\left(\frac{\mathrm{h}_2}{\mathrm{h}_1}\right) = 1 - d_1\left(\phi_1 + \sigma_1 + \frac{\phi_1}{\nu_1}\right),$$

$$\sigma_1\, V\gamma = \frac{\phi_1}{\nu_1} + \frac{\phi_2}{\nu_2} - d_1\phi_1\phi_2\left(\frac{1}{\nu_1} + \frac{1}{\nu_2} + \frac{1}{\nu_1\nu_2}\right) - \frac{d_1\sigma_1\phi_2}{\nu_2} = 0 , \text{ (iv)}$$

where the last equation is best obtained by independently forming the differences.

Proceeding now to the achromatisation of the intercepts it is to be noted that the sign of the coefficient of ϕ_1/ν_1 in the expression for $V\sigma'_2$ is positive for all values of d_1, excepting those intermediate between $\frac{1}{\phi_1 + \sigma_1}$ and $\frac{1}{\phi_1 + \sigma_1 + \frac{\phi_1}{\nu_1}}$. In the latter case the second lens lies between the image-points of the first lens for the first and second colours. Disregarding this special case, in which the power of the second lens is very great in comparison with that of the first, it will be seen that, to obtain an achromatic combination, the focal lengths of the lenses must be of opposite signs, when the values of ν have the same sign, and conversely. It is also possible to achromatise the lenses when $\nu_1 = \nu_2$. In this case the chromatic condition becomes :

$$h_1\mathrm{h}_1\phi_1 = - h_2\mathrm{h}_2\phi_2 , \qquad \ldots \qquad \ldots \qquad \text{(v)}$$

that is to say, the ratio of the deviations for the first colour must be equal to the negative reciprocal ratio of the incidence heights

for the second colour, as in the case of the thick lens. Also, the positive value of the coefficient $\left(\frac{h_1}{h_2}\right)\left(\frac{\mathrm{h}_1}{\mathrm{h}_2}\right)$ in the case of a virtual image-point formed by the first lens, diminishes from $+1$ to 0 as the distance between the lenses increases from 0 to ∞. In the case of a real image-point it increases from $+1$ to ∞ as the second lens changes its position from that of the first lens to that of the image-point due to rays of the first colour when ν is negative, or that due to rays of the second colour when ν is positive; and thence it diminishes again from ∞ to 0 as the second lens moves further away to infinity from the image-point corresponding to the second or first colour. While the value of the coefficient changes from $+1$ to $+\infty$ there are always two positions of the second lens for which the dispersive powers are equal. Let the coefficient be denoted by m^2 and let $\frac{h_2}{h_1}$ and $\frac{\mathrm{h}_2}{\mathrm{h}_1}$ be replaced by the equivalent expressions contained in equations (i) and (iv). The separation of the lenses, omitting the second and higher powers of ϕ_1/ν_1 as being negligibly small in comparison with $(\sigma_1 + \phi_1)$, is accordingly

$$d_1 = \frac{1 \pm \frac{1}{m}}{\sigma_1 + \phi_1}\left(1 - \frac{1}{2}\,\frac{1}{\sigma_1 + \phi_1}\,\frac{\phi_1}{\nu_1}\right). \qquad \ldots \qquad \text{(vi)}$$

When the range of colour is infinitely small, in which case we may neglect the first power, the distance of the second lens in front of or behind the image-point of the first lens will be equal to the $\frac{1}{m}$-th part of its image distance. In order that $V\sigma'_2$ may vanish generally, the dispersive powers of the two lenses for the first colour should be in the negative inverse ratio of the squares of the incidence heights of a ray of a mean colour, since h_1/h_2 changes continuously with the colour.

In systems of this kind, in which the chromatic longitudinal aberration has been compensated, we must now ascertain whether the image-point is real or virtual, *i.e.*, whether σ'_2 is positive or negative. Eliminating ϕ_2 from the expressions for σ'_2 and $V\sigma'_2$ equated to zero, we find

$$\left(\frac{\mathrm{h}_2}{\mathrm{h}_1}\right)(\sigma_1 + \phi_1) - \phi_1\,\frac{\nu_2}{\nu_1} = \left(\frac{h_2}{h_1}\right)\left(\frac{\mathrm{h}_2}{\mathrm{h}_1}\right)\sigma'_2\,. \qquad \text{(vii)}$$

Disregarding the special case, the sign of the coefficient of σ'_2 is positive. When the expressions $\left(\frac{\mathrm{h}_2}{\mathrm{h}_1}\right)(\sigma_1 + \phi_1)$ and $\frac{\phi_1\,\nu_2}{\nu_1}$ have

opposite signs, σ'_2 will have the same sign as the first term ; if they have the same sign, σ'_2 will have the same or the opposite sign according as $\left(\frac{h_2}{h_1}\right) \gtrless \frac{\phi_1}{\sigma_1 + \phi_1} \frac{\nu_2}{\nu_1}$.

The quantities $\gamma\sigma_1$ and Φ have like or opposite signs according as $\frac{h_2}{h_1}$ is positive or negative.

When the combination is required to have a **given magnifying power** (or a given focal length) and if it is to be **achromatic with respect to the position of the image,** ϕ_1 and ϕ_2 can be computed from the equations :

$$\left.\begin{aligned} \phi_1 &= \frac{\nu_1}{\nu_1 - \nu_2\left(\frac{h_1}{h_2}\right)} (\gamma - 1)\,\sigma_1 \\ &\text{and} \\ \frac{h_2}{h_1}\,\phi_2 &= -\frac{\nu_2\left(\frac{h_1}{h_2}\right)}{\nu_1 - \nu_2\left(\frac{h_1}{h_2}\right)} (\gamma - 1)\,\sigma_1 . \end{aligned}\right\} \quad \ldots \quad \text{(viii)}$$

The chromatic longitudinal aberration having been removed in the system of two separated thin lenses, there is still a residual **chromatic difference of magnification.** We may readily obtain its value by modifying the formula which is applicable to a system of many surfaces :

$$\frac{V\gamma}{\gamma} = -\frac{d_1\,\phi_1}{\nu_1}\,\frac{h_1}{h_2} = \frac{V\Phi}{\Phi} \,. \;\ldots \qquad \ldots \quad \text{(ix)}$$

$V\gamma$ cannot vanish when the first lens is single, but it may readily be made to do so if this lens is replaced by an achromatic system of lenses ($\nu_1 = \infty$). The sign of the chromatic difference of magnification changes accordingly when that of ϕ_1, ν_1 and $\frac{h_1}{h_2}$ changes. The sign of $\frac{h_1}{h_2}$ is positive when the image-point of the first lens is virtual, whilst when the latter is real it is positive only when the second lens occupies a position between the first lens and its image-point.

Neglecting meanwhile the correction of the chromatic longitudinal aberration, we shall now determine the **conditions for the removal of the chromatic difference of magnification,** and, to simplify matters, we shall assume that the **colour interval is very**

small, and that the glasses employed in both lenses have the **same disperser** (so that $\nu_1 = \nu_2$). We proceed from equations derived from the expressions (i) and (iv) for $\sigma_1\gamma$ and $\sigma_1 V\gamma$, viz.:

$$\left.\begin{aligned} \phi_1 + \phi_2 - d_1(\phi_1 + \sigma_1)\phi_2 &= (\gamma - 1)\,\sigma_1 \\ \phi_1 + \phi_2 - d_1(2\phi_1 + \sigma_1)\phi_2 &= 0 \end{aligned}\right\}\ldots \qquad \text{(x)}$$

or, by subtracting the equations and eliminating σ_1 from the second equation, we have

$$\left.\begin{aligned} \frac{s_1}{\gamma - 1} &= \frac{f_1 f_2}{d_1} \\ f_1 + f_2 - 2\,d_1 - \frac{d^2_1}{f_2(\gamma - 1)} &= 0. \end{aligned}\right\} \quad \ldots \qquad \text{(xi)}$$

By way of example we may instance the erecting system of a terrestrial eyepiece, where

$$\gamma = -\frac{19}{8}\ ;\ f_1 = -\frac{4}{9}s_1\ ;\ f_2 = -\frac{s_1}{3}\ ;\ d_1 = -\frac{s_1}{2}\ .$$

For an **infinitely distant object** the equations are:

$$\left.\begin{aligned} \phi_1 + \phi_2 - d_1\phi_1\phi_2 &= \Phi \\ \phi_1 + \phi_2 - 2d_1\phi_1\phi_2 &= 0 \\ \text{or} \qquad F &= \frac{f_1 f_2}{2} \\ d_1 &= \frac{f_1 + f_2}{2}\ . \end{aligned}\right\} \quad \ldots \quad \ldots \quad \text{(xii)}$$

As examples, we may instance the Ramsden eyepiece, where

$$f_1 = f_2 = d_1 = F\,,$$

and a Huygens eyepiece, where

$$f_1 = \frac{3\,F}{4}\ ;\ f_2 = \frac{3}{2}F\ ;\ d_1 = \frac{9}{8}F.$$

Certain limiting cases of practical importance known as **Ramsden Combinations,** in which the second lens is situated at the image-point of the first lens for one of the colours, deserve special consideration. As such we may select the first colour without prejudicing the general aspect of the problem. In this case $d_1 = s'_1$

and $s'_2 = s_2 = 0$. Instead of reverting to the general expression for $V\sigma'_2$ we shall proceed from that for s'_2 in order to evaluate the **chromatic longitudinal aberration,** and we find accordingly :

$$\frac{1}{Vs'_2} = \frac{1}{s'_2} = \frac{\sigma_1 + \phi_1 + \dfrac{\phi_1}{\nu_1}}{1 - d_1\left(\sigma_1 + \phi_1 + \dfrac{\phi_1}{\nu_1}\right)} + \phi_2 + \frac{\phi_2}{\nu_2} ,$$

or, noting that $d_1 = s'_1 = 1 / (\sigma_1 + \phi_1)$,

$$\frac{1}{Vs'_2} = -(\sigma_1 + \phi_1) - \frac{(\sigma_1 + \phi_1)^2}{\phi_1}\nu_1 + \phi_2\left(1 + \frac{1}{\nu_2}\right). \qquad \text{(xiii)}$$

The condition for the elimination of the **chromatic difference of magnification,** after suitable cancelling, takes the form

$$\frac{V\gamma}{\gamma} = \frac{\phi_1}{(\sigma_1 + \phi_1)^2}\frac{1}{\nu_1}\left\{\sigma_1 + \phi_1 - \phi_2\left(1 + \frac{1}{\nu_2}\right)\right\} = 0 , \quad \text{(xiv)}$$

from which it will be seen that the power of the second lens for the second colour must be equal to the reciprocal of the image distance of the first lens for the first colour. When this is the case

$$Vs'_2 = -\frac{\phi_1}{(\sigma_1 + \phi_1)^2}\frac{1}{\nu_1} . \quad \dots \quad \dots \quad \text{(xv)}$$

Since $\nu_1 = (n_1 - 1) / Vn_1$, it follows that the chromatic variation of the intercept is directly proportional to that of the refractive index.

197. System of Three Separated Thin Lenses. A system of three lenses separated by two finite distances d_1 and d_2 may be resolved by dividing the intermediate lens into two components so as to form two systems, each consisting of two thin lenses separated by a finite distance, the lenses facing one another having a combined power equal to that of the intermediate lens of the system. We shall assume the component back system to be traversed by the rays in the reverse direction. This expedient is adopted in order to obtain a simple expression for the achromatism of the magnification of a three-lens system corrected as regards the longitudinal chromatic aberration, which may be effected by equating the chromatic difference of magnification in the two components. Without prejudice to the general character of the following investigation, we may suppose the intermediate lens to be divided in such

a manner as to cause the two-lens components of the system to be free from longitudinal chromatic aberration. The condition § 196 (ix) may then be written in the form :

$$\frac{d_1 \phi_1 h_1}{\nu_1} - \frac{d_2 \phi_3 h_3}{\nu_3} = 0 , \quad \ldots \quad \ldots \quad \text{(i)}$$

that is to say, the first and third lenses must be so related that the products of the chromatic variation, the deviation, and the distance from the second lens are equal. Geometrical considerations lead to the same result. In this connection it should be noted that the paraxial rays of the two colours on emergence should coincide. Similarly, by § 194 (iii), a system consisting of two lenses in contact, having finite thicknesses d_1 and d_2 and achromatized as regards both the position of the image and the magnification, should satisfy the condition

$$Q_1 \frac{d_1 h_1}{\mathrm{n}'_1} \frac{V n'_1}{n'_1} = - Q_4 \frac{d_2 h_4}{\mathrm{n}_4} \frac{V n_4}{n_4} . \quad \ldots \quad \text{(ii)}$$

In a system of three separated thin lenses, if the second lens is at the image-point of the first lens for one of the given colours, say the first, and if the longitudinal chromatic aberration has been corrected, there is a simple rule for the removal of the chromatic difference of magnification. For if the second lens be divided in such a manner that the front Ramsden combination is free from any chromatic difference of magnification, the same will apply to the back combination. From this it follows that for rays of the second colour the first and the third lens must be optically conjugate to the second lens, that is to say, for such rays the first lens must be transformed through the second into the third lens.

We shall proceed finally to the computation of a system of three lenses separated by finite distances, in which the images of an infinitely distant object formed by rays of two infinitely near colours are identical in position and size. In this case the leading equation

$$\phi_1 + \phi_2 \frac{h_2}{h_1} + \phi_3 \frac{h_3}{h_1} = \Phi \quad \ldots \quad \ldots \quad \text{(iii)}$$

is associated with the following two equations of achromatism, viz. :

$$\phi_1 + \frac{\phi_2}{\nu_2}\left(\frac{h_2}{h_1}\right)^2 + \frac{\phi_3}{\nu_3}\left(\frac{h_3}{h_1}\right)^2 = 0$$

and $\ldots \quad \ldots \quad \text{(iv)}$

$$\frac{d_1 h_1 \phi_1}{\nu_1} = \frac{d_2 h_3 \phi_3}{\nu_3} .$$

Eliminating d_1 and d_2 from the latter equation with the aid of the relations

$$d_1 = \frac{1 - \frac{h_2}{h_1}}{\phi_1} \; ; \; d_2 = \frac{\frac{h_2}{h_1} - \frac{h_3}{h_1}}{\phi_1 + \phi_2 \frac{h_2}{h_1}} : \quad \dots \quad \text{(v)}$$

and simplifying by means of the leading equation, we obtain the expression

$$\Phi \left(1 - \frac{h_2}{h_1}\right) \frac{1}{\nu_1} = \phi_3 \frac{h_3}{h_1} \left[\left(1 - \frac{h_2}{h_1}\right) \frac{1}{\nu_1} + \left(\frac{h_2}{h_1} - \frac{h_3}{h_1}\right) \frac{1}{\nu_3} \right]. \quad \text{(vi)}$$

To determine the powers of the three lenses when the values of $\frac{h_2}{h_1}$ and $\frac{h_3}{h_1}$ are given it is, therefore, only necessary to solve three linear equations.

If the powers of three lenses are given and are such, for example, as will satisfy the Petzval condition, as will be explained in Chapter VII, the distances separating the lenses may be so chosen as to satisfy the two chromatic conditions. Further, if from the second chromatic equation we eliminate first d_1 and d_2, and then with the aid of the first equation the resulting term $(h_2/h_1)^2$, the two equations will give us h_2/h_1 as a function of h_3/h_1, which appears in a biquadratic equation capable of resolution into two quadratic equations. If we make

$$\frac{h_3}{h_1} = \mathrm{x} \; ; \quad \frac{\phi_1}{\phi_3} \frac{\nu_3}{\nu_1} = \mathrm{a} \; ; \quad \frac{\phi_1 - \phi_2}{\phi_3} \frac{\nu_3}{\nu_1} = \mathrm{b} \; ; \quad \frac{\phi_2}{\phi_3} \frac{\nu_3}{\nu_2} \left(\frac{\nu_1 + \nu_2}{\nu_1}\right)^2 = \mathrm{c} \, ,$$

the equations assume the following abbreviated forms :

$$(\mathrm{c} + 1)\,\mathrm{x}^2 + 2\,\mathrm{bx} + \mathrm{ac} + \mathrm{b}^2 = 0 \; ; \quad \mathrm{x}^2 + \mathrm{a} = 0 \; . \quad \text{(vii)}$$

We then find :

$$\frac{h_2}{h_1} = \frac{(\mathrm{a} + \mathrm{x}^2)(\nu_1 + \nu_2)}{(\mathrm{b} + \mathrm{x})\,\nu_1} \, ,$$

and from the above equations the values of d_1 and d_2. For example, if $n_1 = n_2 = n_3$; $\nu_1 = \nu_3$; $\nu_1 / \nu_2 = 1{\cdot}5$; $\phi_1 = \phi_3 = 1$; $\phi_2 = -2$; the first equation gives the values $d_1 = 0{\cdot}217$ and $d_2 = 0{\cdot}236$. The other root is of no practical use, since the second distance has a negative value. The second equation furnishes in this instance no real roots. These only arise when a is negative, that is to say when the dispersive powers of the first and third lenses have opposite signs. The second chromatic equation then indicates that the only root which yields positive values for both distances is $\mathrm{x} = -\sqrt{\mathrm{a}}$; also that $d_1 / d_2 = - h_1 / h_3$, and therefore the second lens must lie at the image point of the first lens.

C. The Secondary Spectrum.

198. In achromatic systems the required agreement of the factors which determine the formation of the images applies to two colours only. It remains therefore to ascertain the magnitudes of the residual aberrations for a third colour, and to establish the condition for the correction of this residual defect. We shall confine our attention to the longitudinal chromatic aberration, as being the more important defect in practice. Moreover, in a system of thin lenses in contact it is the only defect with which we need concern ourselves. We shall assume also that the system is bounded on both sides by air. It will first be necessary to introduce the following new terms. The excess of the refractive index of the second standard colour over that of the first will be called the **standard dispersion**; the excess of the refractive index of the third colour over that of the first standard colour will be called the **partial dispersion,** and the ratio of the partial dispersion to the standard dispersion will be called the **relative partial dispersion.** The latter we shall denote by the letter θ, or briefly the θ-value for the third colour. The value of ν for the interval between the first and the third colour then becomes $\frac{\nu}{\theta}$. Regarding the colour correction of achromatic systems as a correction of the first order, we may describe the deviation of the intercepts of the pencils composed of rays of the third colour relatively to those of the standard colours as the **secondary spectrum for the third colour.** In what follows we shall suppress, for brevity, the qualifying words "for the third colour."

199. Thin System of Lenses. In an achromatic system of thin lenses in contact the amount of the secondary spectrum is expressed by a formula which is analogous to that for the longitudinal chromatic aberration, viz.

$$\Omega = -s'_k(s'_k + \Omega)\sum_{v=1}^{k}\frac{\theta_v \phi_v}{\nu_v}. \qquad \ldots \qquad \text{(i)}$$

In general, no appreciable error will be introduced by reducing this expression to the simpler form:

$$= -s'^2_k \sum_{v=1}^{k}\frac{\theta_v \phi_v}{\nu_v}. \qquad \ldots \qquad \ldots \qquad \text{(ii)}$$

The secondary spectrum vanishes in all cases when $\theta_1 = \theta_2 = \ldots\ldots \theta_k$. Otherwise the correction for a third colour involves the use of at least three lenses of suitably arranged focal lengths, and it is impossible, even with any number of lenses, to remove the secondary spectrum when

$$\frac{\theta_1 - A}{\nu_1} = \frac{\theta_2 - A}{\nu_2} = \ldots\ldots \frac{\theta_k - A}{\nu_k} = B,$$

where A and B denote any constants, since in this case

$$\Omega = -\frac{s'^2_k}{F}B = -\frac{B}{\gamma\,\sigma_1}\left(1-\frac{1}{\gamma}\right), \quad \ldots \quad \text{(iii)}$$

from which it follows that for infinitely distant objects the secondary spectrum referred to unit focal length is equal to $-B$.

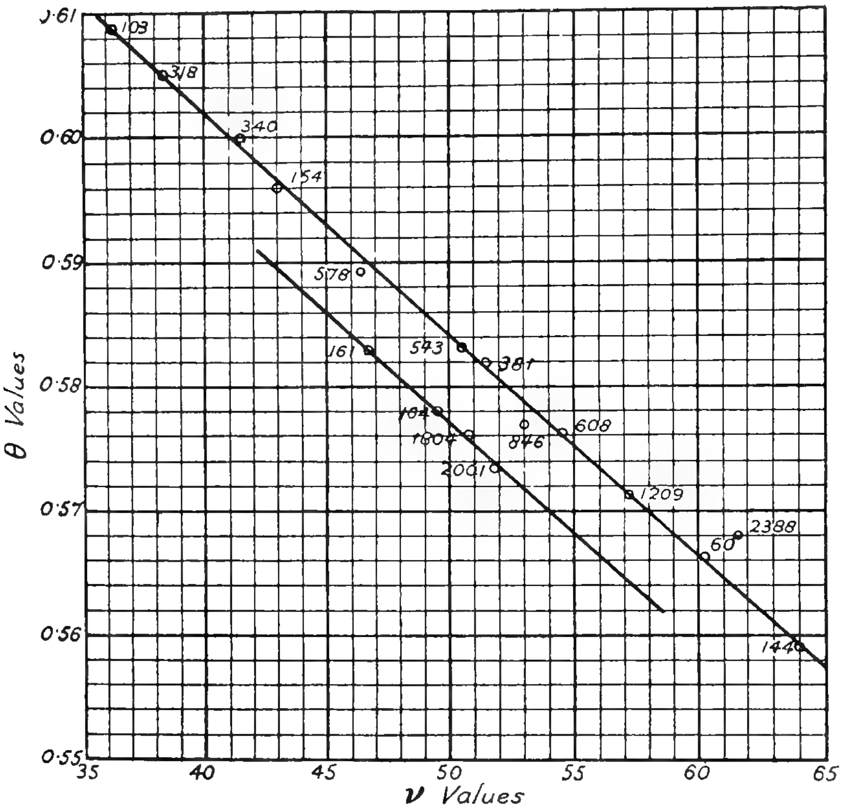

FIG 82.

$$\nu = \frac{n_D - 1}{n_F - n_C}\ ; \ \theta = \frac{n_{G'} - n_F}{n_F - n_C}$$

These values of θ are given in the lists of optical glasses. In the text θ has been defined thus :—

$$\theta = \frac{n_{G'} - n_C}{n_F - n_C} = 1 + \frac{n_{G'} - n_F}{n_F - n_C}$$

Graphic representation of the values of ν and θ of several Optical Glasses.

In the annexed diagram (Fig. 82) the characteristic properties of the most important Jena glasses employed in the construction of lenses are represented graphically in terms of abscissa values of ν for the wave-length interval $C - F$, and of ordinate values of θ for the H_γ line. It will be noticed at once that the points by which the glasses are represented are mostly grouped in two rows. The longer row comprises mainly silicate glasses, the other boro-silicate flint glasses. It will be seen that the glasses of the boro-silicate series have a lower θ-value than those of the silicate series for the same ν-values; whereas a telescopic crown glass, which stands by itself, has a higher value of θ than the silicate glasses of the same value of ν. Within the limits of the silicate series the θ-value conforming to a given ν-value can be calculated with a degree of exactness which is little short of that with which the measurements can be made (the refractive index being accurate within one unit in the fifth decimal). The formula is

$$\theta = 1.674 - 0.0018\,\nu.$$

Similarly, within the boro-silicate series, θ is likewise a linear function of ν, thus

$$\theta = 1.667 - 0.0018\,\nu.$$

The coefficient of ν is again -0.0018, since the two series are represented by two parallel straight lines. From the preceding remarks it follows, therefore, that the secondary spectrum remains unchanged however the aggregate power of the system may be distributed over the component lenses and whatever may be the choice of lenses, so long as we do not employ glasses of one series in conjunction with glasses of the other. We should arrive at a similar result if, in the place of H_γ, we had chosen a different colour within the interval C-H_γ. It is only by combining glasses the representative points of which do not all lie on a straight line inclined to the horizontal axis that the removal of the secondary spectrum becomes possible in a system of thin lenses in contact. It has, therefore, only become practicable by the production of glasses in which the progressive change of the dispersion is of a dual order, so that in a sense the θ-value has been made independent of the ν-value.

In a system of two thin lenses, if we introduce the powers of the component lenses necessary for the achromatism of the system, the quantitative expression for the secondary spectrum takes the form:

$$\Omega = \mp \frac{\theta_1 - \theta_2}{\nu_1 - \nu_2} \frac{s'^2_2}{F}.$$

The ν difference being finite and $\theta_1 = \theta_2$, this expression vanishes when the points corresponding to the two glasses are on a straight

line parallel to the horizontal axis. The smaller the ν-difference the more nearly equal must θ_1 and θ_2 be in order to remove the secondary spectrum.

Systems comprising three or more thin lenses having negligibly small thicknesses and separations can be reduced to a two-lens system. To simplify the discussion we shall replace all the component converging lenses and similarly, all the diverging lenses by an equivalent converging lens and a diverging lens respectively. If l be the number of the converging lenses, then the power Φ_+, the value N_+ of ν, and the value Θ_+ of θ of the equivalent converging lens will be given by the expressions

$$\Phi_+ = \sum_{r=1}^{l} \phi_r$$

$$N_+ = \frac{\sum_{r=1}^{l} \phi_r}{\sum_{r=1}^{l} \frac{\phi_r}{\nu_r}} \quad \ldots \quad \ldots \quad \ldots \quad (\text{v})$$

$$\Theta_+ = \frac{\sum_{r=1}^{l} \frac{\theta_r \phi_r}{\nu_r}}{\sum_{r=1}^{l} \frac{\phi_r}{\nu_r}}.$$

In precisely the same manner the values of Φ_-, N_- and Θ_- may be determined for the equivalent diverging lens. Now the values of N_+ and Θ_+ are intermediate between the minimum and maximum values of ν and θ of the component converging lenses and, similarly, N_- and Θ_- are intermediate between the respective values of the component diverging lenses. If, accordingly, we plot the values of $N_\pm$ and $\Theta_\pm$ for the chosen glasses in a diagram as in Fig. 82, the representative point should lie within the polygon whose angular points represent the extreme points on the curves of existing glasses. Now, in general, there is no special difficulty in producing glasses having properties intermediate between those of existing glasses, so that the use of three or more lenses for the purpose of diminishing the secondary spectrum may be regarded as an expedient for accomplishing the desired result with available glasses; on the other hand, the multiplication of the lenses, apart from colour correction, frequently offers advantages as regards the correction of the spherical aberration which often justifies the expedient.

If in a system of three thin single lenses it happens that the values of θ are not identical, and that the representative points form a triangle of finite area, then the secondary spectrum cannot vanish unless the powers are distributed over the component lenses. Without reference to the actual order of succession of the lenses, which is immaterial, we shall describe as the first lens the one having the

highest value of θ, that having the intermediate value being the second lens, and that having the lowest value the third lens; and we shall refer to the glasses of which they are composed as the first, second, and third glasses respectively. Similarly, let P_1, P_2, P_3 be the representative points. The ratio of the powers of two lenses must now be so chosen as to make the θ-value of the substituted lens equal to the θ-value of the remaining lens. If these two lenses be required to have the same sign, the lenses so chosen will necessarily be the first and third lenses. The representative point of the glass required for the substituted lens is the point of intersection E of the straight line P_1P_3 and a straight line through P_2 parallel to the axis of ν. The condition

$$\theta_2 = \Theta = \frac{\dfrac{\theta_1\phi_1}{\nu_1} + \dfrac{\theta_3\phi_3}{\nu_3}}{\dfrac{\phi_1}{\nu_1} + \dfrac{\phi_3}{\nu_3}} \qquad \ldots \qquad \ldots \qquad \text{(vi)}$$

or

$$\frac{\theta_1 - \theta_2}{\theta_2 - \theta_3} = \frac{\phi_3}{\nu_3} \Big/ \frac{\phi_1}{\nu_1}$$

signifies geometrically that P_1E and EP_3 vary inversely as the dispersive powers of the first and second lenses. When the value of N for the substituted lens is greater than ν_2, then ϕ_1 and ϕ_3 are positive and ϕ_2 is negative. When N is less than ν_2 the signs of ϕ_1, ϕ_2, and ϕ_3 are reversed. The powers of the component lenses also become smaller the greater the difference $N - \nu_2$ in comparison with the values of N and ν_2.

We now proceed to explain the computation of a system of three lenses free from secondary spectrum. Let the three colours be those corresponding to the C, F, and G' lines of the spectrum. For our glasses we shall select those of an objective computed by Dennis Taylor (*1.*), viz.: O.374, O.658, and O.543 having the constants $\nu_1 = 60{\cdot}4$, $\nu_2 = 50{\cdot}2$, $\nu_3 = 50{\cdot}6$, and $\theta_1 = 1{\cdot}5675$, $\theta_2 = 1{\cdot}5767$, $\theta_3 = 1{\cdot}5830$. From the last equation we obtain the value $\phi_3 / \phi_1 = 1{\cdot}224$. Next, we must calculate the value of $N = 54{\cdot}6$. With the aid of the formulæ for the achromatic system of two lenses we obtain finally $\phi_1 = +\ 5{\cdot}58$, $\phi_2 = -\ 11{\cdot}41$, $\phi_3 = +\ 6{\cdot}83$.

These results might also have been obtained directly from the equations

$$\sum_{v=1}^{3} \phi_v = 1\,; \quad \sum_{v=1}^{3} \frac{\phi_v}{\nu_v} = 0\,; \quad \sum_{v=1}^{3} \frac{\theta_v\phi_v}{\nu_v} = 0\,;$$

the previous method however, affords a better insight into the various operations involved in the choice of suitable glasses.

In the above equations for Φ, N and Θ, if we remove the restriction limiting the combination to converging lenses, the following interpretation becomes possible. By suitably choosing the powers of the component lenses, we may vary the values of N and Θ independently from $-\infty$ to $+\infty$, and for this purpose it is sufficient to employ a combination of three lenses. The optician is thus enabled to devise systems which are equivalent to a single lens having any values of ν and θ. Conversely, the case of single thin lenses separated by finite distances, which is described in § 200, includes that of separated systems of thin lenses if appropriate ν and θ values are given to the single lenses. A combination of lenses for which $N = \infty$ and $\frac{\Theta}{N} = 0$, is achromatic and free from secondary spectrum. If a combination consists of lenses whose values of ν and θ are connected by the equation

$$\theta = A + B\nu,$$

this relation holds also for the resulting values Θ and N of the substituted lens. When $N = \infty$, it follows that $\frac{\Theta}{N} = B$.

200. System of Two Separated Thin Lenses.—The expression for the secondary spectrum of a system consisting of two thin lenses (or systems of lenses) separated by a finite distance is analogous to the expression for $V\sigma'_2$ given in § 196 (iii),

$$\Omega = -s'_2(s'_2+\Omega)\left[\frac{\theta_1\,\phi_1}{\nu_1\{1-d_1(\sigma_1+\phi_1)\}^2\left\{1-\frac{d_1\,\phi_1}{1-d_1(\sigma_1+\phi_1)}\frac{\theta_1}{\nu_1}\right\}}+\frac{\theta_2\phi_2}{\nu_2}\right]. \quad \text{(i)}$$

We shall first consider the case in which the front and the back components are individually achromatic, so that $\nu_1 = \nu_2 = \infty$; but we will suppose that $\frac{\theta_1}{\nu_1}$ and $\frac{\theta_2}{\nu_2}$ are finite; in fact, we shall assume that they are equal, and that accordingly, the indices may be omitted. Neglecting Ω in comparison with s'_2 and $\frac{\theta}{\nu}$ in comparison with $\{1 - d_1(\sigma_1 + \phi_1)\}/d_1\,\phi_1$, and eliminating s'_2 with the aid of the expression for σ'_2, that for Ω becomes

$$\Omega = -\frac{\theta}{\nu\,\gamma^2\sigma_1^2}[\phi_1 + \phi_2\{1 - d(\sigma_1 + \phi_1)\}^2] \ldots \quad \text{(ii)}$$

The condition that the secondary spectrum may vanish will then be :

$$\frac{\phi_1}{\{1 - d_1(\sigma_1 + \phi_1)\}^2} = - \phi_2 \dots \qquad \dots \qquad \text{(iii)}$$

The focal lengths of the component lenses will therefore be of opposite sign in all cases. When the image-point of the first lens lies between the two lenses the second lens may occupy two possible positions, viz., on either side of this image-point and at equal distances from it. When the object is situated at infinity, the back focus is always virtual, since

$$\Sigma = \frac{- d_1 \phi_1^2}{(1 - d_1 \phi_1)^2}. \qquad \dots \qquad \dots \qquad \dots \qquad \text{(iv)}$$

Irrespective of this compensation of the secondary spectrum of one of the lenses by means of that of the other, this defect can be greatly diminished in another way, as will be shown in the case of an infinitely distant object. For this purpose the front lens must be given a focal length which is great in comparison with the aggregate focal length, and the image formed by it must be made smaller by a back lens of shorter focal length, it being immaterial whether the second lens has a positive or negative focal length, and whether it is situated close in front of or behind the principal focus of the first lens.

We may now proceed to the general case in which both components of the system have longitudinal chromatic aberration. Assuming Ω to be small in comparison with s'_2 we may, with the aid of the condition of achromatisation established for the entire system, write the expression for Ω in the form :

$$\left.\begin{aligned} \Omega &= - \frac{\phi_1}{\nu_1 \gamma^2 \sigma_1^2} \left[\theta_1 - \theta_2 + \frac{\theta_1^2 \varepsilon}{1 - \theta_1 \varepsilon} - \frac{\theta_2 \varepsilon}{1 - \varepsilon} \right] \dots \\ \text{where} \qquad & \\ \varepsilon &= \frac{d_1 \phi_1}{\{1 - d_1(\sigma + \phi_1)\} \nu_1}. \dots \qquad \dots \end{aligned}\right\} \qquad \text{(v)}$$

This expression consists of two parts, the first of which does not involve the distance d_1, and states the amount of the secondary spectrum which remains when the unaltered first lens is paired with an achromatising lens in contact with it. The second part, which takes account of the finite distance, may be simplified by neglecting ε and $\theta_1 \varepsilon$ as being vanishingly small in comparison with 1, and

noting that the second part is itself proportional to ε. The expression for Ω then becomes :—

$$\Omega = -\frac{\phi_1}{\nu_1 \gamma^2 \sigma_1^2}\left[\theta_1 - \theta_2 + \frac{\theta_1^2 - \theta_2}{\nu_1} \frac{d_1 \phi_1}{1 - d_1(\sigma_1 + \phi_1)}\right]. \quad \text{(vi)}$$

It will thus be seen that d_1 may be so chosen as to cause the secondary spectrum to vanish. We shall illustrate this by an example, in which $\sigma_1 = 0$, the first, second and third colours being C, F and G'. Let the material of the first lens be ordinary crown-glass O.546 with $\nu_1 = 60{\cdot}2$ and $\theta_1 = 1{\cdot}565$; that of the second lens ordinary flint glass O.118 with $\nu_2 = 36{\cdot}9$ and $\theta_2 = 1{\cdot}607$. In order that Ω may vanish we must have $d_1 \phi_1 = 0{\cdot}75$. And in order that the system may be achromatic, as assumed, when $F = -1$, we must have $\phi_1 = +0{\cdot}6328$ and $\phi_2 = -6{\cdot}5312$. This example may be regarded as the dioptric prototype of Schupmann's (*1.*) brachymedial lens, which results when the power of the second lens is of half the amount, whilst the rays are made by reflection to retrace their course within this lens, so as to form a real image-point.

201. Systems of Three Separated Thin Lenses.—Of the systems consisting of three thin lenses (or of three systems of lenses) separated by finite distances we shall consider only those whose intermediate lens is situated at the image-point of the front lens, and which may accordingly be resolved into two Ramsden combinations by the subdivision of the middle lens. These systems include two catadioptric types, viz., one devised by Schroeder (*4.*) and the other, known as the Medial, by Schupmann (*1.*), in so far as we are concerned with its dioptric prototype, which suffices for the consideration of the chromatic aberration. The conditions for the compensation of the longitudinal chromatic aberration of the first order and of the secondary spectrum for these systems are :

$$-(\sigma_1 + \phi_1) + (\sigma_3' - \phi_3) + \phi_2\left(1 + \frac{1}{\nu_2}\right) = \frac{(\sigma_1 + \phi_1)^2}{\phi_1}\nu_1 + \frac{(\sigma_3' - \phi_3)^2}{\phi_3}\nu_3$$

$$-(\sigma_1 + \phi_1) + (\sigma_3' - \phi_3) + \phi_2\left(1 + \frac{\theta_2}{\nu_2}\right) = \frac{\phi_1}{(\sigma_1 + \phi_1)^2}\frac{\nu_1}{\theta_1} + \frac{(\sigma_3' - \phi_3)^2}{\phi}\frac{\nu_3}{\theta_3}. \quad \text{(i)}$$

To ensure, in addition, that the magnification may be the same for all three colours, the left sides of these equations must vanish independently. From this it is apparent that the equations cannot be satisfied unless $\theta_1 = \theta_3$, which will be the case, for instance, when the first and third component systems consist of glass of the same kind. For the middle lens only a colour correction of the first order will be required in so far as the powers for the second and also the third colours must be equal to the sum of the reciprocals

of the distances between the lenses. In this case, if the desired colour correction of the entire system with respect to the position and magnification of the image is required to be of the k^{th} order, it follows that the correction of the middle lens should be of the $(k-1)^{th}$ order. When it is not necessary to correct the chromatic difference of magnification, the above equations may be satisfied even when $\theta_1 \gtreqless \theta_3$.

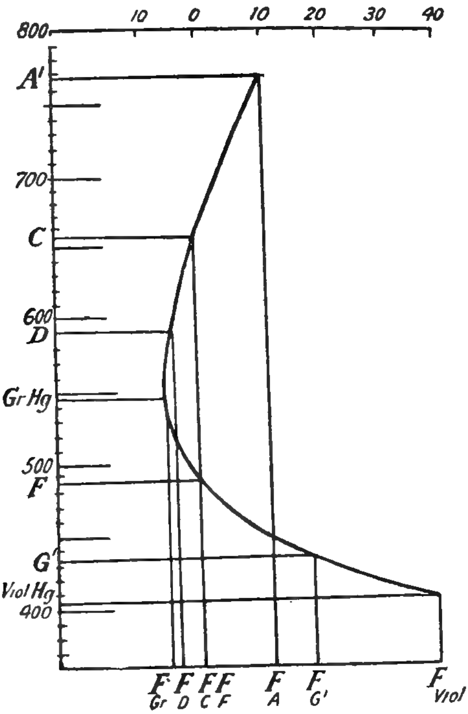

FIG. 83.

Secondary Spectrum of the Combination: O. 1726 and O. 108.
Optical Correction: $F_C = F_F = 100$ mm.

The ordinates represent the wave-lengths in $\mu\mu$. The abscissæ represent $F_\lambda - F_C$ in 1/100ths of a millimetre, plotted from $F_C = F_F$.

D. Choice of the Pair of Wave-lengths for Chromatic Correction.

202. In achromatic systems in which the focus is not constant for all wave-lengths the question arises as to the wave-length for which the intercepts should be made equal. In the first place it is necessary to consider the nature of the light which proceeds from

the object and how the sensitiveness of the light receiving layer (*i.e.*, the retina of the eye or the film of the photographic plate) varies with the wave length. Since this problem lies outside the province of geometrical optics we shall confine ourselves to a few observations respecting the procedure adopted in practical optics. When two colours only are to be combined three cases may be distinguished:

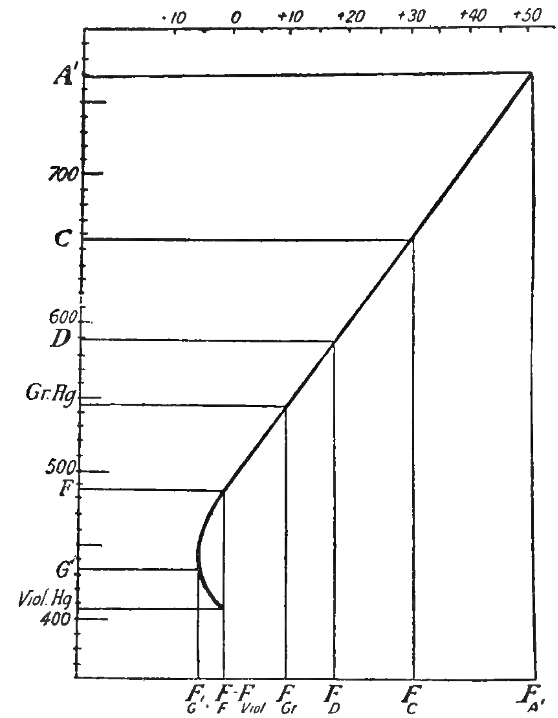

FIG. 84.

Secondary Spectrum of the Combination: O. 1726 and O. 108. Purely actinic correction: $F_F = F_{violet\ Hg} = 100$ mm.

The ordinates represent wave-lengths in $\mu\mu$. The abscissæ represent $F_\lambda - F_F$ in 1/100ths of a millimetre, plotted from $F_F = F_{viol.}$.

If the system is to serve for visual purposes, it is usual to correct it with respect to F and C, or F and a wave-length intermediate between B and C. In systems designed for astrophotographic purposes the wave-lengths chosen are F and $H\delta$; and systems intended for other photographic purposes (*e.g.*, portrait lenses, landscape lenses, photo-micrographic lenses, etc.) are generally corrected with respect to D and $H\gamma$. In the last mentioned case

the best possible combination of the photographically active rays is not insisted upon, *i.e.*, the purely actinic correction aimed at in the achromatisation of astro-photographic combinations. This course is prompted by two considerations. On the one hand, the residual chromatic aberrations in the blue-violet part of the spectrum in systems corrected for visual and actinic rays are not of very serious consequence; on the other hand, in astro-photographic work the necessity for determining the difference between the adjustments for the best visual and the best photographic effects, known as the **difference of focus,** which enables the observer to adjust the photographic plate by visual focussing on the ground glass screen, arises only once, whereas other photographic operations necessitate the use of systems which are achromatised for D and

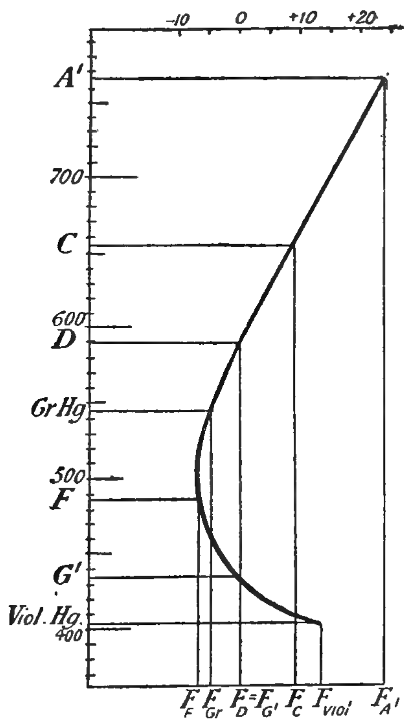

FIG. 85.

Secondary Spectrum of the Combination O.1726 and O.108.

Photographic (actinic) Correction:

$$F_D = F_{G'} = 100 \text{ mm.}$$

The ordinates represent the wave-lengths in $\mu\mu$. The abscissæ represent $F_\lambda - F_D$ in 1/100ths of a millimetre plotted from $F_D = F_{G'}$.

H_γ rays at all distances of the object so as to avoid corresponding adjustments for differences of focus. In practice it is generally considered sufficient to achromatise the system for either the front or the back focus, according as it is required to form a magnified or diminished image; and, moreover, in most cases the correction for the chromatic difference of the lateral magnification implies achromatisation with respect to the focal length.

The annexed diagrams which have been taken from M. v. Rohr's work, "Theorie und Geschichte des photographischen Objektivs" (*3. 61–64*), indicate for three species of achromatisation the variations of the focal lengths with changes of the wave-length in a thin system of ordinary crown and flint glasses. The optical data of the glasses are given in the annexed table. The figures relating to the dispersion indicate the ratio of the partial dispersion and the dispersion for the chromatic interval from C to F.

Refraction and Dispersion of Two Types of Silicate Glass.

O.1726. Silicate Crown.

n_D	C to F	ν	A' to C	C to D	D to *Green Hg*	*Green Hg* to F	F to G'	G' to *Violet Hg*
1·51787	0·00880	58·8	0·00302	0·00259	0·00218	0·00403	0·00501	0·00379
			0·343	0·294	0·248	0·458	0·569	0·431

O.108. Ordinary Silicate Flint.

1·62164	0·01716	36·2	0·00546	0·00488	0·00421	0·00807	0·01045	0·00825
			0·318	0·284	0·245	0·470	0·609	0·481

For the sake of accurate identification we append a table of the wave-lengths in $\mu\mu$ corresponding to the Fraunhofer lines referred to in the above table :—

	A'	C	D	*Green Hg*	F	G'	*Violet Hg*
$\mu\mu$ ...	767·7	656·3	589·3	546·1	486·2	434·1	405·1

2. VARIATION OF THE SPHERICAL ABERRATIONS WITH THE WAVE-LENGTH.

203. We do not propose to establish the somewhat involved formulæ for the differences of the chromatic variation with respect to the five image defects of Seidel, because the chromatic defects with which we are here concerned are of practical importance only when the rays are inclined at considerable angles to the axis. We shall, moreover, confine our attention to the most important defects, viz., the chromatic difference of the spherical aberration, of the sine ratio, and of the distortion.

The first of these defects gives rise to a difference in the spherical aberration of two colours, which increases in general with the angular aperture of the image-forming pencil. As a rule, the aberration for rays of greater refrangibility is in the nature of spherical over-correction. When, for example, the system has been spherically corrected for green rays it will generally be under-corrected for red and over-corrected for blue. This type of correction is advisable in systems designed for visual purposes when it is not considered necessary to correct the chromatic difference of

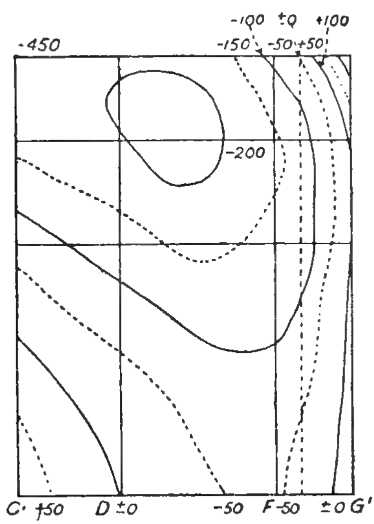

Fig. 86.

Diagram representing the Chromatic Difference of Spherical Aberration occurring in a Petzval Portrait Lens.

$F/3{\cdot}4,\ F_D = 100$ mm.

The abscissæ represent the wave-lengths characterised by the Fraunhofer lines C, D, F, G'. The ordinates represent the incidence heights magnified four-fold.

Isopleth Curves showing the differences in the intercepts for the colour F_D in $\frac{1}{1000}$ths of a millimetre, progressing by steps of 0·050 mm.

spherical aberration; in the case of systems intended for other purposes the spherical correction should apply to the most active rays. This chromatic defect may also be regarded as the variation of the longitudinal chromatic aberration due to changes in the aperture of the system, since for each zone the difference of the intercepts of the two colours changes. If, for example, in any given case chromatic correction has been obtained for rays near the axis, the chromatic over-correction which usually results will increase the nearer the zone to the boundary of the aperture. To ensure chromatic correction over the entire extent of the aperture it is a necessary condition that the chromatic difference of the spherical aberration should be corrected. Where this condition has not been satisfied the question arises as to which zone should be selected for spherical correction. This again is a question which cannot be decided by geometrical optics.

The variations of the intercepts with the wave-length and the incidence heights may be clearly visualised by the method of **isopleths** first employed by M. v. Rohr (*3. 65*). We reproduce his diagram (Fig. 86) for a Petzval portrait lens. This diagram shows that the best combination of rays occurs at an incidence height of 12·5 mm. from the axis for wave-lengths of 540 $\mu\mu$. The spherical aberration exhibits at C under-correction to an extent of about 0·224 mm., and at G' over-correction amounting to 0·125 mm., whilst at $\lambda = 472\ \mu\mu$ the correction is complete for the full aperture. The chromatic aberration at the axis is $+$ 0·08 at C, a minimum $-$ 0·056 at $\lambda = 490\ \mu\mu$, and $+$ 0·016 at G'; whilst in the marginal zone of the lens it is $-$ 0·144 at C, a minimum $-$ 0·188 at $\lambda = 540\ \mu\mu$, and $+$ 0·141 at G'.

To Gauss (*1.*) belongs the credit of having been the first to obtain chromatic correction for both axial and marginal rays in a two-lens objective designed for parallel incident light. Gauss placed the crown lens in front, whereas C. A. Steinheil (*2.*) in the type devised by him placed the flint lens foremost. Abbe (*5.*) investigated the problem of the correction of the chromatic difference of the spherical aberration in microscope objectives of large aperture and evolved two methods of correction. One of these consists in the introduction of cemented surfaces having on the concave side a glass of a similar dispersion but with a lower refractive index than that on the convex side. The second method consists in associating a spherically and chromatically greatly under-corrected front member with a similarly over-corrected back member, the two being separated by a considerable distance. In consequence of this arrangement the blue rays traverse the back member at a smaller distance from the axis than the red rays, and experience there a diminished amount of spherical over-correction, which in its first approximation conforms to the smaller values of the factor $\left(\frac{h_v}{h_1}\right)^4$.

When the second method is employed the blue image experiences a higher magnification than the red. This defect can be corrected either by means of a **compensating eyepiece** having the opposite chromatic difference of magnification, or by a correcting lens situated at a short distance below the image of the objective. This correcting lens is chromatically under-corrected to an appropriate degree and it is **afocal**, that is to say, it is of zero power. It consists of a converging flint lens and a diverging crown lens.

The correction of the spherical aberration and its chromatic difference must necessarily precede any attempt to investigate means of dealing with the chromatic difference of the sine ratio. In systems of moderate aperture in which the spherical aberration and its chromatic difference have been removed, the chromatic variations of the sine ratio which may occur are generally small in amount, so that it is only in systems having very wide apertures that its correction is attended with appreciable results. A system which is **free from secondary spectrum** and which is **aplanatic for two colours** has been called by Abbe an **apochromatic** system.

In systems having a very large field of view it may become necessary to remove the chromatic difference of distortion, which may also be regarded as the variation of the chromatic difference of the lateral magnification. In general, it will be sufficient to correct the chromatic difference of magnification relating to the middle zone instead of that part of the field of view which is near the axis. This zone should be so chosen that the width of the spectrum into which the image of an extra-axial object-point is drawn out may assume similar, but opposite, maximum values as the point moves from its intermediate zone to the axis and from the zone to the margin.

To determine these defects accurately, it is necessary to establish differential formulæ which will enable us to compute for two adjacent rays differing but little in their wave-lengths, the differences in their intercepts and inclinations after traversing an optical system, when the path of one of these two rays has been determined with the aid of the formula given in Chapter II. Denoting these differences in front of and behind the surface by ds, du and ds', du' respectively, we obtain the following set of formulæ :

$$\frac{du}{\tan u} + \frac{ds}{s-r} = \frac{di}{\tan i};$$

$$\frac{di}{\tan i} - \frac{di'}{\tan i'} = \frac{dn'}{n} - \frac{dn}{n};$$

$$di' - di = du - du'$$

$$\frac{di'}{\tan i'} - \frac{du'}{\tan u'} = \frac{ds'}{s'-r}.$$

204. Bibliography. (Sections 1, A and B.)—Newton (*1*.), Euler (*1*.), Dollond and Short (*1*.), Klingenstierna (*1*.), Dollond (*2*.), Clairaut (*1*.), Lambert (*3*.), Ramsden (*1*.), Airy (*1*.), Barlow (*1*.), Rogers (*1*.), Stampfer (*2*.), Seidel (*2*.), Zincken-Sommer (*2*. 78), Kessler (*2*.), Mittenzwey (*1*.), E. v. Hoegh (*1*.), Schroeder (*5*, *6*.), Charlier (*6*.), Dennis Taylor (*5*, *6*.).

(Section 1, C.)—Blair (*1*.), W. Schmidt (*1*.), Abbe (*5*.), Harkness (*1*.), Kramer (*1*.), Czapski (*1*.), Dennis Taylor (*1*. *4*.), Kerber (*6*.), Schroeder (*4*.), Schupmann (*1*.), Charlier (*5*.).

(Section 1, D.)—Fraunhofer (*1*.), Petzval (*3*. *62*), C. A. Steinheil and Seidel (*1*.), W. Schmidt (*1*.), Schroeder (*3*.), Dennis Taylor (*4*.).

(Section 2.)—D'Alembert (*1*.), Gauss (*1*.), C. A. Steinheil (*2*.), Abbe (*5*.), Kerber (*1*, *4*.).

As regards the earlier history of the theory of the chromatic aberrations the reader is referred to Priestley (*2*. *243*, *339*, *520*), Rochon (*1*.), Brewster (*2*. *175*), Barlow (*2*. *408*), Littrow (*2*. *457*), whilst an account of the history of optical glass will be found in M. v. Rohr (*3*. *325*).

Chapter VII.

COMPUTATION OF OPTICAL SYSTEMS IN ACCORDANCE WITH THE THEORY OF ABERRATIONS.

(A. Koenig.)

205. The purpose of an optical system is to form from a given object an image of a certain magnitude and in a given plane. So long as it is only a matter of the Gaussian type of image-formation the problem presents no particular difficulties, and we need not therefore occupy ourselves with this part of the question. Our first task will accordingly be to remove the five image defects of Seidel or one or other of them. For this purpose the radii, thicknesses, distances, and the types of glass of which the system is to be composed (it being assumed throughout this chapter that it is bounded by air on either side) require to be so determined as to reduce to zero the various expressions established for the image defects. The necessary operations resolve themselves accordingly into the solution of algebraical equations, but it should be noted that it is only with very simple systems that we can deal in this way without having recourse to methods involving great analytical complications. Even under this restriction it will be our aim, not so much to arrive at completeness, as to indicate the various methods by which systems may be computed. We shall illustrate the application of these methods by a few numerical examples, and, to simplify matters, the distances of the object will in all cases be supposed to be infinitely great. In many cases we shall, from the outset, introduce additional equations expressing conditions respecting the chromatic correction. In view of the fortunate circumstance that we possess glasses having different dispersion for the same refractive index, the correction of the spherical defects can frequently be undertaken without regard to the chromatic aberrations. This correction may thus be accomplished later by a choice of glasses of suitable dispersions, and in this connection it should be remembered that the existing varieties of glass can be readily supplemented by substituting for a single lens an equivalent combination of several lenses of similar refraction but different dispersions, as explained in § 195. In the numerical examples the chromatic correction always applies to the colours for

which the dispersion is given, whilst the total focal length of the system of lenses is assumed to be unity. Seidel's image defects will be identified by Roman suffixes, as follows :

D_{I} represents the **spherical aberration,**

D_{II} represents the **coma,**

D_{III} represents the **curvature of the image for sagittal rays,**

D_{IV} represents the **curvature of the image for tangential rays,** and

D_{V} represents the **distortion.**

When the defects D_{III} and D_{IV} are simultaneously removed the correction is called **anastigmatic flatness of field.** When this is not the case it is usual in practice to compensate either half the difference of these two defects, viz. $\frac{(D_{IV} - D_{III})}{2}$, *i.e.*, the astigmatism (for the image-point at unit distance from the axis), or half their sum, *i.e.*, the so-called **curvature of the image.** When introducing these derived defects we shall represent them by the symbols D_{IIIa} and D_{IVa}, so that

$$D_{IIIa} \text{ represents } \frac{D_{IV} - D_{III}}{2}$$

$$D_{IVa} \text{ represents } \frac{D_{IV} + D_{III}}{2} \,.$$

Curvature proper of the image cannot enter into consideration until the astigmatism has been removed. It is computed, as indicated in § 142 (i), with the aid of the formula :

$$P = \frac{1}{R} = \sum \frac{1}{r_v} \Delta \frac{1}{n_v} \,.$$

1. THE APPLICATION OF THE PETZVAL CONDITION.

206. In the event of one of the image defects D_{III}, D_{IV}, D_{IIIa}, D_{IVa} being corrected, the others may be removed at the same time by satisfying the condition

$$\sum \frac{1}{r_v} \Delta \frac{1}{n_v} = 0 \,. \quad \dots \quad \dots \quad \text{(i)}$$

This is known as the Petzval condition, he being the first to enunciate it in the form

$$\sum \frac{\phi_v}{n_v} = 0 \,. \quad \dots \quad \dots \quad \dots \quad \text{(ii)}$$

Since the definition of the power of thin lenses was extended by him to include lenses of finite thickness both expressions are identical in form. In view of the definition of the term "power," as given in this work, we shall, however, apply the latter form of the condition only to the case of thin systems. Petzval's expression is independent of the coflexure of the lenses, of which we shall make considerable use later as a means of effecting the correction of image defects. It is, therefore, advisable to devote a separate article to this condition.

A. Thin System of Lenses.

207. We shall first consider a thin system of lenses and we shall satisfy simultaneously the Petzval condition and the condition of achromatism (§ 195, vii), viz.:

$$\sum \frac{\phi_\nu}{\nu_\nu} = 0 .$$

It will accordingly be seen that the result depends on the values of ν and n of the glass employed. We notice further that the equations have the same form as those obtained in our investigation of the secondary spectrum of an achromatic system of thin lenses, the part played by the θ-value being now implied in the value of $\frac{\nu}{n}$. As in the investigation of the secondary spectrum we give a diagram (Fig. 87) in which each glass is represented by a point, whose abscissa is the value of $\bar{\nu}$ and whose ordinate is the value of $\frac{\bar{\nu}}{n}$, the refractive index corresponding with the D line and the dispersion with the interval from D to $H\gamma$. It will be seen that the ordinary silicate glasses range themselves more or less along a straight line drawn through points representing the glasses O. 144 and O. 103, whereas the baryta glasses, which have smaller values of $\frac{\bar{\nu}}{n}$ than other glasses having similar values of $\bar{\nu}$, deviate to one side of the straight line, and the so-called crown glasses of high dispersion deviate towards the other side. In a system of two lenses the value of $\frac{\bar{\nu}}{n}$ must be the same in both glasses in order that Petzval's condition and the chromatic condition may be satisfied concurrently. In a system of several lenses it is only necessary that the glasses should not conform to a straight line in the diagram. By the combination of several glasses we attain nothing more than could be achieved by means of two glasses, if such glasses are available corresponding with all points within the polygon having for its

angular points the extreme positions of points on the curve for actually existing glasses. In so far as the value of $\frac{\bar{\nu}}{n}$ for the ordinary type of silicate glass conforms to the linear relation

$$\frac{\bar{\nu}}{n} = -2{\cdot}71 + 0{\cdot}716\,\bar{\nu}\,,$$

achromatic systems of unit focal length, in which none but these glasses are used, will always satisfy the condition

$$\sum \frac{\phi_v}{n_v} = 0{\cdot}716\,.$$

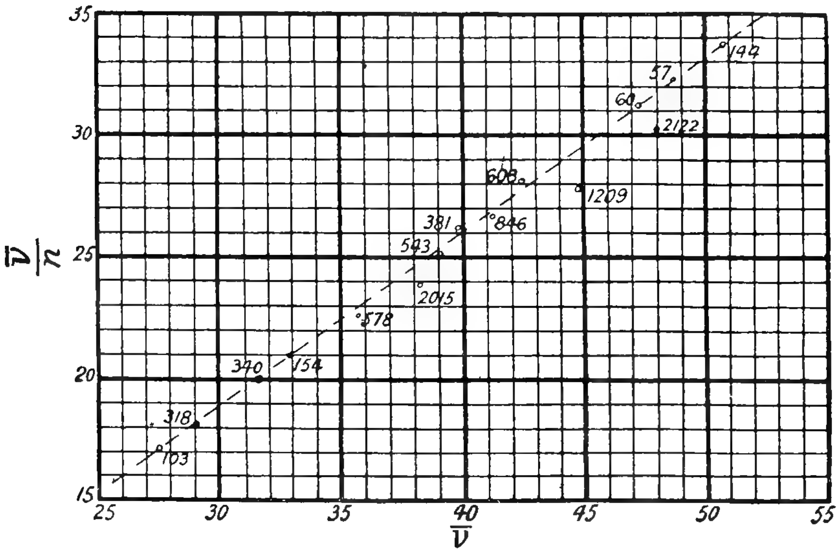

FIG. 87.

$$\bar{\nu} = \frac{n_D - 1}{n_{G'} - n_D}$$

Diagram of the values of $\bar{\nu}$ and $\frac{\bar{\nu}}{n}$ of various types of optical glass.

In converging systems the values become reduced by the introduction of converging lenses of baryta glasses and diverging lenses of crown glasses of high dispersion, the amount by which the value diminishes becoming greater according as the power of the lenses so introduced increases. In achromatic systems of two types of glass if the ν ratio remains unchanged, any change in the refractive indices affects the result to a greater extent the more this ν ratio approaches unity. This is well illustrated in the tables compiled by Harting (**5**.).

B. System of Two Thin Lenses Separated by a Finite Distance.

208. In a system of lenses of finite thicknesses and separated by finite distances the Petzval condition cannot be satisfied by a mere choice of glasses. We shall explain the circumstances by investigating the case of a system consisting of two thin elements placed some distance apart. The curvature at the vertex of the image freed from astigmatism, as formed by this system, is

$$-P = \sum_1 \frac{\phi}{n} + \sum_2 \frac{\phi}{n}, \quad \ldots \quad \ldots \quad \text{(i)}$$

the leading equation is

$$\Phi = \sum_1 \phi + \sum_2 \phi - A \sum_1 \phi \sum_2 \phi, \quad \ldots \quad \text{(ii)}$$

where A is the distance between the two component systems, and the index under the sign of summation relates to the component system for which the summation is to be performed.

Putting

$$\left.\begin{aligned} &\sum_2 \phi = N \sum_1 \phi \\ &\sum_k \frac{\phi}{n} = M_k \sum_k \phi; \qquad k = 1, 2, \\ &A \sum_1 \phi = \bar{A} \\ &P / \Phi = \bar{P}; \\ \text{we obtain:} \\ &\bar{P} = -\frac{M_1 + N M_2}{1 + N(1 - \bar{A})}. \end{aligned}\right\} \quad \ldots \quad \text{(iii)}$$

The choice of the glass affects the curvature $\bar{P}$ in the same manner as has been explained in relation to the system of thin separated lenses. We shall therefore confine ourselves to the case in which $M_1 = M_2 = M$, and we shall proceed to ascertain in what manner the quantity $\bar{P}$ depends upon N and $\bar{A}$, when, with changes in the choice of the glasses the variation of $\bar{P}$ is directly proportional to M.

The investigation will then depend upon the following equations:

$$\left.\begin{aligned} &\bar{P} = -\frac{M(1 + N)}{1 + N(1 - \bar{A})} \\ &\frac{\partial \bar{P}}{\partial N} = -M \frac{\bar{A}}{\{1 + N(1 - \bar{A})\}^2} \\ &\frac{\partial \bar{P}}{\partial \bar{A}} = -M \frac{N(N + 1)}{\{1 + N(1 - \bar{A})\}^2}. \end{aligned}\right\} \quad \ldots \quad \ldots \quad \text{(iv)}$$

It will be sufficient to vary the quantity N from -1 to $+1$, since the value of $\overline{P}$ is the same whether the light is incident from the front or from the back. Within this interval $\frac{N+1}{N}$ undergoes a change of sign conjointly with N. Further, let M be positive; otherwise the substitution of the negative sign would only cause the signs of $\overline{P}$, $\frac{\partial \overline{P}}{\partial N}$ and $\frac{\partial \overline{P}}{\partial \overline{A}}$ to be reversed. We shall divide the range of the combinations of N and $\overline{A}$ into four parts. The subjoined table shows the progressive change of the function $\overline{P}$.

	$\overline{A}$ negative.		$\overline{A}$ positive.	
	$N < \frac{1}{\overline{A}-1}$	$N > \frac{1}{\overline{A}-1}$	$N < \frac{1}{\overline{A}-1}$	$N > \frac{1}{\overline{A}-1}$
$\overline{P}$	neg.	pos.	neg.	pos.
P	pos.	pos.	neg.	neg.
$\frac{\partial \overline{P}}{\partial N}$	pos.	pos.	neg.	neg.
$\frac{\partial \overline{P}}{\partial \overline{A}}$	$\left.\begin{matrix}\text{pos.}\\\text{neg.}\end{matrix}\right\}$ when $N\left\{\begin{matrix}\text{neg.}\\\text{pos.}\end{matrix}\right.$	pos.	neg.	$\left.\begin{matrix}\text{pos.}\\\text{neg.}\end{matrix}\right\}$ when $N\left\{\begin{matrix}\text{neg.}\\\text{pos.}\end{matrix}\right.$

In every section $\overline{P}$ assumes either all positive or all negative values between zero and infinity. When $\overline{A} = \infty$, then $\Phi = \infty$, hence also $\overline{P} = 0$, since P is finite. The relation

$$N = \frac{1}{\overline{A} - 1} \text{ or } \frac{N+1}{N} = \overline{A} \ldots \qquad \ldots \qquad \text{(v)}$$

is a necessary and sufficient condition that the system may be telescopic, and in this case $\Phi = 0$, so that $\overline{P} = \infty$, since P is finite. If the choice of glasses is optional, that is $M \gtreqless 0$, and if the separation of the front and back members is finite, then the Petzval condition can only be fulfilled for a system having a **positive** aggregate focal length when the focal lengths of the component elements are made equal and opposite in sign, so that $N = -1$, and indeed equal to the square root of the resultant focal length multiplied by the separation of the components. The separation should not, therefore, be made too small, if excessively steep curvatures are to be avoided.

With regard to systems in which the lens thicknesses are finite we shall content ourselves with the following observations. A system of this kind is equivalent to a system of thin lenses separated by finite distances as regards the values of Φ and P, the power and the curvature of the image. The latter system can be derived from the former if from each of the thick lenses we imagine a plane-parallel plate of glass having the same refractive index and the same thickness as the lens to be removed and replaced by a plane-parallel plate of air whose thickness is smaller in the ratio of the refractive index to unity.

In conclusion, we shall investigate the case of a system consisting of two separated thin lenses in which the principal focus is achromatised at the same time for an infinitely small colour interval. We shall suppose the object distance to be infinitely great, and the resultant focal length to be positive. The following are the equations which require to be satisfied :

$$\phi_1 + \phi_2 (1 - A \phi_1) = 1$$

$$\frac{\phi_1}{\nu_1 (1 - A \phi_1)^2} + \frac{\phi_2}{\nu_2} = 0 \qquad \ldots \qquad \ldots \qquad \text{(vi)}$$

$$\frac{\phi_1}{n_1} + \frac{\phi_2}{n_2} = 0 .$$

Eliminating the term ϕ_2 from the second equation, with the aid of the third equation, we obtain :

$$1 - A \phi_1 = \pm \sqrt{\frac{n_1 \nu_2}{n_2 \nu_1}} ; \quad \text{or } A \phi_1 = 1 \mp \sqrt{\frac{n_1 \nu_2}{n_2 \nu_1}} ;$$

next, from the first equation, by again eliminating ϕ_2

$$\phi_1 = \frac{1}{1 \mp \sqrt{\frac{n_2 \nu_2}{n_1 \nu_1}}} \qquad \ldots \qquad \ldots \qquad \text{(vii)}$$

and finally from the third equation

$$\phi_2 = - \frac{1}{\frac{n_1}{n_2} \mp \sqrt{\frac{n_1 \nu_2}{n_2 \nu_1}}} . \qquad \ldots \qquad \ldots \qquad \text{(viii)}$$

If the front lens is a diverging one we need only consider the upper sign of the root ; if it is a converging lens we must expressly confine ourselves to one case, *i.e.*, we must suppose that the second lens is disposed between the first lens and its principal focus. If $\frac{n_2 \nu_2}{n_1 \nu_1} = 1$ the powers of the component lenses become infinitely

great. According as this value is less or greater than unity the focal length of the front lens will be positive or negative. The focal lengths of the front lens and of the back lens have always opposite signs. According as the converging lens or the diverging lens is in front, the magnitudes of ϕ_1 and ϕ_2 diminish as $\frac{n_2 \nu_2}{n_1 \nu_1}$ becomes smaller or greater. The converging lens is therefore appropriately made of glasses with high values of n and ν, whilst the diverging lens is made of glasses with low values of n and ν. With any given choice of glasses the disposition of the diverging lens in front results in smaller values of ϕ_1 and ϕ_2. This advantage becomes more pronounced according as the glasses differ in their respective values of $\frac{\nu}{n}$, and the more the one ν value exceeds the other the less is the effect of a difference between the refractive indices. If this difference becomes negligible, or if actually $n_1 = n_2$, then

$$A\phi_1^2 = 1,$$

that is to say, the ratio of A to the front focal length is the same as the ratio of the latter to the resultant focal length. This theorem has already been enunciated in other words in this section.

2. CORRECTION OF SEIDEL'S IMAGE DEFECTS.

A. Thin System of Lenses.

209. In the computation of thin systems of lenses it is usual to employ the expressions for Seidel's image defects, which are obtained by extending the summation over the successive lenses, proceeding from the expressions given in Chapter V (§ 124) for a single lens. In performing the summation it is only necessary to note that

$$\sigma_v = \sigma_1 + \sum_{\lambda=1}^{v-} \phi_\lambda \,. \quad .. \qquad \dots \qquad \dots \quad \text{(i)}$$

It will suffice here to write down the expression for the first image defect, viz., the **spherical aberration**:

$$\left.\begin{aligned} S_I = \sum_{v=1}^{k} \Bigg[& \left(\frac{n_v}{n_v - 1}\right)^2 \phi_v^3 + \frac{3\,n_v + 1}{n_v - 1}\,\phi_v^2 \left(\sigma_1 + \sum_{\lambda=1}^{v-1} \phi_\lambda\right) \\ & + \frac{3\,n_v + 2}{n_v}\,\phi_v \left(\sigma_1 + \sum_{\lambda=1}^{v-1} \phi_\lambda\right)^2 - \frac{2\,n_v + 1}{n_v - 1}\,\phi_v^2\,\rho_{2v-1} \\ & - \frac{4\,n_v + 4}{n_v}\,\phi_v \left(\sigma_1 + \sum_{\lambda=1}^{v-1} \phi_\lambda\right)\rho_{2v-1} + \frac{n_v + 2}{n_v}\,\phi_v\,\rho^2_{2v-1} \Bigg]. \end{aligned}\right\} \quad \text{(ii)}$$

Let S_v denote in abbreviated form the expressions of the summation and let v represent the numerical order of the image defects (D_I, D_{II}, D_{III}, D_{IV}, D_V as the case may be); also, let n and ϕ be numbered in accordance with the lenses, and ρ in accordance with the surfaces.

The advantage of these expressions is that the front curvatures of the component lenses, which we regard as a measure of their coflexure, may be treated as variables. If the requirements regarding the powers of the lenses have already been satisfied with due regard to the chromatic defects or in fulfilment of the Petzval condition, we may proceed to ascertain the amounts of the front curvatures which, when introduced into the expression, reduce to zero the values S_v for the image defect which is to be corrected. None of these S_v expressions are of a higher order than the second with respect to ρ.

If it is required to eliminate two image defects at the same time (excluding the case of a system of thin lenses in which both defects belong to the groups D_{III}, D_{IV}, D_{IIIa}, D_{IVa}), it is only necessary after eliminating one of the defects to equate to zero the difference of the two expressions for the two image defects. These expressions are linear functions of the front curvatures. Their general form for the individual lenses is

$$(G + H\rho_1)(\sigma_1 - \xi_1). \qquad \ldots \qquad \ldots \qquad \text{(iii)}$$

We give here the expressions for G and H as applied to a few important combinations of image defects :

$$D_I \text{ and } D_{II}\,; \quad \left.\begin{aligned} G &= \frac{n}{n-1}\phi^2 + \frac{2n+1}{n}\phi\sigma \\ H &= -\frac{n+1}{n}\phi\,, \end{aligned}\right\} \qquad \text{(iv)}$$

$$D_I \text{ and } D_{IIIa}\,; \quad \left.\begin{aligned} G &= \frac{2n}{n-1}\phi^2 + \frac{3n+2}{n}\phi\sigma + \phi\xi \\ H &= -\frac{2(n+1)}{n}\phi\,, \end{aligned}\right\} \qquad \text{(v)}$$

$$D_{IIIa} \text{ and } D_V\,; \quad \left.\begin{aligned} G &= \frac{n}{n-1}\cdot\phi^2 + \frac{2n+1}{n}\phi\xi \\ H &= \frac{n+1}{n}\phi\,. \end{aligned}\right\} \qquad \text{(vi)}$$

In a **system of thin lenses**, when σ and ξ are numbered in the order of the lenses,

$$\sigma_\nu - \xi_\nu = \sigma_1 - \xi_1\,,$$

so that all that is required is to equate to zero the sum

$$\sum_{v=1}^{k} (G_v + H_v \rho_{2v-1}) \, . \qquad \ldots \qquad \ldots \qquad \ldots \qquad \text{(vii)}$$

In the case of the first combination this sum takes the form

$$\theta_{\mathrm{I,II}} = \sum_{v=1}^{k} \left[\frac{n_v}{n_v - 1} \phi^2_v + \frac{2n_v + 1}{n_v} \phi_v \left(\sigma_1 + \sum_{\lambda=1}^{v-1} \phi_\lambda \right) - \frac{n_v + 1}{n_v} \phi_v \rho_{2v-1} \right] . \quad \text{(viii)}$$

210. Correction of Two Image Defects by an Appropriate Choice of Radii.—Following these preliminary remarks, we shall now proceed to compute an optical system. We shall begin by so determining the radii that the image defects D_{I} and D_{II} will be eliminated. In the first place, we shall suppose the system to consist of two lenses whose powers are such as to render the system achromatic. It will then be necessary to equate to zero the expression for S_I and $\theta_{I,II}$, as given above (taking $k = 2$). In this way we shall obtain two equations for ρ_1 and ρ_3 respectively. The elimination from the first equation of one of these quantities with the aid of the second equation yields a quadratic equation for the other quantity. Since the order of succession of the component lenses is immaterial as regards the chromatic correction, it follows that any chosen pair of glasses may yield either two or four solutions, or the solution may fail. In the case of the pair of glasses O.60 ($n_D = 1.5179$, $n_F - n_C = 0.00860$) and O.103 ($n_D = 1.6202$, $n_F - n_C = 0.01709$) we obtain the results shown in the subjoined table, and in this connection it should be noted that telescope lenses are generally computed in accordance with Type I or III.

Type I. Crown in front :

$\rho_1 = + 1.6445$; $\rho_2 = - 3.2143$; $\rho_3 = - 3.1457$; $\rho_4 = - 0.7006$

Type II. Crown in front :

$\rho_1 = + 7.0089$; $\rho_2 = + 2.1501$; $\rho_3 = + 5.9853$; $\rho_4 = + 8.4304$

Type III. Flint in front :

$\rho_1 = + 2.3198$; $\rho_2 = + 4.7649$; $\rho_3 = + 4.8173$; $\rho_4 = - 0.0415$

Type IV. Flint in front :

$\rho_1 = - 6.8016$; $\rho_2 = - 4.3565$; $\rho_3 = - 0.5413$; $\rho_4 = - 5.4001$.

Within the range of refractive indices which enter into practical consideration (say 1.50 — 1.62) and of the available ν-ratios (say 0.6 — 0.8), the Types I and III have the advantage of introducing relatively shallow curvatures. In particular, the difference of the inner curvatures always remains small. In the case of Type I the

relation to the refractive indices can be expressed by the following empirical formula :

$$\rho_3 - \rho_2 = a + \frac{b}{n_1 + c} + \frac{d}{n_2 + e} \dots \qquad \dots \qquad \text{(i)}$$

$\nu_2 : \nu_1$	a	b	c	d	e
0.6	+ 0.131	0.376	− 1.021	+ 0.292	− 1.098
0.7	+ 0.039	− 0.525	− 1.054	+ 0.511	− 1.064
0.8	− 0.071	− 0.814	− 1.072	+ 0.897	− 1.048

Similarly, ρ_1 can be expressed within the same limits by the following empirical formula :

$$\rho_1 = a + \frac{b}{n_1 + c} + \frac{d\,(n_1 - 1.500) + e}{n_2 + f} \qquad \dots \qquad \text{(ii)}$$

$\nu_2 : \nu_1$	a	b	c	d	e	f
0.6	+ 0.365	+ 0.755	− 0.899	+ 0.200	+ 0.019	− 1.230
0.7	− 0.134	+ 1.017	− 0.830	+ 0.507	+ 0.080	− 1.260
0.8	− 1.227	+ 1.142	− 0.877	+ 1.449	+ 0.205	− 1.319

The connection between the radii and the ν-ratio for the case in which the glass having the higher ν-value has a refractive index 1.52, whilst that of the other is 1.62, has been given for all four types in a table compiled by Charlier (***3***).

In the event of the system being composed of three or more lenses, it is possible to satisfy additional conditions, the form of which determines whether or not the degree of the resulting equation will be raised. This does not arise when the focal lengths of the component lenses are decided upon at the outset and the new conditions made to establish linear relations only between the reciprocals of the radii. A case of this kind investigated by Gleichen (***4***. *318*) has been elaborated by R. Steinheil. The focal lengths of the system containing three lenses of different glasses have been so selected as to obtain achromatisation of the second order, and in addition, the second and third radii are identical so as to admit of the surfaces being cemented together. The radii and focal length are accordingly connected by the relation: $\rho_1 - \rho_3 = \dfrac{\phi_1}{n_1 - 1}$. In view of the fact that the glasses can be

arranged in six different orders of succession there are twelve possible solutions; and with the glasses selected above, all these solutions give real values.

It will be sufficient for our purpose to specify the types of objectives, as given by Gleichen.

Lens	Type of Glass	n_D	$n_D - n_C$	$n_F - n_D$
a	O.543	1·5637	0·00325	0·00790
b	O.164	1·5503	0·00328	0·00786
c	O.374	1·5109	0·00251	0·00593

R. STEINHEIL'S TABLE OF CURVATURES FOR A TELESCOPE TRIPLE OBJECTIVE.

Type	ρ_1	$\rho_2 = \rho_3$	ρ_4	ρ_5	ρ_6	Sequence of Lenses
I	− 15·72	− 22·32	− 8·28	− 4·39	− 14·18	a b c
II	+ 2·60	− 4·00	+ 10·03	+ 10·15	+ 0·36	a b c
III	− 193	− 200	− 210	− 185	− 171	a c b
IV	− 0·14	− 6·74	− 16·39	− 16·29	− 2·25	a c b
V	+ 1·32	− 8·47	− 15·06	− 15·11	− 1·08	c a b
VI	− 31·06	− 40·82	− 47·37	− 51·81	− 37·88	c a b
VII	− 5·62	− 15·41	− 1·38	+ 0·40	− 6·20	c b a
VIII	− 1·42	− 11·21	+ 2·82	+ 3·42	− 3·17	c b a
IX	− 15·67	− 1 64	− 8·24	− 4·55	− 14·35	b a c
X	+ 2·74	+ 16·78	+ 10·17	+ 10·07	+ 0·29	b a c
XI	− 5·97	+ 8·06	− 1·73	+ 1·31	− 5·29	b c a
XII	− 3·05	+ 10·98	+ 1·19	+ 3·41	− 3·19	b c a

We now proceed to investigate the correction of the image defects D_I and D_{II} in two cases of thin systems, in which this correction can be accomplished, not only by a suitable coflexure of the component lenses, but also by a choice of their powers. For the correction of the image defects D_I and D_{II} we shall use the expressions

originally established, viz.: $\sum_{v=1}^{k} Q^2_{sv} \Delta\left(\frac{1}{ns}\right)_v$ and $\sum_{v=1}^{k} Q_{sv} \Delta\left(\frac{1}{ns}\right)_v$ and in doing so we shall treat the cemented surfaces as single surfaces requiring one term of summation only. Nevertheless we shall count the values of σ and ρ in the usual way where the cemented surfaces are regarded as two distinct surfaces. The values of n are reckoned in accordance with the sequence of the lenses.

The first case relates to a system of two cemented lenses in which the chromatic correction has not been ensured at the outset by the choice of the powers of the component lenses. On the contrary, having determined the curvatures from the refractive indices chosen, we shall first find the ν-ratio necessary for chromatic correction, and then ascertain whether there are available types of glass having these refractive indices and the ν-ratio required by the calculation. If this is not the case, the expedient may be resorted to of substituting a combination of two lenses of equal refraction but differing in dispersion. This case has been worked out by Harting (*1*) for infinitely distant objects only, and by von Hoegh (*3*.) for objects at any distance. The latter was particularly fortunate in his choice of the variables (*i.e.*, the power of the front lens and the curvature of the cemented surfaces), and he was able to formulate an explicit self-contained equation of the fifth degree with respect to the first variable. The equation in this form provides a very convenient means of transforming the coefficients to suit other differences of the refractive indices of the two lenses, as well as other distances of the object (in particular the transition from the distance ∞ to $-F$, which implies a reversal of the sequence of the lenses if the infinite distance is to be retained). We shall here content ourselves with showing how the equation of the fifth degree is obtained. At the outset we shall regard $\sigma'_1 = \sigma_2$ and $\sigma'_3 = \sigma_4$ as variables, whereas σ_1 and $\sigma'_4 = \sigma_1 + \Phi$ are constants. We shall then have

$$\left.\begin{aligned}
Q_{s1} &= \frac{n_1}{n_1 - 1}(\sigma_2 - \sigma_1); \quad \Delta\left(\frac{1}{ns}\right)_1 = \frac{\sigma_2}{n_1} - \sigma_1 \\
Q_{s2} &= \frac{n_1 n_2}{n_2 - n_1}(\sigma_4 - \sigma_2); \quad \Delta\left(\frac{1}{ns}\right)_2 = \frac{\sigma_4}{n_2} - \frac{\sigma_2}{n_1} \\
Q_{s3} &= \frac{n_2}{1 - n_2}(\sigma'_4 - \sigma_4); \quad \Delta\left(\frac{1}{ns}\right)_3 = \sigma'_4 - \frac{\sigma_4}{n_2}
\end{aligned}\right\} \ldots \quad \text{(iii)}$$

wherein we may substitute

$$\sigma_2 = (n_1 - 1)(n_2 x + n_1 y)$$
$$\sigma_4 = n_1 (n_2 - 1)(x + y).$$

If now we form expressions for the image defects D_I and D_{II} we should expect the former to assume the general form of the entire

rational function of the third degree in terms of x and y, whilst the second expression should be of the second order, and the resultant equation should be of the sixth degree with respect to one of these variables. Since, however, the factor of x is identical in Q_{s1}, Q_{s2} and Q_{s3}, viz., $n_1 n_2$, and, since further $\sum_{v=1}^{3} \Delta \left(\frac{1}{ns}\right)_v$ is identically equal to Φ and hence constant, the term containing x^3 in the expression for the image defect D_{I}, and the term containing x^2 in the image defect D_{II}, both vanish, causing the final equation to be of the fifth degree. The curvatures, on the other hand, are linear functions of σ, and consequently also of x and y. Harting found from his numerical calculations that of the five resulting roots three were invariably real. The following example has been taken from Harting's paper ($n_1 = 1{\cdot}56$; $n_2 = 1{\cdot}51$):

I;	$\rho_1 =$	$+ 2{\cdot}164$	$\rho_2 = \rho_3 =$	$+ 4{\cdot}863$	$\rho_4 =$	$- 0{\cdot}062$	$\nu_1/\nu_2 =$	$0{\cdot}602$
II;	,,	$+ 48{\cdot}71$	,,	$+ 7{\cdot}76$	,,	$+ 50{\cdot}76$	,,	$1{\cdot}046$
III;	,,	$- 2083$	,,	$- 2276$	,,	$- 2064$	,,	$1{\cdot}009$

Harting (*1.*) appends a table in which are given the radii of Type I for the various combinations of n_1 and n_2.

The second case relates to achromatic systems consisting of three lenses cemented together subject to the restriction that the first and third lenses consist of glass of the same kind ($n_1 = n_3$; $\nu_1 = \nu_3$). Harting (*6.*), who has investigated this case, chose as his variables the curvatures of the front surfaces and of the first cemented surface and formulated an explicit equation for $\rho_2 = \rho_3$. From the condition of achromatism

$$\phi_2 = \frac{\nu_2}{\nu_2 - \nu_1}$$

and hence

$$\sigma_6 - \sigma_2 = \frac{n_2 - n_1}{n_1(n_2 - 1)} \cdot \frac{\nu_2}{\nu_2 - \nu_1} = \tau . \quad \ldots \quad \text{(iv)}$$

We can now write the expressions for Q_s and $\Delta \frac{1}{ns}$ in the forms:

$$\left.\begin{aligned}
Q_{s1} &= \frac{n_1}{n_1 - 1}(\sigma_2 - \sigma_1); & \Delta\left(\frac{1}{ns}\right)_1 &= \frac{\sigma_2}{n_1} - \sigma_1 \\
Q &= \frac{n_1 n_2}{n_2 - n_1}(\sigma_4 - \sigma_2); & \Delta\left(\frac{1}{ns}\right)_2 &= \frac{\sigma_4}{n_2} - \frac{\sigma_2}{n_1} \\
Q_{s3} &= \frac{n_1 n_2}{n_2 - n_1}(\sigma_4 - \sigma_2 - \tau); & \Delta\left(\frac{1}{ns}\right)_3 &= -\frac{\sigma_4}{n_2} + \frac{\sigma_2}{n_1} + \frac{\tau}{n_1} \\
Q_{s4} &= \frac{n_1}{n_1 - 1}(\sigma_2 + \tau - \sigma'_6); & \Delta\left(\frac{1}{ns}\right)_4 &= -\frac{\sigma_2}{n_1} + \sigma'_6 - \frac{\tau}{n_1}.
\end{aligned}\right\} \quad \text{(v)}$$

Now, since the variable parts of Q_s are equal for both the external and cemented surfaces, whilst those of $\Delta\left(\frac{1}{ns}\right)$ are equal and opposite in sign, the expression for the image defect D_I becomes quadratic, whilst for the image defect D_{II} it is linear in terms of σ_2 and σ_4. The final equation thus becomes quadratic in terms of σ_2 and σ_4. In the example instanced by Harting the middle lens consisted of a light flint ($n_D = 1{\cdot}56837$, $n_F - n_C = 0{\cdot}01350$), while the outer lenses were of crown ($n_D = 1{\cdot}50900$; $n_F - n_C = 0{\cdot}00797$). For the type having shallower curvatures he obtained the following values:

$$\rho_1 = +\,1{\cdot}6858\,;\ \rho_2 = \rho_3 = -\,3{\cdot}0672\,;\ \rho_4 = \rho_5 = +\,0{\cdot}3365\,;\ \rho_6 = -\,0{\cdot}6759.$$

When it is desired to correct a system of thin lenses as regards two of Seidel's image defects other than D_I and D_{II}, the procedure is identical with that adopted in the case of D_I and D_{II}, if the distance between the stops remains constant. We shall therefore dispense with a discussion of the analogous cases. It may nevertheless be useful to give a numerical example. We will suppose that it is required to correct spherically and astigmatically a system of two lenses by the method of coflexure, say, for the purpose of using it as one half of a photographic objective of a symmetrical type. The first lens consists of a glass of type O. 569 ($n_D = 1{\cdot}5738$, $n_{G'} - n_D = 0{\cdot}01818$), the second of O. 60 ($n_D = 1{\cdot}5179$, $n_{G'} - n_D = 0{\cdot}01092$). Let the distance of the stop be a constant amount $-\,0{\cdot}03333$. The solution will then be as follows:

$$\text{Type I}\,;\ \rho_1 = +\,0{\cdot}100\,;\ \rho_2 = +\,3{\cdot}567\,;\ \rho_3 = +\,1{\cdot}323\,;\ \rho_4 = -\,4.448.$$

$$\text{Type II}\,;\ \rho_1 = -\,24{\cdot}14\,;\ \rho_2 = -\,20{\cdot}67\,;\ \rho_3 = -\,14{\cdot}58\,;\ \rho_4 = -\,20.35.$$

Type I has been used as one half of a symmetrical objective by Zincken-Sommer [M. v. Rohr (*3.319*)].

211. Correction by Means of the Position of the Stop.—Two spherical image defects, with the exception of the combination of D_I and D_{II}, can be corrected by the method of coflexure if the position of the stop can be suitably arranged. We shall assume that the focal lengths of the component lenses are fixed and that the front curvatures, with the exception of one of them, say the first, are either constant or a linear function of the latter. If one of the image defects be D_I, a quadratic equation for ρ_1 is obtained if we equate to zero the expression for D_I, whilst the expression for the difference of the two image defects enables us to calculate ξ_1 as a linear function of ρ_1. If it is not the image defect

D_I with which we are concerned, it will be necessary, with the aid of the formula giving the finite difference, to eliminate ξ_1 or ρ_1 from the expression for one of the two image defects in order that we may arrive at the quadratic equation. As an example of the former case we shall choose an achromatic system of two cemented lenses which has to be corrected spherically and chromatically. If the flint lens is in front, let the types of glass be O. 60 ($n_D = 1{\cdot}5179$; $n_{G'} - n_D = 0{\cdot}01092$) and O. 2015 ($n_D = 1{\cdot}6041$; $n_{G'} - n_D = 0{\cdot}01579$).

When the flint is in front:

Type I ; $\rho_1 = -0{\cdot}490$; $\rho_2 = \rho_3 = +6{\cdot}417$; $\rho_4 = -3{\cdot}571$; $\xi_1 = -17.77$.

Type II ; $\rho_1 = +8{\cdot}512$; $\rho_2 = \rho_3 = +15{\cdot}419$; $\rho_4 = +5{\cdot}431$; $\xi_1 = +14{\cdot}75$.

When the crown lens is in front, let the glasses be

O. 276 ($n_D = 1{\cdot}5800$; $n_{G'} - n_D = 0{\cdot}01804$)
O. 103 ($n_D = 1{\cdot}6202$; $n_{G'} - n_D = 0{\cdot}02261$).

Then :

Type I ; $\rho_1 = -3{\cdot}94$; $\rho_2 = \rho_3 = -15{\cdot}68$; $\rho_4 = -6{\cdot}31$; $\xi_1 = -14{\cdot}9$.

Type II ; $\rho_1 = +5{\cdot}02$; $\rho_2 = \rho_3 = -6{\cdot}72$; $\rho_4 = +2{\cdot}65$; $\xi_1 = +16{\cdot}0$.

Type I illustrates that of the usual aplanatic halves in particular, the type representative of the second case resembles the half of Steinheil's Aplanatic Lens (M. v. Rohr ***3***.*298*)). If, instead of correcting the system for astigmatism, the endeavour had been to attain correction of the so-called flatness of field we should have obtained $-12{\cdot}66$ and $-10{\cdot}94$ as the reciprocals of the distances of the stops in the systems of Type I. To show the influence of the variation of the ν-ratio and the n-difference upon the position of the stop we shall give two further examples of Type I having the crown lens in front. In the first example let the glass of the second lens be replaced by O. 748 ($n_D = 1{\cdot}6235$; $n_{G'} - n_D = 0{\cdot}02107$).

We now obtain

$\rho_1 = -15{\cdot}265$; $\rho_2 = \rho_3 = -36{\cdot}935$; $\rho_4 = -18{\cdot}381$; $\xi_1 = 42{\cdot}26$

In the second example, the glass of the second lens is replaced by

O. 3269 ($n_D = 1{\cdot}6570$; $n_{G'} - n_D = 0{\cdot}02401$).

We obtain in this case

$\rho_1 = -5{\cdot}883$; $\rho_2 = \rho_3 = -17{\cdot}463$; $\rho_4 = -8{\cdot}762$; $\xi_1 = -19{\cdot}58$

It will be interesting to know whether in the case of a single lens it is possible, by a choice of the distance between the stops and by suitable coflexure of the lenses, to remove two of the image defects D_{II}, D_{IIIa} and D_{V}. In order that the roots of the quadratic equations may be real the correction of the image defects D_{II} and D_{IIIa} in the case of an infinitely distant object, implies that $(2n-1)^2 > 0$; for the correction of D_{II} and D_{V}, it is necessary that $3n < 1$, and for D_{IIIa} and D_{V} that $4n^2(n+1) < 1$. From this it will be seen that with the existing types of glass it is practically possible only to realise the first case.

212. The Independence of a Third Image Defect with respect to the Coflexure of the Lenses.—Whenever a system of thin lenses has been corrected for two spherical image defects, the other defects no longer depend upon the coflexure of the component lenses and their order of succession, as was pointed out by Petzval (*1. 18*) and subsequently proved by C. Moser (*4.*), but solely upon their refractive indices. This includes the case in which the two image defects, instead of being reduced to zero, are required to assume given finite values, as for example in combinations. This is most readily realised by reverting to the formulæ which express the image aberrations given in § 179.

$$
\begin{aligned}
D_{I} &= A \\
D_{II} &= A + D_{xs}B \\
D_{III} &= A + 2\,D_{xs}B + D^2_{xs}C \\
D_{IV} &= 3\,A + 6\,D_{xs}B + D^2_{xs}D \\
D_{V} &= A + 3\,D_{xs}B + D^2_{xs}D + D^3_{xs}E,
\end{aligned}
$$

and in a system of thin lenses in air, where $h_v = h_1$; $y_v = y_1$; $d_{\lambda-1} = 0$:

$$A = \sum_{v=1}^{k} Q^2_{s,v}\Delta\left(\frac{1}{ns}\right)_v = S_I \quad (\S\ 209,\ \text{ii}).$$

$$B = \sum_{v=1}^{k} Q_{sv}\Delta\left(\frac{1}{ns}\right)_v = \theta_{I,II} \quad (\S\ 209,\ \text{iii}).$$

$$C = \sum_{v=1}^{k}\left[\Delta\left(\frac{1}{ns}\right)_v - \frac{1}{r_v}\Delta\left(\frac{1}{n}\right)_v\right] = \sum_{v=1}^{k}\frac{n_v+1}{n_v}\phi_v\,.$$

$$D = \sum_{v=1}^{k}\left[3\,\Delta\left(\frac{1}{ns}\right)_v - \frac{1}{r_v}\Delta\left(\frac{1}{n}\right)_v\right] = \sum_{v=1}^{k}\frac{3\,n_v+1}{n_v}\phi_v\,.$$

$$E = \sum_{v=1}^{k}\frac{1}{Q_{sv}}\left[\Delta\left(\frac{1}{ns}\right)_v - \frac{1}{r_v}\Delta\left(\frac{1}{n^2}\right)_v\right] = -\sum_{v=1}^{k}\Delta\left(\frac{1}{n^2}\right)_v = 0\,.$$

The expressions for the image defects, which are applicable after the correction of two defects, are of the form $D^2_{xs} \sum_{v=1}^{k} M_v \phi_v$. In reality it is scarcely practicable to make these expressions vanish, since with the glasses available the component lenses would assume excessively high powers. We append a table showing the values of the factors M_v, omitting the index appropriate to n. The factors M_v corresponding in each row to the two corrected errors are denoted by zero.

TABLE OF FACTORS M_v.

M_{I}	M_{II}	M_{III}	M_{IV}	$M_{\mathrm{III}a}$	$M_{\mathrm{IV}a}$	M_{V}
0	0	$\frac{n+1}{n}$	$\frac{3n+1}{n}$	1	$\frac{2n+1}{n}$	$\frac{3n+1}{n}$
0	$-\frac{1}{2}$	$\frac{1}{n}$	$\frac{1}{n}$	0	$\frac{1}{n}$	$\frac{3n+2}{n}$
0	$-\frac{2n+1}{4n}$	$\frac{1}{2n}$	$-\frac{1}{2n}$	$-\frac{1}{2n}$	0	$\frac{6n+1}{4n}$
0	$-\frac{3n+1}{3n}$	$-\frac{3n-1}{n}$	$-\frac{3n+1}{n}$	$-\frac{3n+2}{n}$	$-\frac{6n+1}{n}$	0
1	0	$\frac{1}{n}$	$\frac{1}{n}$	0	$\frac{1}{n}$	$\frac{n+1}{n}$
$\frac{2n+1}{2n}$	0	$\frac{1}{2n}$	$-\frac{1}{2n}$	$-\frac{1}{2n}$	0	1
$\frac{3n+1}{2n}$	0	$-\frac{n-1}{2n}$	$-\frac{3n+1}{2n}$	$-\frac{n+1}{2n}$	-1	0
$\frac{3n+2}{n}$	$\frac{n+1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	0	$\frac{1}{n}$	0
$\frac{6n+1}{2n}$	1	$\frac{1}{2n}$	$-\frac{1}{2n}$	$-\frac{1}{2n}$	0	0

B. System of Thin Lenses with One Separation.

213. In a combination of two separated systems of thin lenses, if one of Seidel's image defects is to be removed, it is necessary that an expression of the form $S_v + PS_h$ should equate to zero. S_v (S_h) is the sum expression which ought to vanish in order that the image defect in question may be corrected for the front or back element when either is traversed by a system of rays

identical with that transmitted through the combination. As was shown in § 209, S_v (S_h) may be expressed both as a function of n and σ and as a function of n, ϕ and ρ. The quantity P is a factor which takes account of the incidental fact that the width ($2h$) of the axial pencil and the incidence height (y) of a principal ray inclined to the axis, have different values in the front and back elements respectively, thus

$$P = \{1 - A(\sigma_1 + \Phi_1)\}^{4-m}\{1 - A(\xi_1 + \Phi_1)\}^m,$$

where $m = 0, 1, 2, 3$ according as the system is corrected for the image defect D_I, D_{II}, D_{IIIa} (D_{IVa}) or D_V; A is the distance between the component systems; Φ_1 and Φ_2 are the resultant powers of the front and back elements respectively.

If now the stop occupies the position of one of the two components of the system the expressions for the defects, which are governed by the position of the stop, assume simpler forms. In the case in which the stop is close to the back element the expression is

$$\xi_1 = \frac{1}{A} - \Phi_1, \text{ or } 1 - A(\xi_1 + \Phi_1) = 0.$$

Also for either surface of the back element

$$Q_x = -\frac{\xi_1 + \Phi_1}{1 - A(\xi_1 + \Phi_1)} = \infty;$$

hence

$$\{1 - A(\xi_1 + \Phi_1)\}\ Q_x = -\xi_1 - \Phi_1 = -\frac{1}{A}.$$

When the stop is close to the front element it should be noted that the line of confusion due to the image defect with which we are concerned can be obtained by multiplying the summation given above by a factor containing ξ_1^{-m}, noting that $\xi_1^{-1} Q_x = -1$ for every surface of the front element, thus

$$\xi_1^{-1}\{1 - A(\xi_1 + \Phi_1)\} = -A.$$

is equal to the effective stop distance of the back element.

214. Correction of Two Image Defects by the Coflexure of the Lenses.—We may now proceed to investigate the problem of correcting a combination of two separated systems of thin lenses simultaneously for two, three or four of Seidel's image defects. We do not include the case where two of these defects come under the heading of defects D_{III}, D_{IV}, D_{IIIa} and D_{IVa}, since the simultaneous realisation of the Petzval condition does not, in general, involve special difficulties. For the correction of the

image defects we shall in the first instance have recourse to coflexure of the lens surfaces only, and we shall assume the focal lengths of the lenses and the separation of the component systems to be given.

In the first case we shall assume that two of Seidel's image defects have to be corrected simultaneously, and we shall further assume that one of the front curvatures (ρ_1) is variable in the front member, and similarly a front curvature (ρ_2) in the back member, whilst the remaining front curvatures are either constant or linear functions of the only variable in the respective member. When the expressions have been reduced to a form in which ρ_1 and ρ_2 occur as the only variables, the two equations which result when the two expressions for the two image defects are equated to zero, should be subtracted. ρ_1 will then appear as a quadratic function of ρ_2. Substituting this expression for ρ_1 in one of the original equations, we obtain a biquadratic equation for ρ_2.

To give a numerical example of this case, let the system consist of two single lenses and let the ratios of the focal lengths and the distances between the lenses be such as to satisfy the Petzval condition, and to ensure achromatism of the intercepts. It is required to correct the spherical aberration and the astigmatism. The types of glass chosen are

O. 1811 ($n_D = 1{\cdot}6061$; $n_{G'} - n_D = 0{\cdot}01361$)
O. 1891 ($n_D = 1{\cdot}6061$; $n_{G'} - n_D = 0{\cdot}02100$).

Glasses of equal refractive indices have been chosen in order to shorten the numerical calculation. The reciprocal of the stop-distance is $-22{\cdot}45$ when the crown glass is in front, and $-20{\cdot}00$ when the flint glass is in front. In both cases two of the four roots are real.

With the crown lens in front, the calculation gives the following results :

Type I ; $\rho_1 = -2{\cdot}33$; $\rho_2 = -10{\cdot}85$; $\rho_3 = -15{\cdot}48$; $\rho_4 = -6{\cdot}96$.
Type II ; $\rho_1 = +38{\cdot}73$; $\rho_2 = +30{\cdot}21$; $\rho_3 = -46{\cdot}26$; $\rho_4 = -37{\cdot}74$.

Distance between the lenses $= -0{\cdot}03752$.

Flint lens in front :

Type III ; $\rho_1 = -2{\cdot}107$; $\rho_2 = +4{\cdot}783$; $\rho_3 = +1{\cdot}963$; $\rho_4 = -4{\cdot}927$.
Type IV ; $\rho_1 = +24{\cdot}04$; $\rho_2 = +30{\cdot}93$; $\rho_3 = -18{\cdot}06$; $\rho_4 = -24{\cdot}95$.

Distance between the lenses $= -0{\cdot}05736$.

Type I in this example resembles the component half of a symmetrical photographic objective computed by R. Steinheil (*2.*).

215. Correction of Three or More Image Defects by Coflexure of the Lenses. — We shall now consider the simultaneous correction of a system of two elements for four image defects, viz., D_I, D_{II}, D_{IIIa} and D_V. From the expansions in terms of powers of D_{xs} of the image defects given in § 179 we can draw the following conclusion:—When the whole of the image defects of groups D_I and D_{II}, or D_I, D_{II}, and D_{IV}, or D_I, D_{II}, D_{IV}, and D_V have been removed for one particular position of the stops, *i.e.*, one particular value of D_{xs}, the image defects comprised in any of these groups will vanish for all distances between the stops. The substitution of the image defects D_{III}, D_{IIIa} or D_{IVa} for D_{IV}, does not affect the conditions in the case of the second group. In that of the third group a change in the distance between the stops affects the image defect D_V, but the change caused thereby depends upon the construction of the optical system only in so far as the factor $\sum \frac{1}{r_v} \Delta\left(\frac{1}{n}\right)_v$ is concerned; it is therefore independent of the form of the lenses. In the following equations the stop is supposed to coincide with the back component system. Nevertheless, they may be used when a different position of the stop is required. In the case of equation (v) only the zero value should be replaced by a quantity which is independent of the coflexure of the lenses. The equations embodying the required conditions will then be:

$$(\mathrm{I}) \sum_{v=1}^{i} Q^2_{sv} \, \Delta\left(\frac{1}{ns}\right)_v + (1-A\Phi_1)^4 \sum_{v=i+1}^{k} Q^2_{sv} \, \Delta\left(\frac{1}{ns}\right)_v = 0,$$

$$(\mathrm{II}) \sum_{v=1}^{i} Q_{xv} \, Q_{sv} \, \Delta\left(\frac{1}{ns}\right)_v - \frac{(1-A\Phi_1)^3}{A} \sum_{v=i+1}^{k} Q_{sv} \, \Delta\left(\frac{1}{ns}\right)_v = 0,$$

$$(\mathrm{IIIa}) \sum_{v=1}^{i} Q^2_{xv} \, \Delta\left(\frac{1}{ns}\right)_v + \frac{(1-A\Phi_1)^2}{A^2} \sum_{v=i+1}^{k} \Delta\left(\frac{1}{ns}\right)_v = 0,$$

$$(\mathrm{V}) \sum_{v=1}^{i} \left[Q^3_{xv} \, Q^{-1}_{sv} \Delta\left(\frac{1}{ns}\right)_v - (Q_{xv} - Q_{sv})^2 \, Q_{xv} \, Q^{-1}_{sv} \frac{1}{r_v} \Delta\left(\frac{1}{n}\right)_v \right] = 0,$$

where i is the sequence number of the last surface of the front member. These equations should be solved in two steps, the front member being determined first by means of the last two equations, and the back member by means of the first two equations. The procedure in these two problems is the same as that discussed above in regard to a system of thin lenses and it is applicable to the same types of component systems. There is only the unessential difference that the image defects D_I, D_{II}, D_{IIIa} must not be equated to zero but are required to assume certain finite values. The astigmatism of the front member should be such as to

compensate that inherent in the back member, which is solely a function of its aggregate focal length. The front member having been determined, the back member should be so corrected as to cause the spherical aberration and the coma to be opposite in amount to that prevailing in the front member. We may here enunciate the following rule: *If, in a combination of two separated systems of thin lenses, one of the members is replaced by a new member corresponding to the original member as regards its position, the resultant focal length and the magnitudes of the spherical aberration and the coma (the position of the stop being the same in both cases), the correction of the entire combination remains unchanged as regards astigmatism and distortion.* An example of the application of this rule is to be found in Dallmeyer's modification of the Petzval portrait lens, if we may neglect the thicknesses of the lenses.

When we can dispense with the correction of the distortion it is only necessary to solve the first three equations. The problem presents in this case no new feature. It may, however, be interesting to note a special kind of solution, which consists in separately equating to zero the expressions

$$\sum_{v=1}^{i} Q^2_{sv} \; \Delta\left(\frac{1}{ns}\right)_v ; \quad \sum_{v=1}^{i} Q_{sv} \; \Delta\left(\frac{1}{ns}\right)_v ; \quad \sum_{v=i+1}^{k} Q^2_{sv} \; \Delta\left(\frac{1}{ns}\right)_v ,$$

and $\displaystyle\sum_{v=i+1}^{k} Q_{sv} \; \Delta\left(\frac{1}{ns}\right)_v$ as well as $\displaystyle\sum_{v=1}^{k} \Delta\left(\frac{1}{ns}\right)_v$.

That this method is correct will be realised if we convert the expressions

$$\sum_{v=1}^{i} Q_{xv} \; Q_{sv} \; \Delta\left(\frac{1}{ns}\right)_v \text{ and } \sum_{v=1}^{i} Q^2_{xv} \; \Delta\left(\frac{1}{ns}\right)_v$$

into explicit functions of s_v with the aid of the elimination formulæ given in § 179. The entire system is accordingly free from astigmatism if the two component systems are corrected for the optical conditions under which they are intended to be used with respect to the image defects D_I and D_{II} and if they are given equal and opposite focal lengths. This principle formed the basis of Dennis Taylor's (*3.*) first photographic lens. Kerber (*16.*) has drawn attention to an interesting special case. If the front member is composed of two thin lenses of the same refractive index and so corrected that the image defects D_I and D_{II} vanish, the aberrations due to the image defects D_{III} and D_{IV} can be corrected by a single thin dispersing lens having the same refractive index n, the lens being of the aplanatic type and placed at a distance from the front component equal to $\frac{1}{n}$th the focal length of the latter.

216. Choice of the Distance between the Lenses as a Means of Correction.—We shall now briefly explain a method of effecting corrections by a suitable choice of the distance separating the lenses. Let the system consist merely of two separated thin lenses, and let the focal length of the second lens and the position of the principal focus of the system be given; let the distance of the object be infinitely great; and let the stop be coincident with the first lens, the front surface of which is plane. It is required to correct the spherical aberration and the astigmatism. A system of this kind may be used as one half of a symmetrical triplet designed for photographic purposes. The distance of the principal focus of the system is equal to $\frac{1}{\psi_1 + \phi_2}$, where $\psi_1 = \frac{\phi_1}{1 - A\phi_1}$, thus ψ_1 may be regarded as known. Apart from the variation of the distances between the lenses the front curvature ρ_2 of the second lens may serve for effecting the correction.

Equating to zero the expression for the image defect D_{I} and the difference of the expressions for the image defects D_{I} and D_{IIIa}, we obtain after a few reductions the following equations:

$$\left(\frac{n_1}{n_1 - 1}\right)^2 \psi_1^3 (A\psi_1 + 1) + \left(\frac{n_2}{n_2 - 1}\right)^2 \phi_2^3 + \frac{3\,n_2 + 1}{n_2 - 1}\,\psi_1\,\phi_2^2$$

$$+ \frac{3\,n_2 + 2}{n_2}\,\psi_1^2\,\phi_2 - \frac{2\,n_2 + 1}{n_2 - 1}\,\phi_2^2\,\rho_2 - \frac{4\,n_2 + 4}{n_2}\,\psi_1\,\phi_2\rho_2$$

$$+ \frac{n_2 + 2}{n_2}\,\phi_2\,\rho_2^2 = 0\,, \text{ and}$$

$$\left[\left(\frac{n_1}{n_1 - 1}\right)^2 \psi_1^3 A - \frac{\psi_1 + \phi_2}{A} + \frac{2n_2}{n_2 - 1}\,\phi_2^2 + \frac{3\,n_2 + 2}{n_2}\psi_1\phi_2\right.$$

$$\left. - \frac{2\,n_2 + 2}{n_2}\,\phi_2\,\rho_2\right]\frac{A\psi_1 + 1}{A} = 0\,.$$

Disregarding the case $A\psi_1 + 1 = 0$, which is of no practical importance, the expression obtained from the second equation for ρ_2 should be substituted in the first equation, which thus gives an expression for A in the form of an equation of the 4th order.

As an example, let $\phi_2 = +1$, $\psi_1 = -0{\cdot}72$, and let the refractive index of both lenses be $n_{1\,2} = 1{\cdot}6061$. The focal length of the system is therefore not unity in the first instance, and it will be necessary subsequently to reduce the value obtained. In this particular case $\phi_1 = -2{\cdot}571$, since

$$\frac{\phi_1}{\Phi} = \frac{\psi_1}{\psi_1 + \phi_2}.$$

The four roots of the biquadratic equation are all real in this example, though two of them must be rejected on account of their negative sign. The reduced curvatures of the system corresponding to the remaining roots are :

$$\text{I}\,;\ \rho_1 = 0\,;\ \rho_2 = +\,1{\cdot}243\,;\ \rho_3 = +\,0{\cdot}450\,;\ \rho_4 = -\,4{\cdot}050$$
$$\text{II}\,;\ \rho_1 = 0\,;\ \rho_2 = +\,1{\cdot}243\,;\ \rho_3 = -\,0{\cdot}043\,;\ \rho_4 = -\,5{\cdot}310\,.$$

In the former case the distance between the lenses is 0·121, whilst in the second case it is 0·0462.

If the distance is to serve as one of the means of correction in the case of a combination of two separated systems of thin lenses in which the image defects D_{I}, D_{II}, D_{IIIa} and D_{V} are to be eliminated simultaneously, the problem may be reduced, as before, to that of systems of thin lenses which we have already investigated. No difficulty is occasioned by the introduction of the Petzval condition, in addition to the others, and thus the preceding discussion indicates the method of correcting simultaneously all five image defects of Seidel in a system of three lenses, two of which are close together, whilst the third is at some distance, the correction being effected for all positions of the object.

C. System of Thin Lenses involving Two Finite Separations.

217. We shall now briefly discuss the principles of a combination of three separated systems of thin lenses. Our first task is to correct the image defects D_{I}, D_{II} and D_{IIIa}, and for this purpose we shall do so by applying the principle of coflexure. Since the correction does not depend upon the course of the principal rays we may with advantage place the stop in coincidence with one of the component systems, say the second. We must then satisfy the following equations :

$$(\text{I})\,;\ \left(\frac{h_1}{h_2}\right)^4 F_2^{\text{I}}(\rho_1) + F_2^{\text{I}}(\rho_2) + \left(\frac{h_3}{h_2}\right)^4 F_2^{\text{I}}(\rho_3) = 0\,,$$

$$(\text{II})\,;\ \left(\frac{h_1}{h_2}\right)^3 A_{12}\ F_2^{\text{II}}(\rho_1) - F_1(\rho_2) + \left(\frac{h_3}{h_2}\right)^3 A_{23}\ F_2^{\text{II}}(\rho_3) = 0\,,$$

$$(\text{IIIa})\,;\ \left(\frac{h_1}{h_2}\right)^2 A_{12}^2\ F_2^{\text{IIIa}}(\rho_1) + \Phi_2 + \left(\frac{h_3}{h_2}\right)^2 A_{23}^2\ F_2^{\text{IIIa}}(\rho_3) = 0\,.$$

$F_2^{l}(\rho_i)$ is a quadratic function of ρ_i, whilst $F_1(\rho_2)$ is a linear function of ρ_2. In fact, $F_2^{l}(\rho_i)$ is equal to S_l, as defined in § 209 (ii), and $F_1(\rho_2)$ is equal to $\theta_{\text{I II}}$ (§ 209, viii) with respect to the component system in question, the path of the rays in the

component systems being identical with that prevailing in the combination. From equations (II) and (IIIa) we derive ρ_2 as a linear function of ρ_1 and as a quadratic function of ρ_3. Substituting the expression for ρ_2 in (I) and eliminating ρ_1 with the aid of (IIIa), we arrive at an equation of the 8th degree for the computation of ρ_3.

If it is required also to correct the distortion the above equation should be used in conjunction with the following:

$$(\mathrm{V});\ \left(\frac{h_1}{h_2}\right) A^3_{12}\, F_2^{\mathrm{V}}(\rho_1) + \left(\frac{h_3}{h_2}\right)^3 A^3_{23}\, F_2^{\mathrm{V}}(\rho_3) = 0.$$

We shall assume that in one of the component systems the coflexure of the lenses may be effected in two ways, or that there are two other variables available, and we shall place the stop in coincidence with this component system. In place of $F_2(\rho_2)$ and $F_1(\rho_2)$ we shall then have in the equations (I) and (II), say, the functions $F_2(\rho_{2\,1}) + F_2(\rho_{2\,2})$ and $F_1(\rho_{2\,1}) + F_1(\rho_{2\,2})$. The biquadratic equation resulting from D_{IIIa} and D_{V} then determines ρ_1 and ρ_3 in the manner shown above in § 214. If these values are substituted in (I) and (II) the determination of $\rho_{2\,1}$ and $\rho_{2\,2}$ follows from the quadratic equation. The second step of the procedure will differ according to the variables selected, as in the case of a system of thin lenses.

D. Single Lens of Finite Thickness.

218. In conclusion we shall investigate the case of a single lens of finite thickness (d). For our variables we shall choose:

$$\mathsf{a} = \frac{r_1}{r_2};\ \mathsf{b} = \frac{s'_1}{s'_1 - r_1};\quad \mathsf{c} = \frac{x'_1}{x'_1 - r_1};\quad \mathsf{d} = 1 - \frac{d}{s'_1}.$$

Upon these depend the elements of the expressions of the image defects in the following manner;

$$Q_{1s} = \frac{n}{r_1\,\mathsf{b}};\quad \frac{h_2}{h_1} = \mathsf{d};\quad \frac{h_2}{h_1} Q_{2s} = \frac{n\,\mathsf{M}}{r_1\,\mathsf{b}};$$

$$\Delta\left(\frac{1}{ns}\right)_1 = \frac{n-1}{n}\,\frac{\mathsf{N}}{r_1\,\mathsf{b}};\quad \frac{h_2^2}{h_1^2}\Delta\left(\frac{1}{ns}\right)_2 = \frac{n-1}{n}\,\frac{\mathsf{K}}{r_1\,\mathsf{b}}$$

$$Q_{1x} = \frac{n}{r_1\,\mathsf{c}};\quad \frac{y_2}{y_1} Q_{2x} = \frac{n\,\mathsf{L}}{r_1\,\mathsf{c}};$$

where

$$\mathsf{K} = (n+1)(\mathsf{b}-1)\,\mathsf{d} - n\mathsf{abd}^2;$$

$$\mathsf{L} = \mathsf{ac} + \frac{\mathsf{ab}}{\mathsf{b}-1}(\mathsf{c}-1)(\mathsf{d}-1) - (\mathsf{c}-1);$$

$$\mathsf{M} = \mathsf{abd} - (\mathsf{b}-1);$$

$$\mathsf{N} = n - (\mathsf{b}-1).$$

The terms of the summation for the image defects will then be:

$$\mathrm{D_I} = \frac{n\,(n-1)}{\mathsf{b}^3\, r_1^3}\,(\mathsf{N} + \mathsf{M}^2\,\mathsf{K})\,,$$

$$\mathrm{D_{II}} = \frac{n\,(n-1)}{\mathsf{b}^2\mathsf{c}\, r_1^3}\,(\mathsf{N} + \mathsf{LMK})\,,$$

$$\mathrm{D_{IIIa}} = \frac{n\,(n-1)}{\mathsf{b}\mathsf{c}^2\, r_1^3}\,(\mathsf{N} + \mathsf{L}^2\mathsf{K})\,,$$

$$\mathrm{D_V} = \frac{n\,(n-1)}{\mathsf{c}^3\, r_1^3}\left\{\mathsf{N} + \frac{\mathsf{L}^3\,\mathsf{K}}{\mathsf{M}} + \frac{(\mathsf{b}-\mathsf{c})^2}{\mathsf{b}}\left(1 - \mathsf{a}\,\frac{\mathsf{L}}{\mathsf{M}}\right)\right\}.$$

In order that the astigmatism and the curvature of the image may vanish the following equations must be satisfied:

$$\mathsf{N} + \mathsf{L}^2\,\mathsf{K} = 0\,; \qquad \mathsf{a} = 1\,(r_1 = r_2 = r)\,,$$

whence it follows that

$$\mathsf{c} = 1 - \frac{\mathsf{b}-1}{\mathsf{b}\,(\mathsf{d}-1)}\left\{1 \mp \frac{1}{\sqrt{\dfrac{n\mathsf{b}}{n+1-\mathsf{b}}\,\mathsf{d}\,(\mathsf{d}-1)+\mathsf{d}}}\right\}.$$

We shall discuss this equation with reference only to the case in which $s_1 = \infty$, and hence $\mathsf{b} = n$. We then have

$$\mathsf{c} = 1 - \frac{n-1}{n\,(\mathsf{d}-1)}\left\{1 \mp \frac{1}{\sqrt{\mathsf{d}\,[n^2\,(\mathsf{d}-1)+1]}}\right\}.$$

In order that the roots may not become imaginary d should not be given any values within the limits of zero and $\left(1 - \frac{1}{n^2}\right)$. We append in tabular form a series of real values which arise as d changes from 0.7 to 1.3. In the computation of this table it has been assumed that $n = 1.6$ and that the focal length is 100. The leading equation

$$\frac{r}{(n-1)(1-\mathsf{d})} = 100$$

gives the value of r_1. The magnitude of d can now be found from the expression for d, as defined above, in which we must substitute $s_1' = \dfrac{nr}{n-1}$. The values of x_1 corresponding to the two roots of c can be found with the aid of the expression which defines c and the relation between x'_1 and x_1 involved in the invariant Q_{1x}.

Numerical Data for an Anastigmatic Meniscus.

Assuming $n_D = 1.6$ and $f_D = 100$ mm.

a	+0.70	0.75	0.80	0.85	0.90	0.95	1.00	1.05	1.10	1.15	1.20	1.25	1.30
r	+18	+15	+12	+9	+6	+3	0	−3	−6	−9	−12	−15	−18
d	14.4	10.0	6.4	3.6	1.6	0.4	0	0.4	1.6	3.6	6.4	10.0	14.4
x_1	+6.25 +22.8	+2.94 +21.3	+0.98 +16.5	−0.29 +11.1	−0.73 +7.25	−0.79 +3.31	0 0	+0.93 −2.69	+2.31 −4.74	+3.86 −6.17	+5.58 −7.20	+7.26 −7.35	+10.1 −7.25

The types specified in the 9th and 11th columns, provided with front stops and corrected for finite inclinations of the principal rays, lead to objectives of the type of the Hypergon Double Anastigmatic Lens of Goerz (*1.*) and the single Meniscus of von Hoegh (*4.*).

If it is required to remove the astigmatism and the distortion, the expressions within the brackets of defects D_{IIIa} D_V should be equated to zero. We may in this case choose a and c for our unknown quantities and begin by eliminating c. The second equation, when simplified with the aid of the first, becomes

$$\mathsf{N} - \frac{\mathsf{NL}}{\mathsf{M}} + \frac{(\mathsf{b}-\mathsf{c})^2}{\mathsf{b}}\left(1 - \mathsf{a}\frac{\mathsf{L}}{\mathsf{M}}\right) = 0.$$

The expression which defines L may be transformed thus :

$$\mathsf{L} = \mathsf{M} - \frac{(\mathsf{b}-\mathsf{c})(\mathsf{M}-\mathsf{a})}{\mathsf{b}-1}.$$

Equating the expressions for $(\mathsf{b}-\mathsf{c})^2$ resulting from these two equations, we obtain

$$\frac{\mathsf{bN}(\mathsf{M}-\mathsf{L})}{\mathsf{aL}-\mathsf{M}} = \frac{(\mathsf{M}-\mathsf{L})^2(\mathsf{b}-1)^2}{(\mathsf{M}-\mathsf{a})^2}$$

One of the possible roots of this equation is obviously

$$\mathsf{M} - \mathsf{L} = 0.$$

This condition is fulfilled first, when $\mathsf{b} = \mathsf{c}$ or $s'_1 = x'_1$ or $s_1 = x_1$; secondly, when $\mathsf{M} = \mathsf{a}$, *i.e.* $r_2 = r_1 - d$. From the equation $(D_{IIIa}) = 0$, c is found, and the expression within the brackets in this equation is independent of c, since $\mathsf{L} = \mathsf{M} = \mathsf{a}$. Hence $\mathsf{c} = \infty$, *i.e.* $x'_1 = r_1$. Of these two solutions it is only this latter which has any practical significance.

Multiplying out, we find

$$\frac{\mathsf{b}\mathsf{N}(\mathsf{M}-\mathsf{a})^2}{(\mathsf{b}-1)^2}+\mathsf{M}^2+\mathsf{a}\mathsf{L}^2=\mathsf{L}\mathsf{M}(\mathsf{a}+1).$$

In this equation we shall express a and M as functions of L. Equating D_{IIIa} to zero and substituting the expression symbolised by K, we obtain

$$\mathsf{a}=\frac{n\mathsf{b}-\mathsf{N}}{n\mathsf{b}\mathsf{d}}+\frac{\mathsf{N}}{n\mathsf{b}\mathsf{d}^2\mathsf{L}^2}$$

and

$$\mathsf{M}=\frac{n-\mathsf{N}}{n}+\frac{\mathsf{N}}{n\mathsf{d}\mathsf{L}^2}.$$

The above equation is accordingly of the 6th degree in terms of L.

3. Final Correction by Small Changes of the Radii.

219. In practice it frequently happens that the computation of optical systems can be considerably simplified by reference to an available standard system whose optical data (radii, distances, thicknesses, refractive indices and dispersions) differ by small amounts from the required system, and which have been corrected to suit slightly different requirements. In such cases it is usual to re-correct the system in accordance with the new requirements by small alterations to the radii. We shall discuss here a few examples which lead to problems of this kind: The system may be required to transmit rays of a different range of wave-lengths; or it may be intended to operate at different distances of the object or the image; or the stop may be required to have a different position along the axis; or the aperture, or the field of view may be required to be larger or smaller. In particular, the case may arise in which the system is corrected for an infinitely small aperture and an infinitely small field of view only, in accordance with approximate expressions for the image-defects. As the diameters of the lenses are increased for the purpose of improving this correction it may become necessary to alter the thicknesses and distances. The necessity for such alterations may, however, also be occasioned by other circumstances; in particular, it may happen that the thicknesses and distances may have been provisionally neglected in the computation in view of the analytical difficulties which they involve. Also, it may be necessary to introduce types of glass having slightly different optical constants. Such cases arise when a new glass with improved physical or chemical properties has become available, or when slight changes in the constants of a repeated batch necessitate modifications in the calculation.

To solve problems of this kind the expressions for the image-defects should be expanded by Taylor's theorem, the expansion being terminated at the first differential quotient. We then determine the numerical values of the aberrations of the existing optical system which arise when the new optical data are introduced, and when the path of the rays is modified to suit the new requirements, the size of the aperture, that of the field of view, and, usually also, of the range of wave-lengths being regarded as infinitely small. The amounts of the aberrations to which the system remains subject on this assumption, if it is to be corrected for finite amounts of these factors, should be deducted. The resulting numerical quantities should be regarded as negative and equated to the total first differentials of the expressions for the defects of the same system expanded by Taylor's theorem. In this way we obtain a corresponding number of linear equations for the increments of the variables (*e.g.* the radii), whose number should be chosen equal to that of the defects to be corrected. Since in the elimination formulæ in § 179, the image-defects are expressed in the form of explicit functions of σ, the latter will be chosen as variable; for, when the variations of the σ values have been found, it is easy to derive from them those of the ρ values. We may assume, moreover, that the coefficient h_v/h_1 remains unchanged, notwithstanding the changes of σ, since

$$d_\lambda = \left(\frac{h_{\lambda-1}}{h_\lambda} - 1\right)\frac{1}{\sigma_\lambda},$$

and d_λ varies with σ_λ. Though in the final result this leads to other thicknesses and distances than those previously arrived at, the system with the new thicknesses can, in general, be accepted as final since they only change in the same ratio as σ.

Now, the ratio

$$\frac{\sigma'_{v-1}}{\sigma_v} = \frac{h_v}{h_{v-1}}$$

being given, it follows that

$$\partial\sigma'_{v-1} = \frac{h_v}{h_{v-1}}\,\partial\sigma_v.$$

If, accordingly, a system having k radii be given, and if we keep the positions of the object and the image constant, we can only vary $k - 1$ radii independently of σ. For this purpose we shall choose the unaccented values of σ.

It would now be necessary to form the first partial differential quotients of the defects in terms of $\partial\sigma_v$. It will be sufficient, however, in order to explain the procedure, to do so with respect to the axial spherical aberration, the deviation from the sine condition,

and the chromatic deviation along the axis. Expressing all variable quantities in terms of σ, the equation for the spherical aberration becomes

$$\underline{\mathrm{Sph}} = \sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^4 \left\{ \frac{\Delta\, \sigma_v}{\Delta\left(\frac{1}{n}\right)_v} \right\}^2 \Delta\left(\frac{\sigma}{n}\right)_v.$$

By partial differentiation with respect to σ'_{v-1} and σ_v, and by combination of the two expressions, we obtain

$$\frac{\partial\, \underline{\mathrm{Sph}}}{\partial \sigma_v} = \left(\frac{h_v}{h_1}\right)^4 \left[Q_{sv}(3\, P_{sv} - \sigma'_v) - \left(\frac{h_{v-1}}{h_v}\right)^3 Q_{s,v-1}(3\, P_{s,v-1} - \sigma_{v-1}) \right],$$

where

$$P_{sv} = \frac{\Delta\left(\frac{\sigma}{n}\right)_v}{\Delta\left(\frac{1}{n}\right)_v}.$$

The expression for the deviation from the sine condition, denoted by $\underline{\mathrm{Sinc}}$, is :

$$\underline{\mathrm{Sinc}} = \sum_{v=1}^{k} \left(\frac{h_v}{h_1}\right)^4 \left\{ \frac{\Delta\sigma_v}{\Delta\left(\frac{1}{n}\right)_v} \right\}^2 \Delta\left(\frac{\sigma}{n}\right)_v S_v,$$

where

$$S_v = D_{xs}\left[- \left(\frac{h_1}{h_v}\right)^2 \frac{\Delta\left(\frac{1}{n}\right)_v}{\Delta\, \sigma_v} + h_1^2 \sum_{\lambda=2}^{v} \frac{H_{\lambda-1}}{\sigma_{\lambda-1}} \right],$$

and

$$H_{\lambda-1} = \frac{\frac{h_{\lambda-2}}{h_{\lambda-1}} - 1}{n'_{\lambda-1}\, h_\lambda\, h_{\lambda-1}};$$

and by partial differentiation we obtain

$$\frac{\partial\, \underline{\mathrm{Sinc}}}{\partial \sigma_v} = \left(\frac{h_v}{h_1}\right)^4 \left[Q_{sv}\,(3\, P_{sv} - \sigma'_v)\, S_v \right.$$

$$\left. - \left(\frac{h_{v-1}}{h_v}\right)^3 Q_{s,v-1}\,(3\, P_{s,v-1} - \sigma_{v-1})\, S_{v-1} \right]$$

$$D_{xs} \left(\frac{h_v}{h_1}\right)^2 \left[P_{sv} - \frac{h_{v-1}}{h_v}\, P_{s,v-1} \right]$$

$$- D_{xs}\, h_1^2 \frac{H_v}{\sigma_v^2} \sum_{i=v+1}^{k} \left(\frac{h_i}{h_1}\right) Q^2_{si}\, \Delta\left(\frac{\sigma}{n}\right)_i.$$

By partial differentiation of the expression for the chromatic aberration for a small range of wave-lengths, viz.:

$$V\sigma'_k = +\frac{1}{n'_k}\sum_{v=1}^{k}\left(\frac{h_v}{h_k}\right)^2 \frac{\Delta\sigma_v}{\Delta\left(\frac{1}{n}\right)_v}\Delta\left(\frac{Vn}{n}\right)_v,$$

we obtain

$$\frac{\partial V\sigma'_k}{\partial\sigma_v} = \frac{1}{n'_k}\left(\frac{h_v}{h_k}\right)^2\left(M_v - \frac{h_v}{h_{v-1}}M_{v-1}\right),$$

where

$$M_v = -\frac{\Delta\left(\frac{Vn}{n}\right)_v}{\Delta\left(\frac{1}{n}\right)_v}.$$

4. DISTRIBUTION OF THE OPTICAL FUNCTIONS OF AN INSTRUMENT OVER AN OBJECTIVE AND EYE-PIECE.

220. In the following pages we shall understand by the term **aberrations** those arising in a lateral direction with respect to the axis. They are measured linearly, except when the required aberration refers to an infinitely distant plane, in which case it is expressed in angular measure. As a preparatory step it is necessary to study the influence which a change of dimensions in a single or composite system exercises upon the aberrations. When only a definite magnification of the image is specified, whilst the positions of the object and image are optional, we are at liberty to select a system of any desired focal length. Supposing in one case a system A to have a focal length 1, and in another a system B to have a focal length m, the dimensions of the latter as regards radii, thicknesses and distances being throughout m times greater than those of the first; it will be necessary in the second case to make the object and image distances also m times greater. All linear co-ordinates relating to the course of the rays would accordingly become m times greater, and the aberrations similarly, if the object in the second case were likewise m times greater. Since, however, it is of the same size as in the first case it follows that in the second case, if we neglect the aberrations of the point where the principal rays cross, the trigonometrical tangents of the angles of inclination of the principal rays at corresponding points lying out of the axis are m times smaller. The coefficient of comparison is accordingly

the product of two factors, of which m is one, whilst the other is a measure of the rate at which the image-defect under consideration changes when the distance of the image-point from the axis is m times smaller. Confining the investigation to terms of the third order we find that in system B the spherical aberration is m times greater, the coma of the same amount, the curvature of the field and the astigmatism m times smaller, the distortion m^2 times smaller as compared with system A. The chromatic aberration on the axis is m times greater, whilst the chromatic difference of magnification remains unchanged, within the limits assigned by the Gaussian theory.

We now proceed to investigate the construction of composite systems—*i.e.*, of systems made up of two independent elements having different dioptric functions. We shall distinguish the element which faces the object as the **objective**, the other element being the **eye-piece.** The latter term is usually confined to instruments designed for visual observation. We shall dispense with this restriction, and in this connection refer to the term *projection eye-piece.*

The objective and eye-piece are usually corrected independently, at any rate for several image-defects. The question arises as to how the remaining defects are affected when the total magnification is allotted in various ways to the objective and the eye-piece.

To begin with the telescopic system, which presents the simplest conditions. In this case, if the magnification is to remain unchanged, it will be necessary to alter the respective focal lengths of the objective and eye-piece in the same ratio. If now the entire system is to have all its dimensions increased m times, the same arguments which applied to the system with **finite focal length** and m-fold dimensions, will likewise hold good for the telescopic system with reference to objects at finite distances, assuming that their distance from the system is similarly increased. In the case of objects at an infinite distance the quality of the image does not change at all. It should, however, be noted that the "number of rays" or quantity of light received by the system has now become m^2 times greater. If this quantity is to be the same in both cases, it will be necessary in the enlarged system to make the greatest distance from the axis of the point of incidence of the rays m times smaller, whereby the two systems are provided with objectives of the same apertures. If the systems are compared under these assumptions it will be found that in the enlarged system the aberrations are smaller in the ratio of $m^k : 1$, where k is the power of the ratio connecting the aberrations and the aperture. For example, the chromatic difference of magnification and the distortion do not undergo any change, whilst the chromatic aberration of the axis (to the first degree of approximation), the astigmatism and the curvature of the image are

reduced to the mth part of their original amount, and, if we confine ourselves to terms of the third order, coma is reduced to the m^2th part and the spherical aberration to the m^3th part.

We shall investigate the composite system of finite focal length under the special condition that the objective forms either a greatly magnified or a greatly reduced image, and that the eye-piece forms a greatly magnified image. We shall, in the first place, discuss the path of the rays through a simple system when the object or the image is at a great distance. It will be assumed that certain image-defects have been corrected for parallel incident light. If now it be intended to employ the system in the case of a great finite distance it will be possible, in general, to correct the system for the same defects by small alterations of the radii. We may then assume, without introducing any serious error, that the remaining aberrations are identical in both cases and that the circles of confusion projected back into the object are proportional to the distance of the object. Conversely, if the object lies in the vicinity of the front focal point, in which case the image accordingly recedes to a great distance, it will be assumed that the circles of confusion projected back into the object are equal, and that the circles of confusion in the image may be regarded as proportional to the image distance.

To ascertain the relation between two compound systems whose respective objectives and eye-pieces are of a similar type (neglecting the small alterations of the radii), and in which the magnification of the image is divided unequally over the objectives and eye-pieces, we have only to investigate the influence which a change in the magnification due to the objective and the eye-piece, taken singly, exercises upon the aberrations. Since the magnification of the entire system remains unchanged it does not affect the comparison whether the circle of confusion is referred to the object or to the last image. In making our comparison we assume the angular aperture of the pencils received by the objective, and therefore likewise by the entire system, to be identical in both cases. In the case of an objective or eye-piece of high magnifying power, we shall neglect any change in the angular aperture on the side of the object occasioned by a change in the distance of the latter.

For the sake of brevity, we shall introduce a few symbols. If the scale of the system as well as the size and distance of the object and image are raised to m times the original amount, we shall denote by m_1 the number which indicates the proportional increase of the diameter of the circle of confusion. If in a system of great diminishing power the image distance of the objective is taken m times greater, we shall introduce m_2 to denote the increase of the diameter of the circle of confusion on the object side. Similarly, if in a system of high magnifying power the image distance is m times

greater, we shall denote by m_3 the proportional increase of the diameter of the circle of confusion on the object side, and by m_4 the proportional increase on the image side. We have just found that the value of m_1, m_2 and m_4 is equal to m, whilst that of m_3 is unity. By M_1 we shall denote the number which indicates the ratio in which the diameter of the circle of confusion changes as the diameter of the aperture is increased to m times its original amount, and M_2 stands for this proportional increase when the inclination of the principal ray becomes m times greater. From our previous investigations it follows that these symbols represent different values according to the image-defect under consideration.

The lateral aberration of an image-point of a compound system is partly due to the objective and partly to the eye-piece. When the distribution of the total magnification over the objective and eye-piece is changed each part will experience a separate change. We shall first investigate the part due to the eye-piece. If the magnification produced by the objective is increased m times it follows that the eye-piece will form an m times smaller image of an m times larger objective image by means of pencils having angular apertures which are m times smaller. If we accomplish this by increasing m times the focal length of the eye-piece, the change in the lateral aberration produced by the eye-piece will be expressed by the factor $P_e = \frac{m_1}{m_4 M_1} = \frac{1}{M_1}$. On the other hand, if we obtain the required eye-piece magnification by diminishing the distance of the image, that factor becomes $P_e = \frac{M_2}{m_4 M_1}$. If the objective magnification is to be m times greater in the case of objects at a great distance, this may be accomplished by an increase of the focal length of the objective, without changing its diameter, or by diminishing the distance of the object and the aperture of the objective. In the former case the increase of the lateral aberration due to the objective is $P_o = \frac{m_1}{m_2 M_1} = \frac{1}{M_1}$, whilst in the second case it is $P_o = \frac{M_2}{m_2 M_1}$. In the case of near objects and greatly magnifying objectives, if the magnification produced by the objective be increased m-fold, the procedure is either to diminish in the ratio of $m : 1$ the focal length and the diameter of the objective, or to increase m times the distance of the image formed by the objective. In the former case $P_o = \frac{m_3}{m_1} M_2 = \frac{M_2}{m}$; and in the latter $P_o = m_3 = 1$.

5. HISTORICAL NOTES.

221. On Section 1.

Petzval has treated the condition which bears his name in two papers (*1. 26*) and (*2. 95*), published respectively in 1843 and 1857. Respecting its practical fulfilment he states in the latter paper that in a system of thin lenses, in view of the small difference in the refractive indices of the glasses, the powers of the component lenses assume inconveniently high values relatively to the power of the entire system. A year previously Seidel (*3. 323*) had drawn attention to the fact that in a system of thin lenses composed of the silicate glasses then available, it was not possible at the same time to satisfy the chromatic condition. He said in effect: "It is only in cases in which it is possible to employ lenses of considerable thickness where the apertures are small, as in the case of eye-pieces and possibly microscope objectives, that we may reasonably hope to overcome the difficulty." It will be interesting to note how this difficulty was got over in practice, and we shall first consider objectives made from the older series of glasses. First among those which satisfied both conditions is the Pantoscope of Busch, which has been described by M. v. Rohr (*3. 280*) and which may be regarded as a development of Harrison and Schnitzer's *globe lens*. In view of the special coflexure of the lenses, the separation of the converging outer surfaces from the diverging inner surfaces and the thicknesses could be kept within moderate limits. The objection to this combination is that it does not admit of spherical correction. This was first accomplished in 1893 by Dennis Taylor (*3.*) whose theoretical investigation led him to a type of objective composed of three achromatic combinations of lenses, all consisting of two identical pairs of ordinary silicate glasses, separated by finite distances, the two outer lens combinations having a converging, and the inner combination a diverging effect. In the meantime the highly refracting barium crown glasses of the Jena Glass Works had been placed on the market. These had been employed by Schroeder (*2.*) in his concentric lens which could not be corrected spherically (1887), and by Rudolph* in his spherically corrected anastigmatic lens (Series III[a] of 1891), in both cases as a means of satisfying the Petzval condition as well as the chromatic condition.

222. Section 2.

As regards the subject of the correction of Seidel's five image-defects, the following writers have investigated the defect previously termed Seidel's Image-Defect D_I: Euler (*2.*), Kluegel (*1.*) Stampfer (*1.*), Littrow (*1. 2.*), Steinhaus (*1.*), Grunert (*1.*),

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* Cf. M. v. R. (*3. 365*).

A. Steinheil (*1.*), Hansen (**2.**), W. Schmidt (*1.*), Scheibner (*1.*), H. Kruess (*1.*). The corrections of Seidel's 1st and 2nd image-defects for systems of thin lenses have been investigated by the following writers: Seidel (**3.** *324*), Kramer (*1.*), C. Moser (**3.**), Kerber (**5.**), R. Steinheil (*1.*), Charlier (**4.**), Harting (*1.* **2.**), v. Hoegh (**3.**), Harting (**3.**), Leman (*1.*), Harting (**6.**), Strehl (*1.*). The simultaneous correction of Seidel's image-defects D_I and D_{IIIa} or D_{IVa} in systems of thin lenses has been dealt with by C. Moser (**4.**), Kerber (**5.**), Harting (**4.**), whilst Zinken-Sommer (**2.** *94*) has investigated the correction of Seidel's image-defects D_I, D_{II}, and D_{III}, taking into consideration the thicknesses and distances within the first degree of approximation.

223. Section 3.

The problem discussed in this section has been treated in a manner different from that in this book by Kerber (**9.** **10.**), Harting (**2.**), Kerber (**11.**), Harting (**3.**), Leman (**1.**), Harting (**6.**), Kerber (**13.** **14.** **15.**).

224. Section 4.

Literature published on this subject deals with the problem only with special reference to practical cases. It has been treated more fully by Czapski (**3.** *232*. *254*.).

Chapter VIII.

PRISMS AND SYSTEMS OF PRISMS.

(F. Loewe.)*

1.—TRACE OF A SINGLE RAY THROUGH A PRISM AND A SYSTEM OF PRISMS.

225. Any refracting medium which is bounded by two planes of any extent is known optically as a **Prism.** The angle contained between the two bounding surfaces is called the **refracting angle** of the prism and the straight line in which the planes meet is called the **edge** of the prism. The bounding planes are also called the **faces** of the prism. In what follows we shall always understand by the first face of the prism that plane which is the first to receive the transmitted light, which in our diagram is represented as proceeding from left to right in the plane of the paper. A plane at right angles to the edge of the prism is referred to as a **principal plane** of the prism.

In investigating the chief properties of prisms and systems of prisms we are justified in considering single rays of light, since in the practical use of prisms we are almost exclusively concerned with pencils composed of parallel rays, that is with plane wave fronts, as represented by their normals or rays.

The course of a ray through a prism in obedience to Snell's law of refraction may be traced either graphically or by trigonometrical calculation.

A.—Graphic Method.

226. A graphic method for determining the path of a ray through a prism was first given by Young. Various graphic methods have been given by Herschel (*2.*), Reusch (*2.*), Radau (*1.*), Lommel (*1.*), Kessler (*1.*), Cornu (*2.*), and others. We only give

* This chapter is essentially an extended edition of the corresponding chapter of Czapski's "*Theorie der Optischen Instrumente nach Abbe.*" Of added sections it may suffice to mention those on *homocentric image-formation* and on the *rotation of the image.*

that of Reusch, derived from Young, which is the simplest and is still frequently employed in practice.

(a) Reusch's Graphic Determination of the Path of a Ray in the Principal Section of a Prism.

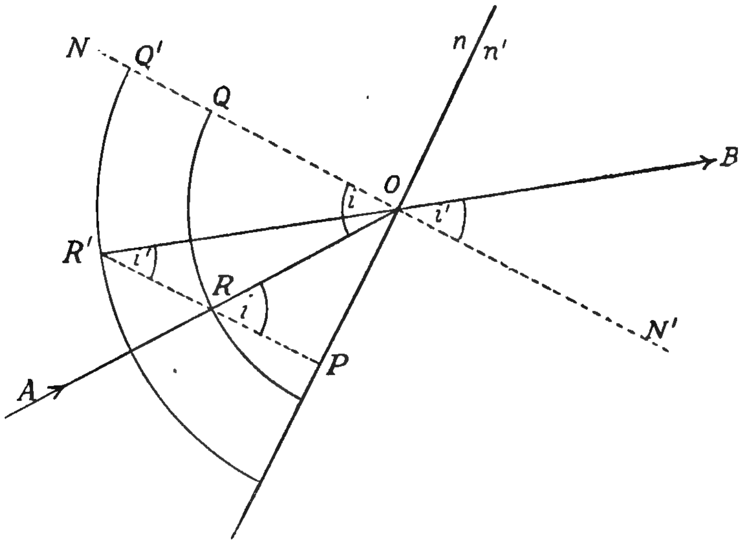

FIG. 88.

AO: Incident ray; OB: Refracted ray. $RO : R'O = n : n'$. Reusch's graphic method of determining the refraction of a ray at a plane.

In Fig. 88 let AO be an incident ray in a medium of refractive index n; let ON be the normal at O to the bounding plane separating the two media (n) and (n'). The direction of the refracted ray OB may then be found in the following manner:

About O as centre describe two circles with the radii $OQ = a \,.\, n$ and $OQ' = a \,.\, n'$, where a is any convenient constant. Let the incident ray AO in the medium (n) meet the circle having the radius $OQ = a \,.\, n$ at R. From R let fall a perpendicular RP upon the interface of the two media and let it be produced to meet the circle whose radius $OQ' = a \,.\, n'$ at R'. Then $R'O$ will be the direction of the refracted ray OB, that is $NOR' = N'OB = i'$.

This follows from the relations:

$$O\hat{R}P = i \quad \text{and} \quad \frac{\sin\ OR'P}{\sin\ ORP} = \frac{OR}{OR'} = \frac{n}{n'}.$$

Regarding the plane of separation of the two media as the first side of the prism GOH (Fig. 89) and retaining the

notation of Fig. 88, $\overrightarrow{OB}$ will be the direction of the ray through the prism. This ray OB will now undergo refraction at the second surface of the prism OH. The direction of the refracted ray may be found by the same graphic construction as shown above, with this difference only, that the ray is now incident in the

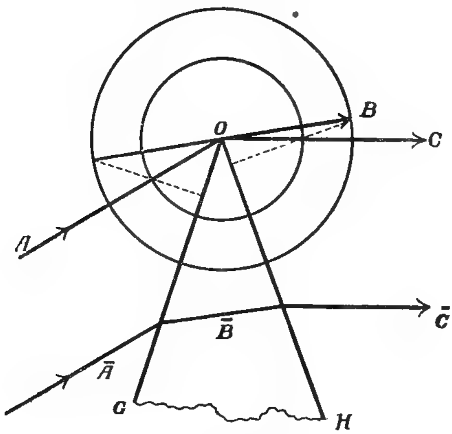

FIG. 89.

AO: Direction of the incident ray; OB: Direction of the emergent ray. Direction of refracted ray; OC: Reusch's graphic method of determining the refraction of a ray in the principal section of a prism.

medium (n') and that the angle of incidence is the angle comprised between the ray and the normal to the second face of the prism. By repeating the graphic method we thus find the direction $\overrightarrow{OC}$ of the refracted ray.

The lines AO, OB, OC accordingly represent the direction of the incident ray, the direction of the ray within the prism, and that of the emergent ray respectively. Lines $\bar{A}$, $\bar{B}$, $\bar{C}$ drawn parallel to these directions represent the path of a ray incident upon the first prism face at a given distance from the refracting edge.

In a **system of prisms** whose refracting edges are parallel to one another, and which have accordingly a common principal section, the path of the ray within this principal section may be ascertained by a continued repetition of Reusch's method. In applying this graphic method it is generally expedient to determine the path of the ray at the refracting edge (say at the first, second, third, &c.), and to represent the actual path of the ray through the prism by drawing parallel lines to these directions.

(*b*) Reusch's graphic method is applicable likewise to a ray traversing a prism **in any secondary section** as well as to a ray

passing through a system of prisms arranged in any desired manner. In this case the circles should be replaced by spheres whose radii should be in the ratio of the refractive indices of the adjoining media. Since the method is purely geometrical it will suffice to refer the reader to Reusch's paper.

B. Trigonometrical Determination of the Path of a Ray through a Prism.

227. Before proceeding with the trigonometrical determination of the path of a ray through a prism we shall adopt a notation and a convention of signs for the angles, in keeping with those of § 9.

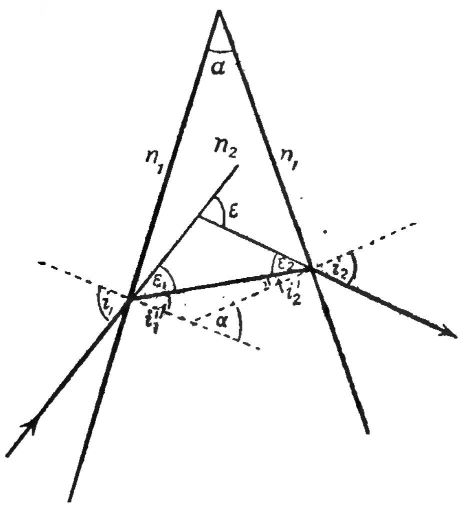

FIG. 90.

The path of a ray in the principal section of a prism surrounded by a less-refracting medium.

Let the angles of incidence and refraction at the v^{th} surface respectively be denoted by i_v and i_v'. Their signs are independent of the sense of the direction in which the light travels. They will be regarded as positive when a rotation of the ray about the point of incidence in a clockwise direction causes it to approach the normal.

The refracting angles a_v of the prisms are reckoned positive when, looking on the plane of the paper, their vertices lie on the left-hand side of an arrow moving with the ray. The sign of the refracting angle depends therefore solely upon the direction of the ray.

The deviation which a ray experiences by refraction at the v^{th} surface is reckoned as positive if the incident ray produced be made to approach the emerging ray when turned in a clockwise direction. It will be seen that the sign of the angle of deviation is governed by the direction of the ray.

228. Path of a Ray within the Principal Section of a Prism.—We shall first consider a ray traversing the prism in a **principal section,** as in Fig. 90.

Let the prism have the refractive index n_2 and let it be surrounded by an optical medium having a refractive index n_1, greater or less than n_2.

The path of a ray will then be determined by the following equations:

$$\left.\begin{aligned} n_1 \sin i_1 &= n_2 \sin i'_1 \\ i_2 &= i'_1 - a \\ n_2 \sin i_2 &= n_1 \sin i'_2 \end{aligned}\right\} \cdots \quad \cdots \quad \cdots \quad (1)$$

$$\left.\begin{aligned} \varepsilon_1 &= i_1 - i'_1 \\ \varepsilon_2 &= i_2 - i'_2 \\ \varepsilon_1 + \varepsilon_2 &= \varepsilon = i_1 - i'_2 - a \end{aligned}\right\} . \quad \cdots \quad \cdots \quad (2)$$

In equations (1), if we denote the refractive index of the prismatic medium relative to the surrounding medium, namely $\frac{n_2}{n_1}$, by n, noting that $n \gtrless 1$, we can write the equation (1) in the following form:

$$\sin i'_1 = \frac{1}{n} \sin i_1, \quad i_2 = i'_1 - a, \quad \sin i'_2 = n \sin i_2 . \qquad (3)$$

The total deviation $\varepsilon_1 + \varepsilon_2 = \varepsilon$, which the ray experiences in its transmission through the prism is completely determined as a function of i_1, a, and n, and cannot vanish so long as $n \gtrless 1$ and $a \gtrless 0$. On the other hand, every value of the deviation ε has two values of the angle of incidence i_1 corresponding to it.

If we consider two rays, one having an angle of incidence $i_1 = a$ and an angle of emergence $i'_2 = b$, whilst the other has an angle of incidence $(i_1) = -b$ and an angle of emergence $(i'_2) = -a$, then the deviation of the first ray will be:

$$\varepsilon = i_1 - i'_2 - a = a - b - a ,$$

and the deviation of the second ray will be:—

$$(\varepsilon) = (i_1) - (i'_2) - a = -b + a - a ,$$

so that

$$(\varepsilon) = \varepsilon .$$

It will thus be seen that a ray which enters a prism at an angle i_2' and another ray which emerges from the prism at the same angle reckoned as negative, experience the same deviation.

Such a pair of double direction rays includes a ray for which $i_1 = (i_1) = -i'_2$. This ray passes symmetrically through the prism. It traverses at right angles the bisecting plane of the prism angle and it has the further remarkable property that the deviation which the prism imparts to the ray is a minimum value, as we shall presently show.

Let the suffix $_0$ be added to the symbols of the angles which occur when a ray passes **symmetrically** through the prism.

229. In order that the deviation ε_0 may be a minimum it must satisfy the conditions

$$\left(\frac{\delta\varepsilon}{\delta i_1}\right)_0 = 0 \text{ and } \left(\frac{\delta^2\varepsilon}{\delta i_1^2}\right)_0 > 0 \,. \quad \dots \quad (4)$$

By (2)

$$\left(\frac{\delta\varepsilon}{\delta i_1}\right)_0 \equiv \left(\frac{\delta\,[i_1 - i_2' - a]}{\delta i_1}\right)_0 = 1 - \left(\frac{\delta i'_2}{\delta i_1}\right)_0 = 0$$

and thus

$$\left(\frac{\delta i'_2}{\delta i_1}\right)_0 = 1 \,. \quad \dots \quad \dots \quad (5)$$

Differentiating equations (3) it follows that

$$\frac{\delta i_1'}{\delta i_1} = \frac{\tan i'_1}{\tan i_1}, \quad \delta i'_1 = \delta i_2 \,, \quad \frac{\delta i'_2}{\delta i_2} = \frac{\tan i'_2}{\tan i_2} \,;$$

hence

$$\left(\frac{\delta i'_2}{\delta i_1}\right)_0 = \left(\frac{\delta i'_2}{\delta i_2} \cdot \frac{\delta i'_1}{\delta i_1}\right)_0 = \left(\frac{\tan i'_2}{\tan i_2} \cdot \frac{\tan i'_1}{\tan i_1}\right)_0 = 1,$$

or

$$\left(\frac{\tan i_1}{\tan i'_1}\right)_0 = \left(\frac{\tan i'_2}{\tan i_2}\right)_0 . \quad \dots \quad \dots \quad (6)$$

By virtue of equation (3) the left side of equation (6) represents a single-value function of i_1 in the same way that the right side is a function of i'_2. We accordingly have the alternative equations

$$\begin{aligned} i_1 &= +\,i'_2 \\ i'_1 &= +\,i_2 \end{aligned} \quad \text{or} \quad \begin{aligned} i_1 &= -\,i'_2 \\ i'_1 &= -\,i_2 \,. \end{aligned}$$

The first pair of equations would make $a = 0$, and therefore cannot be fulfilled by a prism. The only realisable conditions are therefore represented by the relations :

$$\begin{aligned} i_1 &= -\,i'_2 \\ i'_1 &= -\,i_2 \end{aligned} \,, \quad \dots \quad \dots \quad \dots \quad (7)$$

and, as we have seen above, these are the values which occur when the path of the rays is symmetrical with respect to the prism.

It follows therefore that the deviation cannot be a minimum unless the path of the ray through the prism is symmetrical.

To determine the sign of $\frac{\delta^2 \varepsilon_0}{\delta i_1^2}$ we proceed from the relation contained in equation (2), and we find

$$\frac{\delta \varepsilon}{\delta i_1} = \frac{\delta (\varepsilon_1 + \epsilon_2)}{\delta i_1} = \frac{\delta \varepsilon_1}{\delta i_1} + \frac{\delta \varepsilon_2}{\delta i'_2} \cdot \frac{\delta i'_2}{\delta i_1},$$

$$\frac{\delta^2 \varepsilon}{\delta i_1^2} = \frac{\delta^2 \varepsilon_1}{\delta i_1^2} + \frac{\delta^2 \varepsilon_2}{\delta i'_2 \, \delta i_1} \cdot \frac{\delta i'_2}{\delta i_1} + \frac{\delta^2 \varepsilon_2}{\delta i'_2} \cdot \frac{\delta^2 i'_2}{\delta i_1^2}.$$

For the symmetrical path we found that $\left(\frac{\delta \varepsilon_2}{\delta i'_2}\right)_0 = 0$ and $\delta i_1 = \delta i_2'$.

Hence

$$\left(\frac{\delta^2 \varepsilon}{\delta i_1^2}\right)_0 = \left(\frac{\delta^2 \varepsilon_1}{\delta i_1^2}\right)_0 + \left(\frac{\delta^2 \varepsilon_2}{\delta i'^2_2}\right)_0.$$

Also, since ε_1 is the same function of i_1, that ε_2 is of i'_2 it follows that

$$\left(\frac{\delta^2 \varepsilon_1}{\delta i_1^2}\right)_0 = \left(\frac{\delta^2 \varepsilon_2}{\delta i'^2_2}\right)_0$$

and

$$\left(\frac{\delta^2 \varepsilon}{\delta i_1^2}\right)_0 = 2 \left(\frac{\delta^2 \varepsilon_1}{\delta i_1^2}\right)_0.$$

The sign of $\left(\frac{\delta^2 \varepsilon}{\delta i_1^2}\right)_0$ is accordingly the same as that of $\left(\frac{\delta^2 \varepsilon_1}{\delta i_1^2}\right)_0$.

For the calculation of $\left(\frac{\delta^2 \varepsilon_1}{\delta i_1^2}\right)_0$, we have by equation (2)

$$\left(\frac{\delta \varepsilon_1}{\delta i_1}\right)_0 = \left(\frac{\delta (i_1 - i_1')}{\delta i_1}\right)_0 = 1 - \frac{\delta i'_{01}}{\delta i_1} = 1 - \frac{\tan i'_{01}}{\tan i_{01}},$$

whence

$$\left(\frac{\delta^2 \varepsilon_1}{\delta i_1^2}\right)_0 = \frac{\delta}{\delta i_1}\left(1 - \frac{\tan i_1'}{\tan i_1}\right)_0 = - \frac{\delta}{\delta i_1}\left(\frac{\sin i_1' \cos i_1}{\sin i_1 \, \cos i_1'}\right)_0$$

$$= - \frac{1}{n} \frac{\delta}{\delta i_1} \left(\frac{\cos i_1}{\cos i_1'}\right)_0$$

$$\left(\frac{\delta^2 \varepsilon_1}{\delta i_1^2}\right)_0 = + \frac{1}{n} \left(\frac{\sin (i_1 - i_1')}{\cos^2 i_1'}\right)_0 = \frac{1}{n} \cdot \frac{\sin \varepsilon_{01}}{\cos^2 i'_{01}} \cdot \quad \ldots \qquad (8)$$

In the expression for $\left(\frac{\delta^2 \varepsilon_1}{\delta i_1{}^2}\right)_0$ in equation (8) all terms are positive with the exception of $\sin \varepsilon_{01}$. The sign of $\left(\frac{\delta^2 \varepsilon_1}{\delta i_1{}^2}\right)_0$, and hence also that of $\frac{\delta^2 \varepsilon_0}{\delta i_1{}^2}$, is therefore the same as that of $\sin \varepsilon_{01}$.

So far we have not considered the sign of $n = \frac{n_2}{n_1}$. Now, in discussing the sign of $\frac{\delta^2 \varepsilon_0}{\delta i_1{}^2}$ we have to distinguish the two cases in which $n > 1$ and $n < 1$.

230. First Case, in which $n > 1$, *i.e.* $n_2 > n_1$; that is to say, the prism is surrounded by a medium whose refractive index is less than that of the prism.

In this case $i_{01} > i'_{01}$, $i_{01} - i'_{01} \equiv \varepsilon_1 > 0$, and $\frac{\delta^2 \varepsilon_0}{\delta i_1{}^2}$ is positive, so that we have here the condition of minimum deviation.

231. Second Case in which $n < 1$ (Fig. 91), that is, the prism is less dense than the surrounding medium.

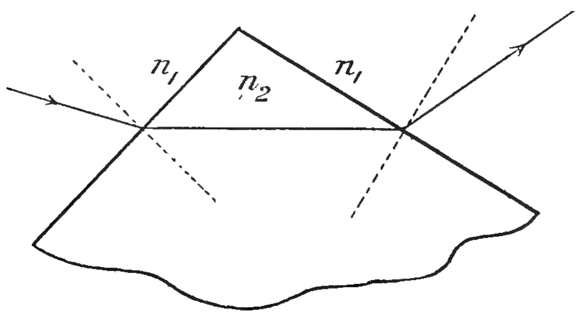

FIG. 91.

Trace of a ray in the principal section of a prism surrounded by a medium whose refractive index is greater than that of the prism.

In this case $i_{01} < i'_{01}$, and $\frac{\delta^2 \varepsilon_0}{\delta i_1{}^2}$ is negative, and the **deviation** therefore reaches its **maximum** value. This somewhat remarkable result will be readily understood if we consider that in this case the deviation has a negative value according to our convention of signs, viz.

$$\varepsilon_0 = 2\,\varepsilon_{01} = 2\,(i_{01} - i'_{01}) < 0\,.$$

When the path of a ray through a prism is symmetrical the negative deviation attains accordingly its greatest value, *i.e.* its absolute amount is a minimum.

232. Case of a Prism surrounded by air.—If now we apply these results to the case which occurs most frequently in practice, viz. when the prism is surrounded by air as the outer medium, we have $n_1 = n_3 = 1$ and $n_2 \equiv n > 1$. The path of a ray is then determined by the equations (3) and (2), *i.e.*,

$$(3) \quad \begin{array}{l} \sin i'_1 = \frac{1}{n} \sin i_1 \\ i_2 = i'_1 - a \\ \sin i'_2 = n \sin i_2 \end{array} \qquad (2) \quad \begin{array}{l} \varepsilon_1 = i_1 - i'_1 \\ \varepsilon_2 = i_2 - i'_2 \\ \varepsilon_1 + \varepsilon_2 \equiv \varepsilon = i_1 - i'_2 - a \, , \end{array}$$

and the minimum deviation, in particular, has the following values:

$$\begin{array}{l} i_{01} = - i'_{02} \\ i'_{01} = - i_{02} \end{array} \quad \text{and} \quad \left\{ \begin{array}{l} \varepsilon_0 = 2\, i_{01} - a \\ i_{01} = \dfrac{\varepsilon_0 + a}{2} \end{array} \right. \quad \ldots \quad (9),$$

and since $\sin i_{01} = n \sin i'_{01}$, we obtain finally

$$\sin \frac{\varepsilon_0 + a}{2} = n \sin \frac{a}{2} . \quad \ldots \quad \ldots \quad (10)$$

Equation (10) embodies the principle upon which Fraunhofer based his method of determining the refractive indices of prisms.

If we cause the angle of incidence to increase gradually from the value i_{01} conforming to the condition of minimum deviation, the deviation ε will likewise increase, and, as was shown by the corollary in § 17, it does so at a greater rate than i_1. The greatest possible angle of incidence $i_1 = +90^\circ$ has accordingly corresponding to it the greatest value of ε, *i.e.* the ray which enters the prism at grazing incidence in the direction towards the refracting edge, experiences its greatest deviation $\bar{\varepsilon} = 90^\circ - \bar{i}'_2 - a$. In § 228 we saw that a similar maximum deviation is experienced by a ray which enters the prism at an angle of incidence $i_1 = -\bar{i}'_2$ and which leaves it at grazing emergence. As regards the path of a ray which enters the prism by grazing incidence at an angle $i_1 = -90^\circ$, that is to say in a direction from the edge towards the base of the prism, nothing can be stated at present.

In deriving the preceding conclusions from the equations (2), (3), (9) and (10) we have tacitly supposed that every ray so considered may traverse the prism in any desired manner, whereas in reality this is by no means the case.

In general, when light passes from a less refracting into an optically denser medium every angle of incidence in the less refracting medium has a real angle of refraction corresponding to it in the denser medium. This statement is not, however, reversible, since in the transition from the denser to the less refracting medium an angle of incidence i conforming to the relation

$$n \sin i = \sin 90^\circ = 1 \quad \ldots \quad \ldots \quad (11)$$

has already corresponding to it the greatest angle of refraction which it is possible to conceive, viz., 90°, as represented by the ray B in Fig. 92. Any ray which meets the face of the prism at a greater angle of incidence ($\bar{i} > i$) does not undergo refraction at this face of the prism but is **totally reflected** into the substance of the prism. This angle i is accordingly called the **critical angle of total reflection.** According to equation (11) this angle is $\sin^{-1}\frac{1}{n}$. In practical optics it supplies an important means of determining refractive indices. Total reflectometers embodying this principle have been devised by Kohlrausch (*1.*), Abbe (*3.*), Pulfrich (*1. 2.*).

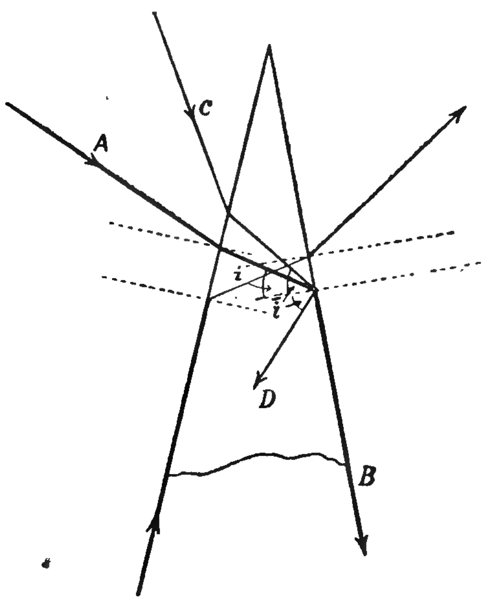

FIG. 92.

Trace of a ray which enters the prism at grazing incidence and leaves it at grazing emergence, and is totally reflected at the second prism face.

233. Special Cases.—Whereas, in the case of a prism surrounded by air all rays of a pencil meeting the first face of the prism necessarily enter the prism, the second face of the prism will transmit only those whose angles of incidence i_2 are less than, or equal to, the critical angle according to equation (11).

In the special case in which a ray enters the prism at grazing incidence and leaves it again at grazing emergence, the values of both i'_1 and i_2, irrespective of their sign, are equal to that of the critical angle $\mathrm{i} = \sin^{-1}\frac{1}{n}$. Owing to the symmetry of the path of the ray the deviation is a minimum, and therefore

$$\bar{a} = 2\,\mathrm{i} = 2\sin^{-1}\frac{1}{n}$$

or

$$\sin\frac{\bar{a}}{2} = \frac{1}{n}. \quad \dots \quad \dots \quad \dots \quad (12)$$

This value of $\bar{a}$, which appears as a function of n, is thus the greatest refracting angle which a prism of refractive index n may be given in order that the light may pass through the prism by pure refraction at the two faces of the prism, that is more particularly, without undergoing internal reflection.

The deviation $\bar{\varepsilon}$ in this case is $\bar{\varepsilon} = 180° - \bar{a}$.

The subjoined Table I gives for a series of refractive indices the greatest values which the refracting angle $\bar{a}$ may assume in accordance with equation (12) and the corresponding deviations $\bar{\varepsilon}$.

TABLE I.

$n = 1{\cdot}3$	1·4	1·5	1·6	1·7	1·8	1·9	2·0
$\bar{a} =$ 100° 34′	91° 10′	83° 37′	77° 22′	72° 4′	67° 30′	63° 30′	60° 0′
$\bar{\varepsilon} =$ 79° 26′	88° 50′	96° 23′	102° 38′	107° 56′	112° 30′	116° 30′	120° 0′

Owing to the great loss of light by reflection with which every oblique incidence is attended it is usual to give the prisms much smaller refracting angles than the calculated limiting values.

Another interesting special case is that in which the refracting angle of the prism is equal to the critical angle of total reflection corresponding to the material of the prism, *i.e.* $a = \sin^{-1}\frac{1}{n}$ or

$\sin \alpha = \frac{1}{n}$. In this case the ray which enters the prism at right angles to the entrance face leaves it at grazing emergence, and conversely (Fig. 93).

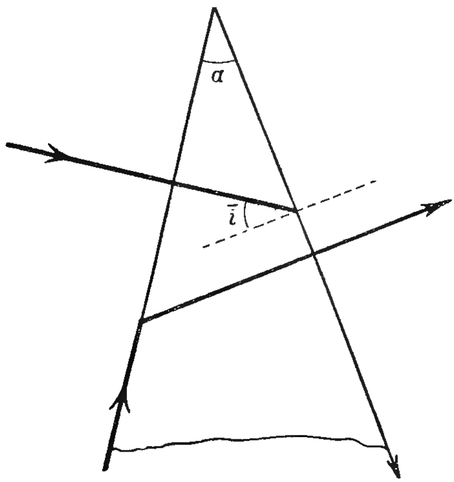

FIG. 93.

Grazing incidence and normal emergence.

This case embodies the principle of a method originated by Pulfrich (*3.*), by which refractive indices of fluids can be measured without restriction as regards the magnitudes of the refractive indices.

The subjoined Table II. illustrates the changes of the deviation in the case of a prism of refractive index $n = 1{\cdot}6$ and having a refracting angle $\alpha = 30°$.

TABLE II.

	$n = 1{\cdot}6$.			$\alpha = 30°$.	
Angle of incidence i_1	$+ 90°$	$+ 53° \; 8'$	$+ 24° \; 28'$	$\pm 0°$	$- 13° \; 59'$
Deviation ε	$+ 46° \; 1'$	$+ 23° \; 8'$	$+ 18° \; 56'$	$+ 23° \; 8'$	$+ 46° \; 1'$
Angle of emergence i_2'	$+ 13° \; 59'$	$\pm \; 0°$	$- 24° \; 28'$	$- 53° \; 8'$	$- 90°$

It will be seen that this prism will transmit those rays only whose angles of incidence i_1 range from $+ 90°$ to $- 13° \; 59'$. All the rays

whose angles of incidence range from $-90°$ to $-13° 59'$ are totally reflected at the second face of the prism. This applies in particular to a ray which, proceeding in a direction away from the refracting edge of the prism, enters the latter at grazing incidence. Moreover, it may easily be shown that this is a general property of the prism and that it is immaterial whether the prism is situated in a medium of a higher or a lower refractive index than its own.

If, in accordance with Heath (*2. 32*), we assume the refracting angle α of a prism to be so small that its sine may be replaced by the arc and that $\cos \alpha$ may be taken equal to unity, it follows that the deviation ε is likewise small, and since

$$\varepsilon = i_1 - i'_2 - \alpha ,$$

it follows that

$$\sin i'_2 = \sin [i_1 - (\varepsilon + \alpha)] = \sin i_1 - (\varepsilon + \alpha) \cos i_1 ;$$

and, on the other hand,

$$\sin i'_2 = n . \sin i_2 = n . \sin (i'_1 - \alpha) = n . \sin i'_1 - n . \alpha . \cos i'_1 .$$

Equating the right sides of these equations, we obtain accordingly

$$\sin i_1 - (\varepsilon + \alpha) \cos i_1 = \sin i_1 - n . \alpha . \cos i'_1$$

and, finally, for finite angles of incidence

$$\varepsilon = \alpha \left(n \frac{\cos i'_1}{\cos i_1} - 1 \right) .$$

If now we restrict the consideration to very small angles of incidence, in which case $\cos i'_1 = \cos i_1 = 1$, the approximate value of the deviation becomes

$$\varepsilon = \alpha (n - 1) , \qquad \dots \qquad \dots \qquad (13)$$

and it will be seen that it is independent of the angle of incidence.

234. Trace of a Ray in the Principal Section of a System of Prisms.—We now consider a system of *k* refracting planes, *i.e.*, of *k—1* prisms, and we shall suppose that the principal sections of all prisms coincide, or, which comes to the same thing, that all the prism edges are parallel, as represented in Fig. 94.

Retaining the notation adopted in equations (1) and (2) as well as the conventions respecting the signs given at the beginning of this

chapter, we can determine the path of a ray within the principal section by means of the following set of equations, as given by Gleichen (*3.*) and Czapski (*3.137*) :

$$\left.\begin{aligned} n_2 \sin i'_1 &= n_1 \sin i_1 \\ i_2 &= i'_1 - a_1 \\ n_3 \sin i'_2 &= n_2 \sin i_2 \\ i_3 &= i'_2 - a_2 \\ \cdot\ \cdot\ \cdot\ &\cdot\ \cdot\ \cdot\ \cdot \\ n'_k \sin i'_k &= n_k \sin i_k \end{aligned}\right\} (14) \qquad \left.\begin{aligned} \varepsilon_1 &= i_1 - i'_1 \\ \varepsilon_2 &= i_2 - i'_2 \\ \cdot\ \cdot\ &\cdot\ \cdot\ \cdot \\ \varepsilon_k &= i_k - i'_k . \end{aligned}\right\} (15)$$

Hence the resulting deviation is

$$\varepsilon^{(k)} = \sum_{v=1}^{k} \varepsilon_v = i_1 - i'_k - \sum_{v=1}^{k} a_v$$

where $\sum_{v=1}^{k} a_v$ is the algebraical sum of all the refracting angles and is therefore nothing more nor less than the angle comprised between the first and the last surfaces of the prism, which we denote by $a_{1\,k}$, that is $\varepsilon^{(k)} = i_1 - i'_k - a_{1\,k}$.

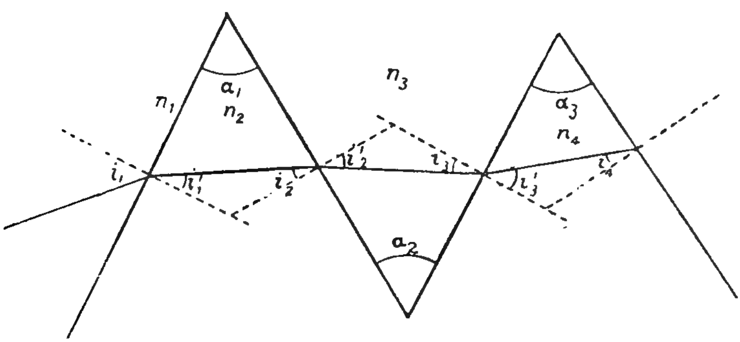

FIG. 94.

Trace of a ray in the principal section of a system of prisms.

The resulting deviation $\varepsilon^{(k)}$ is a function of the angle of incidence at the first surface, of the refractive indices n_1 to n'_k, and the refracting angles a_1 to a_{k-1}.

Assuming that $\frac{\delta^2 \varepsilon^{(k)}}{\delta i_1^2} > 0$, the condition

$$\frac{\delta \varepsilon^{(k)}}{\delta i_1} = 0 \text{ or } \delta i_1 = \delta i'_k \qquad \dots \qquad \dots \qquad (16)$$

defines the angle of incidence for which the deviation $\varepsilon^{(k)}$ is a minimum.

A set of equations which results from the differentiation of equations (14) will give us $\delta i'_k$ as a function of δi_1 thus:

$$\delta i'_1 = \frac{n_1}{n_2} \cdot \frac{\cos i_1}{\cos i'_1} \cdot \delta i_1 ,$$

$$\delta i'_2 = \frac{n_2}{n_3} \cdot \frac{\cos i_2}{\cos i'_2} \cdot \delta i_2 , \text{ where } \delta i_2 = \delta i'_1 , \quad \ldots \quad (17)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\delta i'_k = \frac{n_k}{n'_k} \cdot \frac{\cos i_k}{\cos i'_k} \cdot \delta i_k , \text{ where } \delta i_k = \delta i'_{k-1} ;$$

hence finally,

$$\delta i_k' = \frac{n_1}{n'_k} \cdot \frac{\cos i_1}{\cos i'_1} \cdot \frac{\cos i_2}{\cos i'_2} \cdot \ldots \cdot \frac{\cos i_k}{\cos i'_k} \cdot \delta i_1 . \quad \ldots \quad (18)$$

In this expression $\frac{\cos i_v}{\cos i'_v}$, the ratio of the cosines of the angles of incidence and refraction at any of the successive surfaces, is a function of the angle of incidence i_v and the relative refractive index of the two media, *i.e.* $\frac{n_v}{n'_v}$. Denoting the products by the symbol $\prod$ and noting that $\delta i_1 = \delta i'_k$ at minimum deviation, by equation (16),

$$n_1 \cdot \prod_{v=1}^{k} \cos i_v = n'_k \prod_{v=1}^{k} \cos i'_v ,$$

or, when the first and the last media are air, and therefore $n_1 = n'_k = 1$,

$$\prod_{v=1}^{k} \cos i_v = \prod_{v=1}^{k} \cos i'_v . \quad \ldots \quad \ldots \quad (19)$$

When $k = 2$, *i.e.* in the case of a single prism, the expression

$$\cos i_1 . \cos i_2 = \cos i'_1 . \cos i'_2 \ldots \quad \ldots \quad (19a)$$

becomes the condition of minimum deviation, and by reason of the law of refraction this agrees with the condition which we have previously established for minimum deviation in a single prism, viz.

$$\tan i_1 . \tan i_2 = \tan i'_1 . \tan i'_2 . \quad \ldots \quad (6)$$

235. Case of Two Crossed Prisms.—We now describe by an example the condition under which it is possible and advantageous to obtain the effect of a single prism by the use of two. The general case of prisms in air occupying any positions relatively

to one another is difficult to follow. We shall therefore confine our attention to the following special case which is of some practical importance.

Let two prisms similar in form and of similar substance (*see* Fig. 95) be separated by a thin plane parallel stratum of air, and let it be assumed that they are capable of being rotated about the normal c to the two adjacent prism sides. Let ρ be the angle of rotation in any given position of the two prisms, between the refracting edge of prism I and the vertical projection of the refracting edge of prism II upon the inside face $O_1\,Q_1$ of the first prism. The relation between ρ and the angle α contained between the outer prism sides $O_1\,P_1$ and $O_2\,P_2$ may then be derived in the following manner:

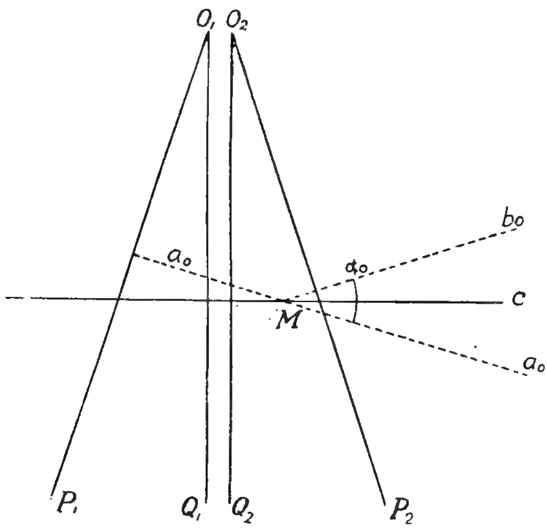

FIG. 95.

A system of prisms consisting of two similar prisms capable of being rotated about an axis c. The refracting angle is capable of being varied continuously from 0 to $\alpha_0 = 2\ P_1O_1Q_1$, whilst the refracting edge remains stationary.

Let the prism edges O_1 and O_2 be initially parallel to one another and let them be situated on the same side of the axis c. From any point M on the axis let fall perpendiculars a_0, b_0 upon the prism sides O_1P_1 and O_2P_2 respectively, and let them include an angle $\alpha_0 = 2\ P_1O_1Q_1 = 2\ P_2O_2Q_2$, which is accordingly bisected by c. Retaining prism II in position, suppose prism I to be turned about c. Then, as is shown in Fig. 96, the point A_0 on the normal a_0

of prism I will describe a circle having its centre on c. This circle appertains to the sphere which can be described about the centre M with the radius $MA_0 = MB_0$. Its pole is the point of intersection C of the axis of rotation c, and the points of intersection A_0 and B_0 of the normals a_0 and b_0 are on one and the same meridian. The resulting refracting angle of the prismatic system in the initial position, viz. a_0, is the angle subtended at the centre by the arc A_0CB_0. When prism I has been rotated through an angle A_0CA, denoted by ρ, the angle a subtended by the arc AB_0

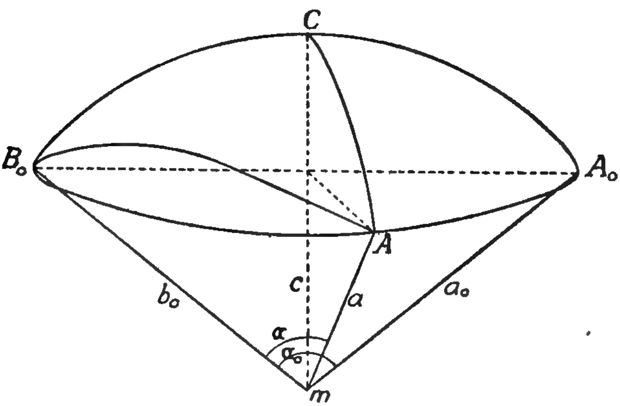

FIG. 96.

Diagram representing the connection between the angle of rotation ρ and the resulting refracting angle a. The plane of the paper is the principal section of the stationary prism. MC is the axis of rotation, MA is the normal to the plane P_1O_1 of Fig. 95.

will then be the angle included between the normal a to the prism side O_1P_1 and the normal b_0 to the prism side O_2P_2, *i.e.* it will be the new resulting refracting angle.

In the spherical triangle ACB_0

$$\cos AB_0 = \cos CB_0 \cos CA + \sin CB_0 \sin CA \cos B_0CA,$$

or, since

$$CB_0 = CA = \frac{a_0}{2}, \text{ and } A\hat{C}B_0 = 180° - A\hat{C}A_0 = 2\,\mathrm{R} - \rho,$$

$$\cos a = \cos^2\frac{a_0}{2} - \sin^2\frac{a_0}{2}\cos\rho\,.$$

In the initial position $\rho = 0$ and $a = a_0$. When the prism has been turned through 180°, $a = 0$, *i.e.* the prism combination will then act as a plane parallel plate.

If prism II be turned out of its initial position through an angle ρ with respect to prism I it follows from the conditions of symmetry that the resulting refracting edge will be turned through an angle $\frac{\rho}{2}$. If it be required that this refracting edge shall retain its initial position it will be necessary to turn both prisms from the initial positions by equal amounts in opposite directions.

An arrangement of two identical prisms capable of being symmetrically rotated, especially within the region in the neighbourhood of the position corresponding to $\rho = 180°$ and $a = 0$, serves to impart to a pencil of rays deflections of any desired sign within a certain plane. It is equivalent to a prism whose refracting angles can be continuously varied within the limits zero and a_0, whilst the refracting edge remains stationary, a_0 having double the value of the refracting angle of a component prism.

All the properties of the prism and combinations of prisms considered so far, have been derived on the assumption that the ray considered lies in the principal section of the prism or of the prismatic combination.

In what follows we shall abandon this restriction, in order to deal with the more general problem of a ray lying outside the principal plane.

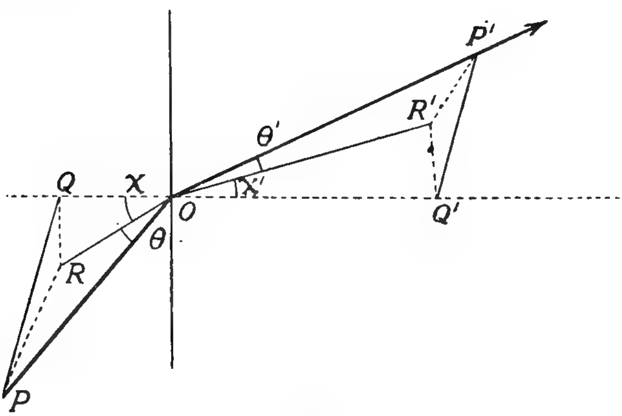

FIG. 97.

POP is the trace of a ray not in the principal section which coincides with the plane of the paper. *ROR'* is the projection of the ray upon the principal section. *PQ*, *P'Q'* are perpendiculars let fall upon the normals of incidence.

236. The Trace of a Ray not lying in the Principal Section of a Prism.—We shall again suppose the principal section of one prism to be in the plane of the paper, and we shall further suppose that the latter contains the point of incidence O, as

represented in Fig. 97. If now the path of the incident ray PO lies below the plane of the paper, it follows that the refracted ray OP' will pass above the plane of the paper. Make $PO = 1$ and $OP' = n$, and from P, P' let fall the perpendiculars PQ, $P'Q'$ upon the normal of incidence. Then the angle $QOP = i_1$, $Q'OP' = i'_1$, and

$$PQ = PO \sin i_1 = \sin i_1,$$

$$P'Q' = OP' \sin i_1' = n \sin i_1',$$

i.e.

$$PQ = P'Q'.$$

These two perpendiculars to the normal of incidence are situated in the plane of incidence, which is inclined to the principal plane, and they are parallel to one another and of equal magnitude.

Now, let P and P' be projected upon the principal plane and let the projections of P and P' be denoted by R and R'. Then the intervals QR and $Q'R'$, being the projections of PQ and $P'Q'$, are equal.

Let θ and θ' denote the acute angles ROP and $R'OP'$ which the incident and refracted rays form with the principal plane, as in § 15.

Then

$$PR = PO \,.\, \sin\theta = \sin\theta$$

and

$$P'R' = OP' \,.\, \sin\theta' = n \sin\theta',$$

and, since $PR = P'R'$, we have finally

$$\sin\theta = n \sin\theta',$$

which signifies that the angles which the incident and the refracted rays form with the principal section obey the law of refraction.

Also, let the acute angles QOR and $Q'OR'$ which the projections of the incident and the refracted rays form with the normals (denoted by ϕ, ϕ' in § 15) be represented by χ and χ'. Then

$$OR = \frac{QR}{\sin\chi} = OP \,.\, \cos\theta = \cos\theta$$

$$OR' = \frac{Q'R'}{\sin\chi'} = OP' \,.\, \cos\theta' = n\cos\theta'$$

and, since $QR = Q'R'$

$$\cos\theta \sin\chi = n \cos\theta' \sin\chi'$$

or

$$\frac{\sin\chi}{\sin\chi'} = n \,.\, \frac{\cos\theta'}{\cos\theta}\,.$$

The projections RO of the incident ray and OR' of the refracted ray are subject to a law which is analogous to the law of refraction, the normal ratio of the refractive indices being now replaced by

the coefficient $n \frac{\cos\theta'}{\cos\theta}$, that is, by a value which depends upon the inclination of the incident ray to the principal section.

The refraction of a ray which meets the first prism side at O, and makes an angle i_1 with the normal and an angle θ_1 with the principal section, and whose projection upon the principal section is inclined to the normal at an angle χ_1, is determined by the following equations :

$$\left.\begin{aligned} \sin i'_1 &= \frac{1}{n} \sin i_1 \\ \sin \theta'_1 &= \frac{1}{n} \sin \theta_1 \\ \sin \chi'_1 &= \frac{1}{n} \sin \chi_1 \frac{\cos\theta_1}{\cos\theta_1'} . \end{aligned}\right\} \quad \ldots \quad (20)$$

With respect to the second prism side let the principal section of the prism be the plane of projection. Then $\theta_2 = \theta'_1$; and, since

$$\sin \theta'_2 = n \,.\, \sin \theta_2 = n \,.\, \sin \theta'_1,$$

it follows that

$$\theta'_2 = \theta_1.$$

That is to say, the ray emerging from the prism has the same inclination to the principal section as the incident ray.

Let this angle of inclination be denoted accordingly by θ with respect to air and θ' with respect to the substance of the prism. Then the trace of the projection of the ray upon the principal section within the latter is determined by the following equations :

$$\sin \theta' = \frac{1}{n} \sin \theta$$

$$n \,.\, \sin \chi'_1 \,.\, \cos \theta' = \sin \chi_1 \,.\, \cos \theta \quad \ldots \quad (21)$$

$$\chi_2 = \chi'_1 - \alpha \quad \ldots \quad \ldots \quad (21a)$$

$$\sin \chi'_2 \,.\, \cos \theta = n \sin \chi_2 \,.\, \cos \theta' . \quad \ldots \quad (21b)$$

We may now determine the **deviation** E which the ray experiences in traversing the prism, as well as the **deviation** η imparted to the projection of the ray upon the principal section.

To this end, as we are solely concerned with the direction of the rays and not with the lengths of their paths within the prism, let $\overrightarrow{OP}$ (Fig. 98) be drawn parallel to the incident ray through a point O in the refracting edge $\overleftrightarrow{JK}$ of the prism, and, similarly, $\overrightarrow{OP'}$ parallel to the emerging ray. Also, let the plane

which is defined by JK and OP' be the plane of the paper. About O as centre let an auxiliary sphere be described which is intersected by $\overleftrightarrow{JK}$ at J and K, and by $\overrightarrow{OP}$ and $\overrightarrow{OP'}$ at P and P' respectively.

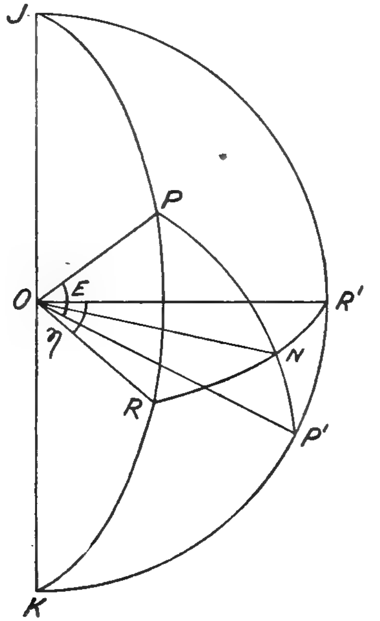

FIG. 98.

Deviation E of a ray lying out of the principal section. η is the projection of E upon the principal section ROR', JK being the refracting edge.

Then the angle POP' represents the deviation E which the prism imparts to the ray. Now, through O let the principal section (or the equatorial plane) be described at right angles to KJ, and at R let it intersect the meridian KPJ passing through P, and at R' the meridian passing through P'. Then ROR' represents the projection η of the deviation E upon the principal section, and $POR = P'OR'$ represents the inclination θ to the principal plane of the ray in air.

Let the great circle which joins P and P' meet the equator RR' at N. Then, since the right-angled spherical triangles PRN, $P'R'N$ are equal in every respect, it follows that

$$RN = R'N = \frac{1}{2} RR' = \frac{1}{2}\eta$$

and

$$PN = P'N = \frac{1}{2} PP' = \frac{1}{2} E.$$

The relation obtaining in this triangle PRN

$$\cos \widehat{PN} = \cos \widehat{RN} \, . \cos \widehat{RP}$$

may now be written thus :

$$\cos \frac{1}{2} E = \cos \frac{1}{2} \eta \, . \cos \theta \, . \quad \ldots \qquad \ldots \qquad (22)$$

The **deviation** η, **within a principal section** attains its minimum value η_0 in the same way as the deviation of an actual ray, when

$$\chi_1 = -\chi'_2 = \frac{1}{2}(\eta_0 + a) \text{ and } \chi_1' = -\chi_2 = \frac{a}{2}.$$

Substituting these values in equation (21*b*), we obtain accordingly

$$\sin \frac{1}{2}(\eta_0 + a) = n \, . \sin \frac{a}{2} \, . \frac{\cos \theta'}{\cos \theta} \, . \qquad .. \qquad (23)$$

When $\theta = 0 = \theta'$, equation (23) and all the angles contained therein will be the same as in the case of a ray within the principal section ; that is

$$\sin \frac{1}{2}(\eta_0 + a)_{\theta = 0} = n \sin \frac{a}{2} = \sin \frac{1}{2}(\varepsilon_0 + a) \ldots \qquad (24)$$

or, for a finite value of θ, if we substitute in equation (23) the value of $n \sin \frac{a}{2}$, as given in equation (24),

$$\sin \frac{1}{2}(\eta_0 + a) = \sin \frac{1}{2}(\varepsilon_0 + a) \frac{\cos \theta'}{\cos \theta} \, . \ldots \qquad (25)$$

Now, when $n > 1$, $\theta' < \theta$ in all cases, so that $\frac{\cos \theta'}{\cos \theta} > 1$, and hence always

$$\sin \frac{1}{2}(\eta_0 + a) > \sin \frac{1}{2}(\varepsilon_0 + a) \, ,$$

that is to say, the **minimum deviation** E_0 **of a ray lying out of a principal section,** in its projection η_0 upon the principal section, attains a **greater value than the minimum deviation** ε_0 of a ray contained within the principal section and incident at the same angle. This applies, therefore, *a fortiori* to the minimum deviation E_0 itself, as has also been shown by Heath and Reusch.

When $n < 1$ the investigation is similar to that given in § 231, where the case considered was that of a ray passing through a prism within its principal section.

The path within the principal section of the projection of a ray which is incident at an angle θ to the principal section follows by Reusch's investigation from the relation :

$$n_\theta = n \frac{\cos \theta'}{\cos \theta} \, .$$

Cornu (*2.*) by introducing $\sin\,\theta' = \frac{1}{n} \cdot \sin\,\theta$, writes it in the alternative form:

$$n_\theta = \sqrt{n^2 + (n^2 - 1) \cdot \tan\,\theta'} \,.$$

237. Curvature of the Spectrum Lines. As a consequence of this dependence of the refractive index n_θ upon the inclination θ of the ray to the principal section, all rays proceeding from the points of a straight line parallel to the edge of the prism and traversing the prism at different angles to the principal section of the prism experience different deviations. This is, for instance, the case when these rays all cross at a point, such as the centre of the observer's pupil. The ray which undergoes minimum deviation is that which lies in the same principal section as the crossing point referred to, whilst the deviation of the other rays increases with the magnitude of the angle contained between the ray and the principal section at incidence. When these rays are the axes of pencils whose vertices are situated on the straight line parallel to the prism edge, the image of the straight line will appear as a curved line in the field of view; it forms, in fact, an arc whose vertex lies within the principal section which passes through the crossing point of the principal rays (as for instance, the pupil of an eye looking through the prism).

A more accurate computation of this image curve by Ditscheiner (*1.*) and v. Hepperger (*2.*) has shown that it is a parabola. The **radius of curvature** ρ at the vertex of the curve under the conditions of **minimum deviation** has been already given by Bravais (*1.*), as follows:

$$\rho = \frac{n f \sqrt{1 - n^2 \sin^2 \frac{a}{2}}}{2\,(n^2 - 1) \sin \frac{a}{2}} = \frac{n^2 f}{2\,(n^2 - 1)} \tan i_1 \,,$$

where f is the focal length of the telescope. Proceeding to other regions of the spectrum we find, as shown by Kayser (*1*, *320*), that $\frac{n^2 f}{2\,(n^2 - 1)}$, the first factor of ρ, changes but slowly, that is to say, when the deviation is a minimum the radius of curvature is proportional to the tangent of the angle of incidence, from which it follows that the curvature of the image curve diminishes as the angle of incidence increases. The works of Christie (*1.*), Simms (*1.*) and Crova (*1.*) on the same subject should be consulted.

2. The Formation of Images by a Prism or a Combination of Prisms. Astigmatism.

238. The modifications which a wide or narrow pencil of rays proceeding from a luminous point experiences by refraction at normal or oblique incidence on a system of planes, such as a combination of prisms, can be directly derived from the results of our investigations respecting refraction at spherical surfaces, it being only necessary to make all their radii infinite. The conditions under which it is possible to form images by these refractions, as well as their limits and defects, are directly deducible from the general investigations respecting spherical surfaces.

239. Image Formation due to a Plane.—Proceeding first to the investigation of the formation of an image by a plane, we find that a thin pencil whose principal ray is obliquely incident upon the plane experiences the astigmatic modification described in §§ 92 and 93.

The resulting image may be regarded as due to two separate image-formations occurring in two planes at right angles to each other. The image formed in the sagittal plane, which is at right angles to the plane of incidence, conforms to other laws than the image due to the tangential rays which are contained within the plane of incidence. The focal line of the sagittal rays (the second image-point) is an infinitely small portion of the normals of incidence, whilst the focal line of the tangential pencil is an infinitely small arc of a circle intersecting the plane of incidence perpendicularly at a point (viz. the first image-point). The geometrical aspect of this problem has been investigated in detail by Reusch (*2.*), Bauer (*2.3.*), Matthiessen (*13.*) and Zech (*3.*).

As before, let the intercept of the incident rays on the axis referred to the vertex of the refracting surface be denoted by f in the sagittal plane, and by t in the tangential plane; and, similarly, let the intercepts of the refracted rays be denoted by f' in the sagittal plane and by t' in the tangential plane. Also let the convention of signs respecting the intercepts and angles be the same as that adopted in § 227. We shall then be able to investigate the refraction of pencils of rays at a plane with the aid of the equations derived in § 100 for an infinite value of r and, in our present case, if we dispense with the distinction between i and j, we obtain the following set of equations:

In the sagittal section:

$$f'_1 = f_1 \frac{n'_1}{n_1},$$

and

$$\gamma_f = \frac{dv'_1}{dv_1} = \frac{f_1}{f'_1} = \frac{n_1}{n'_1}.$$

In the tangential section:

$$t'_1 = t_1 \cdot \frac{n'_1}{n_1} \frac{\cos^2 i'_1}{\cos^2 i_1}$$

and

$$\gamma_t = \frac{du'_1}{du_1} = \frac{t_1}{t'_1} \frac{\cos i'_1}{\cos i_1} = \frac{n_1}{n'_1} \frac{\cos i_1}{\cos i'_1}. \quad (26)$$

It will thus be seen that a homocentrically incident pencil ($f_1 = t_1$) is rendered astigmatic by a single refraction at a plane. In fact, the astigmatic difference is a function of the angle of incidence i_1 and is expressed by the equation:

$$t'_1 - f'_1 = f_1 \frac{n'_1}{n_1}\left(\frac{\cos^2 i'_1}{\cos^2 i_1} - 1\right).$$

The astigmatic difference vanishes only when $i_1 = i'_1 = 0$, that is only in the case of a normally incident homocentric and infinitely thin pencil of rays, which alone remains homocentric after refraction at a plane.

To trace the astigmatic difference as a function of the angle of incidence, it is necessary to distinguish the two cases $n_1 < n'_1$ and $n_1 > n'_1$.

(*a*) **Transmission of a homocentric pencil from an optically less refracting to an optically denser medium,** that is when $n_1 < n'_1$. In this case f'_1 is independent of the angle of incidence, $t'_1 > t_1$ and increases as the angle of incidence i_1 increases, until at grazing incidence where $i_1 = 90°$, $t'_1 = \infty$; that is to say, the astigmatic difference is positive and increases from zero at normal incidence to infinity at grazing incidence.

(*b*) **Transmission of a homocentric pencil from an optically denser to a less refracting medium,** that is when $n_1 > n'_1$. In this case $t'_1 < t_1$ and diminishes as the angle of incidence increases, until $t'_1 = 0$ at grazing incidence, that is to say,—

The astigmatic difference $t'_1 - f'_1$ occasioned in a homocentrically incident pencil is negative and diminishes as the angle of incidence increases from zero until total reflection occurs (at $\bar{i}_1 = \sin^{-1}\frac{n'_1}{n_1}$), when the magnitude of the astigmatic difference becomes $-f_1\frac{n'_1}{n_1}$.

A. The Formation of an Image in the Principal Section of a Prism surrounded by Air.

240. The two cases defined in the preceding article arise in succession when a pencil of rays passes through a prism surrounded by air, in which case $n_1 = n'_2 = 1$, $n'_1 = n_2 = n$. The refraction at the first surface imparts to the pencil a negative astigmatic difference, and the second refraction a positive astigmatic difference. We are thus led to expect that for any given intercepts t_1, f_1 an angle of incidence can be found which will cause the negative astigmatic difference arising at the first surface to be equal to the positive astigmatic difference which occurs at the second surface.

In a system of pencils proceeding in conformity with this condition the astigmatism prevailing in a pencil before it enters the prism is not affected by the prismatic refraction. In particular, a homocentric pencil, after refraction through the prism, will again be homocentric.

We shall now determine by calculation the path of the pencil of rays through a prism in air.

By equation (26), if we substitute for n_i its special values, we obtain the following expressions for f'_1 and t'_1, viz :

$$f'_1 = n \cdot f_1 ; \qquad t'_1 = n \cdot t_1 \cdot \frac{\cos^2 i'_1}{\cos^2 i_1} .$$

Denoting by d the length of the path of the light within the prism, we may obtain the following relations with respect to the second surface of the prism :

$$f_2 = f'_1 - d ; \qquad t_2 = t'_1 - d ;$$

$$f'_2 = \frac{1}{n} \cdot f_2 = \frac{1}{n}(f'_1 - d) ; \qquad t'_2 = \frac{1}{n} t_2 \frac{\cos^2 i'_2}{\cos^2 i_2} = \frac{1}{n}(t'_1 - d) \frac{\cos^2 i'_2}{\cos^2 i_2} ;$$

$$f'_2 = f_1 - \frac{d}{n} ; \qquad t'_2 = \frac{\cos^2 i'_2}{\cos^2 i_2}\left(t_1 \frac{\cos^2 i'_1}{\cos^2 i_1} - \frac{d}{n}\right) . \qquad (27)$$

An incident pencil of rays, whose astigmatic difference has the value $t_1 - f_1$, acquires by refraction through the prism the new astigmatic difference :

$$t'_2 - f'_2 = t_1 \frac{\cos^2 i'_1 \cos^2 i'_2}{\cos^2 i_1 \cos^2 i_2} - f_1 - \frac{d}{n}\left(\frac{\cos^2 i'_2}{\cos^2 i_2} - 1\right) . \qquad (28)$$

An incident **homocentric** pencil ($f_1 = t_1$), after traversing the prism, will accordingly have the following astigmatic difference :

$$t'_2 - f'_2 = t_1 \left(\frac{\cos^2 i_1' \cos^2 i_2'}{\cos^2 i_1 \cos^2 i_2} - 1\right) - \frac{d}{n}\left(\frac{\cos^2 i_2'}{\cos^2 i_2} - 1\right) . \qquad (29)$$

It is a notable fact that prior to the publication of a paper by Burmester (*1.*) it was erroneously held that the rays of an infinitely thin pencil proceeding under the conditions of minimum deviation from any chosen point through the extreme edge of the prism, furnished the only instance where it was possible for rays to meet at a point after traversing the prism ; and, conversely, that an incident homocentric pencil of rays, after passing through the prism could not

again become homocentric under any circumstances. This erroneous view arose from the fact that in the application of Helmholtz' equation (*1.*) to the transmisson of light through a prism the investigation was confined to the single case in which the length of the path of the light within the prism (*i.e.*, d by our notation) could be neglected in comparison with the length of the ray measured from the luminous point to the prism (*i.e.*, the intercept f_1 or t_1, as the case may be).

Burmester, on the contrary, has shown that pencils remain homocentric under other circumstances, and has formulated three theorems to this effect:

Theorem I: Any point can be made the focus of a thin converging or diverging pencil of rays which after transmission through the prism again becomes homocentric.

Theorem II: The homocentricity of these co-ordinated pencils extends to any number of prisms, provided that their edges are parallel.

Theorem III: There is a certain category of pencils which after traversing the prisms in planes other than the principal section retain their homocentric properties.

Burmester in his paper shows the construction of the homocentric pencils traversing the prism and proves by geometrical investigations the statements contained in the three theorems.

Extending Burmester's investigations, Wilsing (*1.*) has pointed out that the first two theorems of Burmester may be deduced from the equations of Czapski (*3. 159*), and Gleichen (*3.*), and has furnished an analytical proof of Burmester's third theorem, proceeding from the principle of the reduced path, as applied in Helmholtz' equations.

241. Homocentric Image-formation. — Burmester's first theorem will now be deduced from equation (29) of the preceding article.

The condition that an incident homocentric pencil may remain homocentric after traversing the prism is satisfied by reducing equation (29) to zero. We thereby obtain the following relation between t_1, the intercept of the incident pencil, and i_1, the angle of incidence at the first prism surface, namely:

$$t_1 = \frac{d}{n} \cdot \frac{\cos^2 i_1 (\cos^2 i'_2 - \cos^2 i_2)}{\cos^2 i'_1 \cos^2 i'_2 - \cos^2 i_1 \cos^2 i_2} \; . \quad \cdots \quad (30)$$

Equation (30) thus determines t_1 as a single-valued function of i_1 and d. This signifies that **on every principal ray,** which

here is defined by the angle of incidence i_1 at the first prism surface and by a value of d, **there is only one point of which a prism can form an image by homocentric pencils.**

Since d is proportional to the distance a of the point of incidence P from the refracting edge O it follows that t_1 varies as a, and when $a = 0$, then $t_1 = 0$, that is

The geometrical position of all points of which homocentric images are formed by a family of infinitely thin pencils having parallel principal rays contained within the principal section and inclined at any angle to the first prism surface, is in a plane passing through the edge of the prism.

We now proceed to determine, for any given angle of incidence of the pencil, the angle ψ_1 contained between the homocentric plane $L_1\,O$ of the image-points and the first prism side $\overrightarrow{O\,P}$, as represented in Fig. 99.

Notation :

$$OP \equiv a\,;\ L_1P \equiv t_1\,;\ PQ \equiv d\,;\ O\hat{L}_1P \equiv \xi_1\,;\ L_1\hat{O}P \equiv \psi_1\,;$$
$$L_1PO \equiv \zeta_1 = 90^\circ + i_1.$$

Then, in the triangle L_1PO

$$\frac{\sin \xi_1}{\sin \psi_1} = \frac{a}{t_1} = \frac{d}{t_1}\frac{\cos i_2}{\sin a}.$$

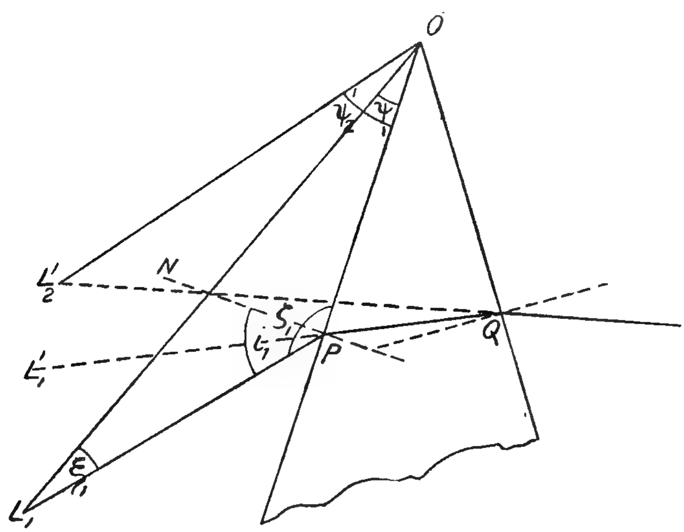

FIG. 99.
Image formed by homocentric pencils refracted at a plane.

Introducing the equivalent expression for $\frac{d}{t_1}$ from equation (30), we obtain

$$\frac{\sin \xi_1}{\sin \psi_1} = \frac{n}{\sin a} \frac{\cos i_2 (\cos^2 i_1' \cos^2 i'_2 - \cos^2 i_1 \cos^2 i_2)}{\cos^2 i_1 (\cos^2 i_2' - \cos^2 i_2)} . \quad \ldots \quad (31)$$

Also, $\xi_1 = 180^\circ - (\zeta_1 - \psi_1)$, and hence

$$\frac{\sin \xi_1}{\sin \psi_1} = \frac{\sin (\zeta_1 - \psi_1)}{\sin \psi_1} = \cot \psi_1 \, . \, \sin \zeta_1 - \cos \zeta_1 . \quad \ldots \quad (32)$$

Equating the right sides of equations (31) and (32), we obtain:

$$\cot \psi_1 \sin \zeta_1 - \cos \zeta_1 = n \frac{\cos i_2}{\sin a} \frac{\cos^2 i_1' \cos^2 i'_2 - \cos^2 i_1 \cos^2 i_2}{\cos^2 i_1 (\cos^2 i_2' - \cos^2 i_2)} ,$$

and, noting that $\zeta_1 = 90^\circ + i_1$, *i.e.*, $\sin \zeta_1 = \cos i_1$, and $\cos \zeta_1 = -\sin i_1$, we derive from this the equation:

$$\cot \psi_1 = n \frac{\cos i_2}{\sin a} \frac{\cos^2 i_1' \cos^2 i_2' - \cos^2 i_1 \cos^2 i_2}{\cos^3 i_1 (\cos^2 i_2' - \cos^2 i_2)} - \tan i_1 . \quad (33)$$

Equation (33) gives accordingly ψ_1, the angle between the plane of homocentric image-points OL_1 and the first prism side as a function of the prism angle a, the refractive index n of the substance of the prism and the angle of incidence i_1 of the principal rays of the image-forming pencils.

The plane defined by ψ_1 is homocentrically transformed by the successive refractions at the two sides of the prism. We are not here concerned with the character of the two "images" of the plane OL_1, which are formed by the tangential and sagittal rays by refraction at the first surface of the prism. It is sufficient for us to know that these two images are reduced to a homocentric image OL'_2 of the plane OL_1 by the refraction at the second surface of the prism.*

To determine the magnitude of the angle ψ'_2 between the plane OL'_2 and the first prism surface $\overrightarrow{OP}$ it is necessary to consider the triangle L'_2QR (Fig. 99).

Notation :

$$L'_2Q = t'_2 , \; RQ = d', \; Q\hat{R}O = \beta'_2 , \; R\hat{Q}O = \gamma'_2 .$$

---

* Note the remark in § 240 on equal and opposite astigmatic differences.

It was assumed that $t_1 = f_1$ and $t'_2 = f'_2$; hence, by equation (27),

$$t'_2 = t_1 - \frac{d}{n},$$

and if for t_1 we substitute its value from equation (30), we have

$$t'_2 = \frac{d}{n}\left\{\frac{\cos^2 i_1 (\cos^2 i'_2 - \cos^2 i_2)}{\cos^2 i'_2 \cos^2 i'_1 - \cos^2 i_2 \cos^2 i_1} - 1\right\}$$

$$t'_2 = \frac{d}{n} \cdot \frac{\cos^2 i'_2 (\cos^2 i_1 - \cos^2 i'_1)}{\cos^2 i'_2 \cos^2 i'_1 - \cos^2 i_2 \cos^2 i_1}. \qquad \ldots \qquad (34)$$

Equation (34) determines t'_2 as a single-value function of i_1, n, a and d_1.

Also

$$d' = d\,\frac{\sin \beta_1}{\sin \beta_2'} = d\,\frac{\cos i'_1}{\cos (a - i_2')}$$

and

$$RO = RQ\,\frac{\sin a}{\sin \gamma'_2} = d'\,\frac{\sin a}{\cos i_1'}.$$

In the triangle L'_2RO the angle $OL'_2R = \beta'_2 - \psi'_2$, and $L'_2R = t'_2 + d'$, hence ψ'_2 is determined by the following equation:

$$\frac{\sin (\beta'_2 - \psi'_2)}{\sin \psi'_2} = \frac{OR}{L_2'R} = \frac{d' \sin a}{\cos i'_2 (t'_2 + d')} = \frac{\sin a}{\cos i'_2 \left(\frac{t'_2}{d'} + 1\right)},$$

whence

$$\cot \psi'_2 = \cot \beta'_2 + \frac{\sin a}{\sin \beta'_2 \,.\, \cos i'_2 \left(\frac{t_2'}{d'} + 1\right)},$$

or, since $\beta'_2 = 90° - (a - i'_2)$,

$$\cot \psi'_2 = \tan (a - i'_2) + \frac{\sin a}{\cos (a - i_2') \cos i'_2 \left(\frac{t_2'}{d'} + 1\right)} \qquad (35)$$

where

$$\frac{t'_2}{d'} = \frac{1}{n} \cdot \frac{\cos (a - i'_2) \cos^2 i_2' (\cos^2 i_1 - \cos^2 i'_1)}{\cos i_1 (\cos^2 i'_1 \cos^2 i'_2 - \cos^2 i_1 \cos^2 i_2)}.$$

Equations (33) and (35) may be interpreted in the following words:

To every incident pencil of parallel rays there is a corresponding plane passing through the edge of the prism of which that pencil forms a homocentric image.

This theorem does not, however, hold without exception. When the pencil passes through the prism under the conditions of minimum deviation the denominator in equation (30) reduces to zero, since $i_1 = -i'_2$, $i'_1 = -i_2$, that is $t_1 = \infty$ for all finite values of d.

From this we conclude that no homocentric image can be formed of any point situated on any ray which passes symmetrically through the prism, unless it happens to coincide with the edge of the prism.

Helmholtz (*1. 259*) considers another special case, viz. that of a pencil of rays traversing the prism close to its refracting edge, where the path d within the prism can be neglected in comparison with the intercepts f_i and t_i. Equations (27) may then be written in the form

$$f'_2 = f_1, \quad t'_2 = t_1 \frac{\cos^2 i'_1 \cos^2 i'_2}{\cos^2 i_1 \cos^2 i_2},$$

or

$$t'_2\left(\frac{n^2-1}{\cos^2 i'_2} + 1\right) = t_1\left(\frac{n^2-1}{\cos^2 i_1} + 1\right). \quad \ldots \quad (27a)$$

Hence in a symmetrically transmitted pencil, where $i'_2 = -i_1$, it will be seen that $t'_2 = t_1$ in the neighbourhood of the edge. In the case of the rays passing symmetrically through the prism, it follows from equation (27a) that the intercepts are greater for those rays which enter the prism at a smaller angle of incidence than that of the symmetrical ray.

Straubel (*2.*) has investigated the special case of the transformation by homocentric pencils of a plane inclined to a pencil of parallel rays into a plane normal to the pencil, and thus affords a method of erecting, by means of a prism, a plane object field inclined to the optical axis of a system but otherwise optically corrected. (See also § 244.)

Burmester, moreover, has investigated the case of the transformation by a prism of a plane parallel to the homocentric image of a plane.

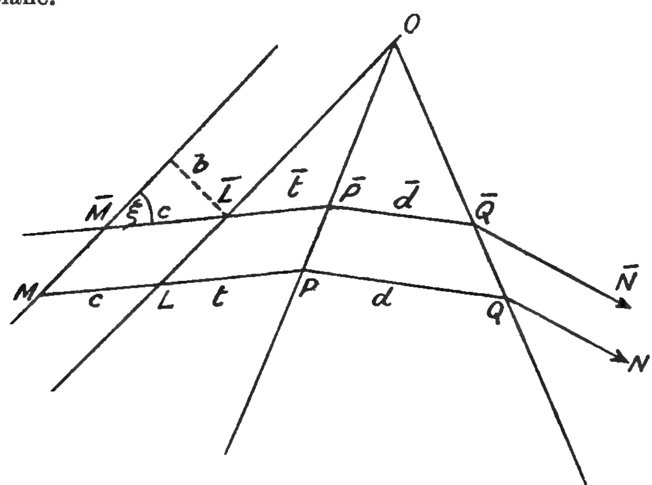

FIG. 100.

The transformation of a plane $M\bar{M}$ parallel to the homocentric plane LO.

In Fig. 100, let b be the distance between two parallel planes measured along their common normal. Then the distance between the two points where the incident ray intersects the two planes is $c = \dfrac{b}{\sin \xi}$.

Consider the two rays $\overline{M}\overline{P}\overline{Q}\overline{N}$ and $MPQN$ and let them cut the homocentrically transformed plane at $\overline{L}$ and L. Then

$$\overline{M}\overline{P} = \overline{S}_1 = \overline{f}_1 + c$$
$$MP = S_1 = f_1 + c.$$

Also,

$$\overline{M}\overline{P} = \overline{T}_1 = \overline{t}_1 + c$$
$$MP = T_1 = t_1 + c$$

$$\overline{S}'_2 = \overline{S}_1 - \frac{\overline{d}}{n}; \qquad \overline{T}'_2 = \overline{T}_1 \; \frac{\cos^2 i'_1 \cos^2 i'_2}{\cos^2 i_1 \cos^2 i_2} - \frac{\overline{d}}{n} \frac{\cos^2 i'_2}{\cos^2 i_2},$$

$$\overline{S}'_2 = \overline{f}_1 - \frac{\overline{d}}{n} + c; \quad \overline{T}'_2 = (\overline{t}_1 + c) \, . \, \frac{\cos^2 i'_1 \cos^2 i_2'}{\cos^2 i_1 \cos^2 i_2} - \frac{\overline{d}}{n} \frac{\cos^2 i'_2}{\cos^2 i_2},$$

$$S'_2 = f_1 - \frac{d}{n} + c; \quad T'_2 = (t_1 + c) \frac{\cos^2 i'_1 \cos^2 i'_2}{\cos^2 i_1 \cos^2 i_2} - \frac{d}{n} \frac{\cos^2 i'_2}{\cos^2 i_2}.$$

Noting that $t_1 = f_1$ and $\overline{t}_1 = \overline{f}_1$, we derive from this the following equations :

$$\overline{T}'_2 - \overline{S}'_2 = \overline{t}_1 \left(\frac{\cos^2 i'_1 \cos^2 i'_2}{\cos^2 i_1 \cos^2 i_2} - 1 \right) - \frac{\overline{d}}{n} \frac{\cos^2 i'_2}{\cos^2 i_2}$$
$$+ \frac{\overline{d}}{n} + c \frac{\cos^2 i'_1 \cos^2 i'_2}{\cos^2 i_1 \cos^2 i_2} - c,$$

and, similarly,

$$T'_2 - S'_2 = t_1 \left(\frac{\cos^2 i'_1 \cos^2 i'_2}{\cos^2 i_1 \cos^2 i_2} - 1 \right) - \frac{d \cos^2 i'_2}{n \cos^2 i_2}$$
$$+ \frac{d}{n} + c \frac{\cos^2 i'_1 \cos^2 i'_2}{\cos^2 i_1 \cos^2 i_2} - c.$$

In the last two equations, in conformity with equation (29), the sum of the last three terms on the right side is equal to zero, so that the remainder

$$\overline{T}'_2 - \overline{S}'_2 = c \left(\frac{\cos^2 i'_1 \cos^2 i'_2}{\cos^2 i_1 \cos^2 i_2} - 1 \right) = T'_2 - S'_2 \,. \quad (36)$$

Translated into words, this means that :

All points of a plane parallel to an image-plane formed by homocentric pencils and from which homocentric pencils proceed in

a particular direction, when transformed by a prism, acquire the same astigmatic difference, which is proportional to the distance c between the parallel planes measured in the direction of the image-forming rays.

B. Formation of Images in a Principal Section of a System of Prisms.

242. Equation (26), which gives the position of the image, enables us to determine the astigmatism which a pencil acquires by its successive refractions through a system of k refracting surfaces presented by $k - 1$ prisms having parallel refracting edges. It is in this case more convenient to determine this astigmatism in terms of the difference of the last two intercepts, instead of employing for this purpose the difference of the reciprocals of f and t, as we did in our previous investigations, the latter being the better course under the usual conditions.

We have for this purpose two sets of $2k$ equations, viz., for

Sagittal Rays: and Tangential Rays:

$$\left.\begin{aligned} f'_v &= \frac{n_{v+1}}{n_v} f_v \\ f_{v+1} &= f'_v - d_v \end{aligned}\right| \begin{aligned} v &= k \\ v &= 1 \end{aligned} \qquad \left.\begin{aligned} t'_v &= \frac{n_{v+1}}{n_v} \cdot t_v \frac{\cos^2 i'_v}{\cos^2 i_v} \\ t_{v+1} &= t'_v - d_v \end{aligned}\right| \begin{aligned} v &= k \\ v &= 1, \end{aligned} \tag{37}$$

where d_v is the length of the path of the corresponding principal ray within the v^{th} prism, and f_1, t_1, n_1 refer to the first medium which bounds the system of prisms and f'_k, t'_k, n'_k to the last medium.

If we assume that $n_1 = n'_k = 1$, which is the case when the system of prisms is bounded on either side by air, the solution of these equations gives us the relations

$$f'_k = f_1 - \sum_{v=2}^{k} \frac{d_{v-1}}{n_v},$$

which does not contain the terms i_v and i'_v, and

$$t'_k = t_1 \prod_{v=1}^{k} \left(\frac{\cos^2 i'_v}{\cos^2 i_v}\right) - \sum_{v=2}^{k} \frac{d_{v-1}}{n_v} \prod_{v=v}^{k} \left(\frac{\cos^2 i'_v}{\cos^2 i_v}\right).$$

A homocentrically incident pencil ($t_1 = f_1 = C$) acquires accordingly by refraction through a system of prisms the astigmatic difference

$$t'_2 - f'_2 = C\left[\prod_{v=1}^{k}\left(\frac{\cos^2 i'_v}{\cos^2 i_v}\right) - 1\right] - \sum_{v=2}^{k} \frac{d_{v-1}}{n_v}\left[\prod_{v=v}^{k}\left(\frac{\cos^2 i'_v}{\cos^2 i_v}\right) - 1\right]. \tag{38}$$

In the case of minimum deviation considered above

$$\prod_{v=1}^{k} \left(\frac{\cos^2 i'_v}{\cos^2 i_v}\right) = 1,$$

that is the factor of C vanishes. The astigmatic difference then becomes independent of the distance of the luminous point; it is, however, largely governed by the values of d_v and vanishes with these, as has been shown by the analogous investigations of Gleichen. For finite magnitudes of d_v the amount of the astigmatism becomes more and more insignificant in comparison with the distances of the object and image, according as the latter increases, and vanishes when the objects are at an infinite distance. This follows also directly if we consider that pencils of parallel rays, or telecentrical pencils, undergo no modifications excepting such as affect their direction and cross section. It is for this reason that it is a decided advantage to employ parallel pencils (or collimators) for all spectroscopic investigations.

The subject of the formation of images by homocentric pencil transmitted by a system of prisms is dealt with in the last chapter of Burmester's papers, to which we have already referred.

The image of a vertical spectroscope-slit parallel to the refracting edge of the prisms, remains sharply defined when the telescope is appropriately focussed with respect to the slit, though the astigmatism may be uncorrected. The horizontal end lines only of the slit will appear blurred. In investigations involving measurements the use of telecentrical pencils has the further advantage that under these conditions the observation relates to directions and their changes only, and does not involve any considerations respecting the relative distances separating the prisms, the collimator and the telescope.

Moreover, the astigmatism which arises in the case of homocentric pencils may be made a means of adjusting the collimator and telescope for infinity with the aid of a good prism, and without having recourse to a Gauss eyepiece. If, from the position of minimum deviation of the prism, we proceed to greater angles of incidence there will be introduced a corresponding change in the position of the image due to tangential pencils, so that it will become necessary to extend the eye-piece of the telescope in order to restore the sharp image of the vertical slit of the collimator when the latter is at an excessive distance from the objective, and to push it in if it should happen to be nearer to the objective than the principal focus. The converse takes place if we diminish the angles of incidence (cf. Helmholtz (*1, 257*)). A similar method has been proposed by Schuster (*2.*).

243. Magnifying Effect of Prisms. — The **apparent length** of the image of the slit at right angles to the principal section of the system of prisms, as seen from the latter, is the same

as that of the slit itself; for, as we have seen, every ray, including the principal ray of the image forming pencil, emerges from the system of prisms at the same angle θ'_k with respect to the principal section at which it entered the latter. The **apparent width** of the slit, on the other hand, as seen through the prisms, undergoes a change in general; that is, the angular width $\delta i'_k$ of the slit image as seen from the point of emergence at the last surface of the system, differs in general, from the apparent width δi_1 of the slit itself, as seen from the point of incidence of the pencil at the first surface. The connection between these two quantities is expressed by equation (18) in § 234. In the case of a system of prisms surrounded by air, where $n_1 = n'_k = 1$, this relation is established by the equation:

$$\delta i'_k = \delta i_1 \prod_{v=1}^{k} \left(\frac{\cos i_v}{\cos i'_v}\right).$$

From this equation it follows that when the principal ray traverses the prism system under the conditions of minimum deviation, the apparent width of the slit in the image is the same as its actual width. For in this case we have

$$\prod_{v=1}^{k} (\cos i_v) = \prod_{v=1}^{k} (\cos i'_v).$$

In other positions of the prisms of the system the apparent width of the slit can be diminished as well as increased. It is infinitely small in any position of the prisms in which the angle of incidence $i_v = \frac{\pi}{2}$, and it is always infinitely great when any one of the angles of emergence is $i'_v = \frac{\pi}{2}$, excepting when these two conditions arise simultaneously.

In the case of a single prism in air $k = 2$ and

$$\delta i'_2 = \delta i_1 \frac{\cos i_1 \cos i_2}{\cos i'_1 \cos i'_2} \qquad \dots \qquad \dots \qquad (39)$$

or

$$\frac{\delta i'_2}{\sqrt{\dfrac{n^2 - 1}{\cos^2 i'_2} - 1}} = \frac{\delta i_1}{\sqrt{\dfrac{n^2 - 1}{\cos^2 i_1} + 1}}.$$

Incidentally we conclude that in the case of a prism in air

$$\delta i'_2 \,/\, \delta i_1 = \sqrt{t_1} \,/\, \sqrt{t'_2}.$$

The apparent width of the slit, as seen through a single prism, increases accordingly continuously from the position in which the principal ray enters at grazing incidence and where its value is zero, until a position is reached where the principal ray leaves the

prism at grazing emergence, in which case the apparent width of the slit becomes infinitely great.

The astigmatic refraction due to transmission through a plane parallel plate or a system of such plates has been considered in §§ 101 and 138.

244. Rotation of the Image formed by Prisms.—Following Straubel's (*2.*) investigations of the formation of the image of a plane by prisms, we shall first consider the particular case in which the image of an object-plane parallel to the refracting edge of the prism is formed by parallel infinitely thin pencils having their principal rays within the principal section of the prism.

Let the normal to the object-plane and the principal rays of the incident tangential pencil contain an angle μ_1, and let the angle included between the normal and the principal rays of the sagittal pencil be μ_2. Also, let the normal to the image-plane and the principal rays of the emerging tangential and sagittal pencils of rays contain the angles μ'_1 and μ'_2 respectively. Straubel calls μ_v the inclination of the object and μ'_v the inclination of the image. The connection between the inclinations of the object and the image, which are governed by the angle of incidence at the first surface of the prism, as well as by the refractive index and the refracting angle of the prism, is given by the following equations:

For the tangential pencils:

$$\tan\mu'_1 \frac{\cos i_2}{\cos i'_2} - \tan\mu_1 \frac{\cos i'_1}{\cos i_1} = \frac{(n^2-1)\sin a}{n\cos i'_1 \cos i_2} \cdot \frac{1-\sin i_1 \sin i'_2 \cos(i'_1 - i_2)}{\cos^2 i_1 \cos^2 i'_2},$$

and for the sagittal pencils:

$$\tan\mu'_2 \frac{\cos i'_2}{\cos i_2} - \tan\mu_2 \frac{\cos i_1}{\cos i'_1} = \frac{(n^2-1)\sin a}{n\,.\cos i'_1 \cos i_2}.$$

From these equations we arrive at the following conclusions:

(1) The inclinations of the object and image are different for the tangential and sagittal pencils. There is, however, a pair of conjugate planes corresponding to either species of pencils and any inclinations of the rays (i_1, i'_1, i_2, i'_2) whose inclinations conform to a definite ratio. When the deviation attains its least value the condition that the inclination of the object and image may be equal corresponds with the case of planes which are parallel to the image-forming pencils.

(2) An object normal to the image-forming pencils has in no case a normal image corresponding to it.

Straubel has also investigated in the case of the tangential and sagittal pencils the condition which causes the inclination to increase or diminish, *i.e.*, $\tan\mu'_1 > \tan\mu_1$. In this investigation the minimum deviation is distinguished from all other paths of the rays

by the fact that the increase or decrease of the inclination cannot be attained by any choice in the position of the object; whatever the position of the latter, the inclination can only be increased, apart from the special case considered in *Theorem I* relating to the object and image of identical inclination. The reader is referred to Straubel's paper in which the progressive changes in the inclination of the image are fully discussed (loc. cit. pp. 67 *et seq.*) and where will also be found an investigation of the limit of the practical applicability of a few of the cases dealt with.

We now proceed to the **general case**, viz., the formation of an image of a plane object parallel to the edges of the prism by convergent or divergent pencils, whose principal rays proceed within the common principal section of the prismatic system.*

We shall accordingly assume that these pencils or their principal rays intersect at a single point A (Fig. 101) and we shall confine ourselves to so small a segment EE of the object that we may ascribe to these principal rays a common direction. If A were situated at an infinite distance it would be permissible to extend our investigation to finite objects.

On the above assumptions the image is, in general, inclined at a different angle to the principal rays which go to form the image as compared with the angle comprised between the object and the principal rays proceeding from it. This angular displacement of the image differs in the cases of the sagittal and tangential images, which are in general dissimilar.

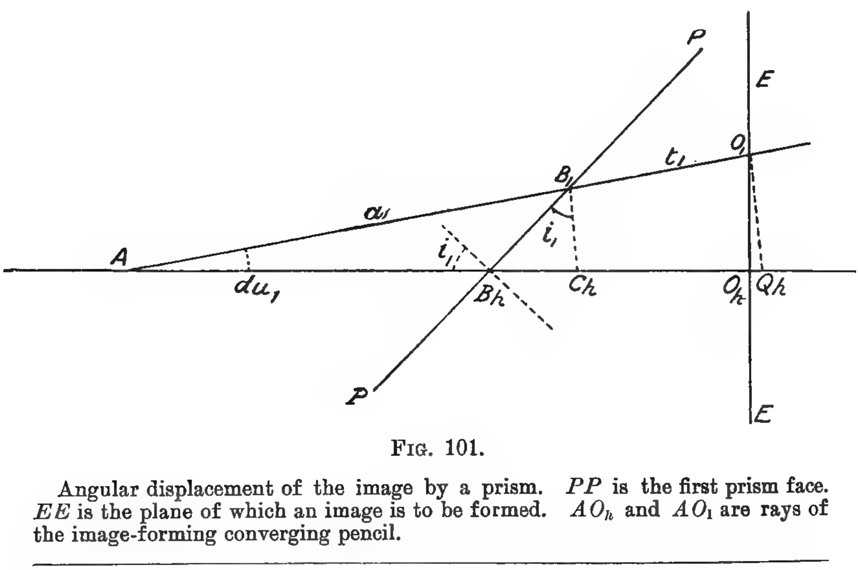

Fig. 101.

Angular displacement of the image by a prism. PP is the first prism face. EE is the plane of which an image is to be formed. AO_h and AO_l are rays of the image-forming converging pencil.

* Due to Dr. A. Koenig and in conformity with the exposition given by Eppenstein.

Let an image of the object plane EE (Fig. 101) be formed by the pencil of rays proceeding from A (or converging at A), and let PP be the first face of the prism. Let the leading principal ray be denoted by the suffix h, and the neighbouring ray by the suffix 1. Let B_h and B_1 be the points where the two rays intersect the first prism surface and let O_h and O_1 be the corresponding points of intersection in the plane of the object. Let the small angle $O_h A O_1$ be denoted by du_1 and the small difference $O_h Q_h = A O_1 - A O_h$ by Δ.

Then Δ is the **difference of focal adjustment** for either object point of an optical instrument which is symmetrical with respect to the axis, such as a magnifying lens, having its axis on the principal ray $AB_h O_h$ and focussed for an object at its principal focal plane.

In Fig. 101, if we put $B_h O_h = t_h$; $B_1 O_1 = t_1$; $B_h A = a_h$; $B_1 A = a_1$, we then have the following equations in which, for the present, we ignore the surface indices:

(i) For the **tangential pencils**:

$$t_1 = t_h + a_h \,.\, du_1 \tan i_1 + \Delta \quad \ldots \quad \ldots \quad (40)$$

and, similarly, after refraction at the plane PP:

$$t'_1 = t'_h + a'_h \,.\, du'_1 \tan i'_1 + \Delta' \,. \quad \ldots \quad (40a)$$

Now, equation (26) establishes the general relation:

$$t'_h = \frac{n'_1 \cos^2 i'_1}{n_1 \cos^2 i_1} t_h \,,$$

and since the adjacent ray $A_1 B_1 O_1$ has an angle of incidence $i_1 + du_1$ and an angle of refraction $i'_1 + du'_1$, we shall have, generally, for the adjacent rays

$$t'_1 = \frac{n'_1}{n_1} \frac{\cos^2 (i'_1 + du'_1)}{\cos^2 (i_1 + du_1)} t_1$$

$$= \frac{n'_1}{n_1} \frac{(\cos i'_1 - du'_1 \sin i'_1)^2}{(\cos i_1 - du_1 \sin i_1)^2} t_1 \,;$$

or, if we confine ourselves to the first powers of the infinitesimals du_1 and du'_1,

$$t'_1 = \frac{n'_1}{n_1} \,.\, \frac{\cos^2 i'_1 - 2\, du'_1 \sin i'_1 \cos i'_1}{\cos^2 i_1 - 2\, du_1 \sin i_1 \cos i_1} t_1 \,. \quad \ldots \quad (41)$$

To compute du'_1 we have

$$n_1 \sin (i_1 + du_1) = n'_1 \sin (i'_1 + du'_1) \,,$$

whence

$$du'_1 = \frac{n_1 \cos i_1}{n'_1 \cos i'_1} du_1 \,.$$

Substituting this value in equation (41), we get

$$t'_1 = \frac{n'_1}{n_1 \cos^2 i_1} \frac{\cos^2 i'_1 - 2\frac{n_1}{n'_1} \sin i'_1 \cos i_1 \, du_1}{1 - 2\, du_1 \tan i_1},$$

or, since

$$\frac{1}{1 - 2\, du_1 \tan i_1} = 1 + 2\, du_1 \tan i_1,$$

after simplification,

$$t'_1 = \frac{n'_1 \cos^2 i'_1}{n_1 \cos^2 i_1}\left(1 - 2\frac{n_1^2 - n'^2_1}{n'^2_1}\frac{\tan i_1}{\cos^2 i'_1} du_1\right) t_1, \quad (42)$$

where t_1 is to be replaced by its value from equation (40). From equation (40a) we find

$$\Delta' = t'_1 - t'_h - a'_h \, du'_1 \tan i'_1. \quad \ldots \quad (43)$$

The last term on the right side of equation (43) may be written in the form :

$$a'_h \, du'_1 \tan i'_1 = a_h \frac{\sin i'_1}{\cos i_1} du_1.$$

Substituting in equation (43) this value as well as that of t'_1 from equation (42) and that of t'_h we obtain the equation :

$$\Delta' = \frac{n'_1 \cos^2 i'_1}{n_1 \cos^2 i_1}\left(1 - 2\frac{n_1^2 - n'^2_1}{n'^2_1}\frac{\tan i_1}{\cos^2 i'_1} du_1\right)(t_h + a_h \, du_1 \tan i_1 + \Delta)$$
$$- \frac{n'_1 \cos^2 i'_1}{n_1 \cos^2 i_1} t_h - a_h \frac{\sin i'_1 \, du_1}{\cos i_1}. \quad \ldots \quad (44)$$

If, as before, we confine ourselves to the first powers of the infinitesimals du_1 and Δ, we may write equation (44) in the form

$$\Delta' = \frac{n'_1 \cos^2 i'_1}{n_1 \cos^2 i_1}\Delta - 2\, t_h \frac{n_1^2 - n'^2_1}{n_1 n'_1}\frac{\tan i_1}{\cos^2 i_1} du_1 + a_h \frac{n'_1 \cos^2 i'_1}{n_1 \cos^2 i_1} \tan i_1 \, du_1$$
$$- a_h \frac{\sin i'_1}{\cos i_1} du_1$$

or, after simplification,

$$\Delta' = \frac{n'_1 \cos^2 i'_1}{n_1 \cos^2 i_1}\Delta + \frac{n'^2_1 - n_1^2}{n_1 n'_1 \cos^2 i_1} \tan i_1 \, du_1 (2\, t_h + a_h). \quad (45)$$

Similarly, Δ_v, the change in the difference of focal adjustment at the vth surface, may be written in the form

$$\Delta'_v = \Delta_{v+1} = \frac{n_{v+1} \cos^2 i'_v}{n_v \cos^2 i_v}\Delta_v + \frac{n^2_{v+1} - n^2_v}{n_v \cdot n_{v+1}}\frac{\tan i_v}{\cos^2 i_v} du_v (2\, t_{hv} + a_{hv}).$$

By repeated application of this formula we obtain the following expression for the change of Δ after k refractions:

$$\Delta'_k = du'_k \left[\frac{\Delta_1}{di_1} \left(\frac{n'_k}{n_1}\right)^2 \prod_{v=1}^{k} \frac{\cos^3 i'_v}{\cos^3 i_v} \right.$$

$$\left. + \sum_{v=1}^{k} \left\{ (2\,t_{hv} + a_{hv}) \frac{n^2_{v+1} - n^2_v}{n_v^2} \frac{\tan i_v \cos i'_v}{\cos^3 i_v} \prod_{\mu=v+1}^{k} \left(\frac{\cos^3 i'_\mu}{\cos^3 i_\mu} \cdot \frac{n'^2_\mu}{n_\mu^2} \right) \right\} \right]. \quad (46)$$

Denoting by B_1 the width $O_1 Q_h$ of the pencil of principal rays at the position of the object, we obtain on reference to Fig. 101 the following expression for the inclination of the object:

$$\tan O_1 \hat{O}_h Q_h = \frac{B_1}{\Delta_1}\,.$$

Now, $B_1 = (t_{h1} - a_{h1})\,du_1$, and hence, after k refractions:

$$B'_k = (t'_{hk} - a'_{hk})\,du'_k = \frac{n'_k}{n_1} \prod_{v=1}^{k} \frac{\cos^2 i'_v}{\cos^2 i_v} (t_{h1} - a_{h1})\,du'_k\,. \quad (47)$$

Dividing B'_k by Δ'_k, we obtain the inclination of the image with respect to the tangential pencils after k refractions.

(ii) For the **sagittal pencils.**

By analogy to equations (40) and (40a) we can write for the sagittal pencils:

$$f_1 = f_{h1} + a_{h1} \cdot du_1 \tan i_1 + \Delta_1$$

and after refraction at the plane PP:

$$f'_1 = f'_{h1} + a'_{h1} \cdot du'_1 \tan i'_1 + \Delta'_1\,.$$

Also, since the pencil of the principal rays is identical with the tangential pencil,

$$a'_{h1}\,di'_1 \tan u'_1 = a_1 \frac{\sin i'_1}{\cos i_1} du_1\,,$$

$$f'_1 = \frac{n'_1}{n_1} f_1 = \frac{n'_1}{n_1} (\Delta_1 + f_{h1} + a_{h1} \cdot du_1 \tan i_1)$$

$$\Delta'_1 = \frac{n'_1}{n_1} \Delta_1 + \frac{n'_1}{n_1} a_{h1}\,du_1 \tan i_1 \left(1 - \frac{n_1 \sin i'_1}{n'_1 \sin i_1}\right),$$

or finally, since

$$1 - \frac{n_1 \sin i'_1}{n'_1 \sin i_1} = \frac{n'^2_1 - n_1^2}{n'^2_1}\,,$$

$$\Delta'_1 = \frac{n'_1}{n_1} \Delta_1 + \frac{n'^2_1 - n_1^2}{n_1 n'_1} a_{h1}\,du_1 \tan i_1\,.$$

We obtain accordingly the following expression for the change in the focal adjustment at the v^{th} refraction, namely:

$$\Delta'_{1v} = \frac{n'_v}{n_v} \cdot \Delta_{1v} + \frac{n'^2_v - n_v^2}{n_v\, n'_v} a_{hv} \cdot du_v \tan i_v \,;$$

and, after k refractions,

$$\Delta'_{1,k} = du'_k \left[\frac{\Delta_{1,1}}{di_1} \left(\frac{n'_k}{n_1}\right)^2 \prod_{v=1}^{k} \frac{\cos i'_v}{\cos i_v} \right.$$

$$\left. + \sum_{v=1}^{k} \left\{ \frac{n'^2_v - n_v^2}{n'^2_v} a_{hv} \tan i_v \prod_{\mu=v}^{k} \frac{\cos i'_\mu}{\cos i_\mu} \; \frac{n'^2_\mu}{n^2_\mu} \right\} \right]. \quad (48)$$

The fraction B'_k / Δ'_{1k} gives us the inclination of the image for the sagittal pencils. As was to be expected, the inclination of the image formed by the tangential pencils is in general different from that due to the sagittal pencils.

3. SPECTRA FORMED BY A PRISM OR A SYSTEM OF PRISMS.

A. General Properties of a Prismatic Spectrum.

245. Having investigated the laws which govern monochromatic pencils of rays traversing a prism, we now consider pencils composed of rays of different wave-lengths.

We are no longer able to regard the refractive indices n_i of the prisms as constants; on the contrary, it is necessary to take into consideration their dependence upon the wave-lengths of the component coloured rays, that is, the dispersion due to the substance of the prisms.

For example, if a pencil of rays of several colours proceeding from a slit passes through a prism, the image of the slit does not merely undergo deviation amounting in magnitude to a single angle ε, as determined in accordance with the equation given in § 228. There will be as many deviations of the image of the slit as there are colours in the incident pencil. The image formed by a prism of a slit through which light of different colours passes consists accordingly of a series of single-coloured images placed side by side, the aggregate of which is known as a **spectrum**. Without paying attention to the details of the spectra, we shall proceed to investigate the conditions affecting a spectrum which will be suitable for optical measurements.

246. The Intensity of the Spectrum is governed by the cross-section of the image-forming pencil of rays, the width of the slit, and the intensity of the light which enters the slit. The intensity is a point of particular importance in the case of spectra which are intended for visual observation, whereas in photographic reproductions of the spectrum any deficiency in the intensity of the spectrum can generally be overcome by a suitable extension of the time of exposure.

247. The Purity of the Spectrum.—A spectrum, to be absolutely pure, would require to be composed of the images of infinitely narrow slits; each slit-image corresponding to a particular wave-length; it is essential also that the slit-images due to rays of adjacent wave-lengths should not overlap. The theory of diffraction teaches, however, that even the image of an infinitely narrow slit transmitting monochromatic light of wave-length λ is not in itself an infinitely narrow slit, but that it has a finite width depending upon λ and the width of the image-forming pencil of rays. From this it follows that an absolutely pure spectrum is physically unattainable.

In an actual spectrum we see therefore that the slit-images due to two adjacent wave-lengths always overlap; also, that the purity of the spectrum increases continually as we diminish the number of the wave-lengths whose slit-images overlap from either side of the slit-image due to a single wave-length.

(*a*) **Helmholtz' Expression for the Purity *R* of any Region of the Spectrum.**—The number of overlapping slit-images becomes clearly smaller and smaller, and the purity of the spectrum correspondingly greater, the more the slit-images due to two given wave-lengths are drawn apart by the dispersion of the prismatic system, and also the narrower the individual monochromatic slit-images are in themselves.

Helmholtz (*I*. *259*) expresses accordingly the purity of any region of the spectrum comprised between the wave-lengths λ_1 and λ_2 by the ratio

$$R = \frac{\text{Length of the spectrum between slit-images due to } \lambda_1 \text{ and } \lambda_2}{\text{Width of the slit-image due to the wave-length } \lambda = \frac{1}{2}(\lambda_1 + \lambda_2)},$$

where the length of the spectrum and the width of the slit-image are stated in angular measure. Since the linear dimensions of these two constituents are proportional to the focal lengths of the objectives employed in the spectroscopic apparatus, it follows that the focal lengths disappear in the quotients, and thus the expression by which Helmholtz defines the purity of a prismatic spectrum, apart

from the width of the slit, is a magnitude which is governed solely by the prismatic system and cannot be increased with the magnifying power of the telescope.

Helmholtz' expression for the purity of the spectrum enables us to calculate the **purity** R which is actually attained in any given case **in accordance with the rules of geometrical optics.** It will be seen that R contains in its denominator the width of the slit-image and is accordingly governed by the width of the slit itself and not exclusively by the properties of the prism. For this reason the purity of a spectrum, as defined by Helmholtz, is not adapted for a comparison of the optical performances of prisms and combinations of prisms. Such a comparison requires reference to its resolving power.

(*b*) **Resolving Power** r **of a Prism with respect to an infinitely narrow slit,** according to Rayleigh (*2.*). We arrive at the definition of r in the following manner. The purity of a spectrum at any point corresponding to the wave-length λ increases in the inverse order of the difference $d\lambda$ of two wave-lengths λ and $\lambda + d\lambda$, the strictly monochromatic slit-images of which are just discernible **to the human eye** as two distinct lines. Rayleigh was led by his experiments to the conclusion that a band of unit intensity, *e.g.*, a spectrum line, cannot be recognised as consisting of two bands (or lines) unless the whole band is divided in the middle by a "dark" strip the intensity of which is 0·81 or less. With the aid of this empirical coefficient Rayleigh was able to determine the value of $d\lambda$ for the general case of a pencil of finite width transmitted by a prism. The ratio $\lambda/d\lambda = r$, which applies to the special case of the symmetrical transmission of a pencil through an **infinitely narrow slit** is called by Rayleigh the **resolving power of a prism.**

We shall show later that the magnitude of the quantity r, as here defined, is equal to the product of the base of the prism and the **characteristic dispersion** $\frac{dn}{d\lambda}$. From this it follows that r is a quantity **characteristic of the prism** and entirely independent of all the other constituents of the spectroscope. It indicates the highest degree of purity which a spectrum of an infinitely narrow slit may attain when the pencil proceeding from it traverses the prism symmetrically. The resolving power of a prism is therefore a conclusive criterion of its optical performance.

To adapt the investigation more closely to the conditions as they exist in practice, in which an infinitely narrow slit is an unattainable ideal, Schuster (*1.*) has calculated the resolving power of a prism in the case of a **finite slit** transmitting **homogeneous light.** Wadsworth (*1. 2. 3.*) carried the investigation to the final stage

by calculating the actual purity of the spectrum due to a slit of finite width transmitting light of wave-lengths extending from λ to $\lambda + d\lambda$. For a full account of this investigation the reader is referred to the discussion of Wadsworth's excellent investigations in Kayser's treatise (*1. 553*). Wadsworth bases his deductions likewise upon the fundamental experiments of Rayleigh; in particular, he accepts the results of Rayleigh's experiments, showing the human eye to be capable of perceiving with certainty differences of 20% in the intensity of the light. By the investigations of Wadsworth the problem of the purity of the spectrum has reached the stage of practical utility.

248. Derivation of Rayleigh's Resolving Power of a Prismatic System from the Principles of the Undulatory Theory.—Before proceeding to discuss the various factors which determine the general properties of a spectrum we must consider, in terms of the undulatory theory, Rayleigh's (*2.*) investigation of the relation of the resolving power of a system of prisms to the dispersion and the cross-section of the image-forming pencil.

By refraction through a system of prisms P, let the plane wave-front A_0B_0 of the incident light be translated into the position AB. Then the path of the light from A_0 to A as well as that extending from B_0 to B are minimum values, and both are equal in magnitude, viz.:

$$\int_{A_0}^{A} ndl = \int_{B_0}^{B} ndl.$$

A wave of a different length $\lambda + d\lambda$ is translated during the same interval of time into the position $A'B'$ (Fig. 102). The paths described by the rays of these waves differ from the paths corresponding to the wave-length λ; but differences of the reduced paths of the light, owing to the fact that the reduced paths $A_0 \ldots . A$ and $B_0 \ldots . B$ represent minimum values, vanish if we neglect infinitesimals of higher orders in comparison with the paths themselves. The reduced paths from A_0 to A' and from B_0 to B' for the wave-length $\lambda + d\lambda$ may, therefore, be measured along the same geometrical paths as those corresponding to the wave-length λ.

The difference of the optical lengths extending from A_0 and B_0 to A and B is accordingly represented for $\lambda + d\lambda$ by the equation

$$\int_{B_0}^{B} dndl - \int_{A_0}^{A} dndl = L,$$

and this quantity L divided by the cross-section of the emergent pencil $AB = q'$ is equal to the angle comprised between the two wave-fronts corresponding to the wave-lengths λ and $\lambda + d\lambda$, *i.e.* it is equal to the dispersion di', and therefore

$$L \mid q' = di' .$$

The smallest angular distance between the two spectrum lines, *i.e.* the smallest value di', at any given cross-section q' where it is still possible to distinguish the two lines one from the other, has been shown by Rayleigh to be

$$di' \geqq \frac{\lambda}{q'}, \quad \dots \quad \dots \quad \dots \quad (45)$$

where λ is the mean wave-length of the two colours. It follows accordingly that

$$di' . \, q' \geqq \lambda ,$$

and that $L \geqq \lambda$.

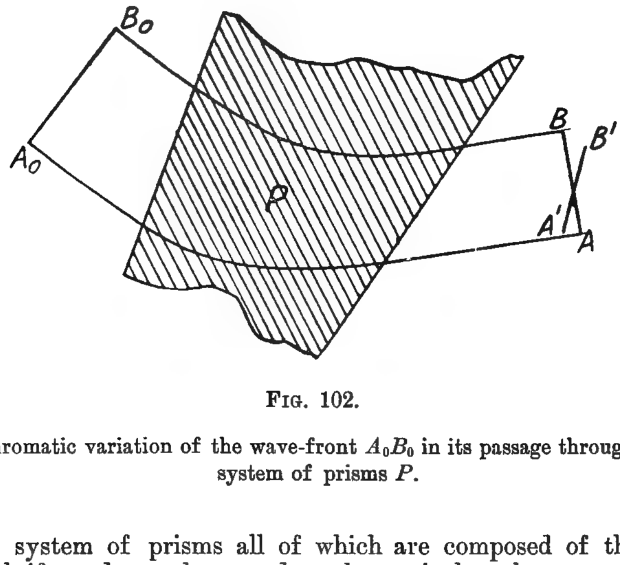

Fig. 102.

Chromatic variation of the wave-front A_0B_0 in its passage through a system of prisms P.

In a system of prisms all of which are composed of the same material, if we denote by e_1 and e_2 the optical paths corresponding to the extreme marginal rays of the pencil within the glass, then

$$L = dn\,(e_2 - e_1) . \quad \dots \quad \dots \quad \dots \quad (46)$$

In order that the two spectrum lines conforming to a difference of wave-length $d\lambda$ and a corresponding difference dn in the two

refractive indices of the glass may be just capable of being resolved, we must satisfy the condition :

$$dn\,(e_2 - e_1) \geqq \lambda \,. \quad \dots \quad \dots \quad \dots \quad (47)$$

Supposing that one of the rays passes exclusively through the prism edges, we have $e_1 = 0$; and substituting e for e_2, and dividing both sides of the last equation by $d\lambda$, we then have

$$\frac{\lambda}{d\lambda} \geqq \frac{dn}{d\lambda} \cdot e \,,$$

or, if we denote by $\frac{\lambda}{d\lambda}$ the smallest admissible value of r,

$$r = \frac{\lambda}{d\lambda} = \frac{dn}{d\lambda} \cdot e \,. \quad \dots \quad \dots \quad (48)$$

Rayleigh's resolving power r of a system of prisms is thus expressed in the form of the product of the optical path through glass and the characteristic dispersion $\frac{dn}{d\lambda}$ of the glass, that is, in terms of two qualities of the system of prisms itself, independently of the number of the refracting angles of the component prisms and the other constituents of the spectroscope.

When all the prisms occupy positions of minimum deviation, it is clear that e is equivalent to the sum of the thicknesses of the prisms at the base. For example, in order that a flint prism in which $n_D = 1{\cdot}6504$ and $n_D - n_C = 0{\cdot}00552$ may be capable of resolving the principal double line of sodium light (*i.e.* $d\lambda = 6{\cdot}10^{-7}$ mm.) a prism arranged in the position of minimum deviation should have a thickness of 10·2 mm. according to Rayleigh's formula.

B. The Various Factors which determine the Properties of the Spectrum.

249. Dispersion of a Prism in Air.—As we have already seen, a spectrum arises from the dissimilar displacement of the slit-images due to rays of different colours. In establishing, accordingly, an expression for the variation of the deviation from equation (3) we must regard the refractive index as variable, as well as the angles of incidence and refraction, the refractive index being, in fact, a function of the wave-length.

From equation (3) in § 228,

$$\sin i_1 = n.\sin i'_1$$

$$i_2 = i'_1 - \alpha$$

$$\sin i'_2 = n \sin i_2 \,,$$

we derive thus by differentiation the following set of equations:

$$\cos i_1 \delta i_1 = n \cos i'_1 \delta i'_1 + \sin i'_1 \delta n$$
$$\delta i_2 = \delta i'_1$$
$$\cos i'_2 \delta i'_2 = n \cos i_2 \delta i_2 + \sin i_2 \delta n\,,$$

whence we derive the following expressions for $\delta i'_k$

$$\delta i'_1 = \frac{1}{n}\left(\frac{\cos i_1}{\cos i'_1}\delta i_1 - \delta n \tan i'_1\right) = \delta i_2 \quad \dots \quad (49)$$

and

$$\delta i'_2 = \frac{\cos i_1 \cos i_2}{\cos i'_1 \cos i'_2}\delta i_1 - \frac{\sin a}{\cos i'_1 \cos i'_2}\,.\,\delta n\,.$$

In most cases there is no dispersion when the pencil of rays enters the prism, and therefore $\delta i_1 = 0$. In such cases, whatever the position of the prism, the dispersion at emergence is

$$\delta i'_2 = -\,\delta n\,.\,\frac{\sin a}{\cos i'_1 \cos i'_2} \quad \dots \quad \dots \quad (50)$$

and evidently $\delta i'_2$ cannot vanish so long as a and δn have finite values, whence it follows that a single prism having a finite refracting angle and surrounded by air can never be achromatic.

Under the condition of minimum deviation, where $i'_1 = -\,i_2 = a$ and $i_1 = -\,i'_2$,

$$\delta i'_{02} = \delta i_{01} - \frac{\sin 2i_{01}\,\delta n}{\cos i'_{01} \cos i_{01}}\,,$$

or

$$\delta i'_{02} - \delta i_{01} = -\,\frac{2\delta n}{n}\tan i_{01}\,, \quad \dots \quad (51);$$

that is to say, a pencil of rays passing through a prism under the conditions of minimum deviation experiences an increment $\delta i'_{02} - \delta i_{01}$ in its dispersion, which is independent of that present in the incident pencil.

The investigation of equation (50) by Mousson (*1.*) and Thollon (*1.*) with respect to the connection of the dispersion and the angle of incidence i_1 of the mean ray, shows that the dispersion increases continuously with the angle of emergence from a certain minimum value which is intermediate between grazing incidence and minimum deviation, and that at grazing incidence ($\cos i'_2 = 0$) it attains its maximum value, which is infinite.

Owing to the very great losses of light caused by reflection, it is impossible practically to employ a high dispersion in the region near grazing emergence. If, accordingly, for any given

purpose the dispersion of a prism be insufficient in the positions approximating to that of minimum deviation, the dispersion should be increased by causing the pencil of rays to traverse the prism several times or by the employment of a system of several prisms.

250. Dispersion due to a System of Prisms.—Reverting to the investigations by Gleichen (*2.*) and Czapski (*3. 145*), let a system of k refracting surfaces or $k-1$ prisms be considered, and at the outset let the first and last refracting medium be air and let their dispersion be neglected. By the differentiation of equation (14), if we put $n_1 = n'_k = 1$ and $\delta n_1 = \delta n'_k = 0$, we then obtain the following set of equations:

$$\delta i'_1 = \frac{1}{n_2}\frac{\cos i_1}{\cos i'_1}\,\delta i_1 - \frac{\delta n_2}{n_2}\tan i'_1\,; \qquad \delta i_2 = \delta i'_1,$$

$$\delta i'_2 = \frac{n_2}{n_3}\frac{\cos i_2}{\cos i'_2}\,\delta i_2 + \frac{\delta n_2}{n_3}\frac{\sin i_2}{\cos i'_2} - \frac{\delta n_3}{n_3}\tan i'_2\,; \qquad \delta i_3 = \delta i'_2\,, \quad (52)$$

$$\delta i'_2 = \frac{1}{n_3}\frac{\cos i_1 \cos i_2}{\cos i'_1 \cos i'_2}\,.\,\delta i_1 - \frac{\delta n_2}{n_3}\frac{\sin a_1}{\cos i'_1 \cos i'_2} - \frac{\delta n_3}{n_3}\tan i'_2\,,$$

$$\vdots$$

$$\delta i'_v = \frac{n_v}{n_{v+1}}\frac{\cos i_v}{\cos i'_v}\,.\,\delta i_v + \frac{\delta n_v}{n_{v+1}}\frac{\sin i_v}{\cos i'_v} - \frac{\delta n_{v+1}}{n_{v+1}}\tan i'_v\,; \quad \delta i_{v+1} = \delta i'_v\,,$$

and, finally, $\delta i'_k$, the variation of the angle of emergence i'_k at the last, or k^{th}, surface of the prisms, is given by

$$\delta i'_k = \frac{n_k}{n'_k}\frac{\cos i_k}{\cos i'_k}\,\delta i_k + \frac{\delta n_k}{n'_k}\frac{\sin i_k}{\cos i'_k} - \frac{\delta n'_k}{n'_k}\tan i'_k\,,$$

or, if we consider that $n_k' = 1$, $\delta n'_k = 0$,

$$\delta i'_k = \frac{\cos i_1 \cos i_2 \ldots\ldots \cos i_k}{\cos i'_1 \cos i'_2 \ldots\ldots \cos i'_k}\,\delta i_1 - \sum_{v=2}^{k}\frac{\cos i_{v+1}\,.\,\cos i_{v+2}\ldots\ldots\cos i_k}{\cos i'_{v-1}\,.\,\cos i'_v\ldots\ldots\cos i'_k}\,\delta n_v\,.\,\sin a_{v-1}.$$

In this expression, if we denote by $\prod$ the product of the cosines of the angles of incidence and by $\prod'$ that of the cosines of the angles of refraction, we may write

$$\delta i'_k = \frac{\prod_1^k}{\prod_1'^k}\,\delta i_1 - \sum_{v=2}^{k}\frac{\prod_{v+1}^k}{\prod_{v-1}'^k}\sin a_{v-1}\,\delta n_v\,, \quad \ldots \quad (53)$$

or

$$\delta i'_k \cdot \textstyle\prod_1'^k = \delta i_1 \cdot \prod_1^k - \sum\limits_{v=2}^{k} \delta n_v\,.\,\sin a_{v-1} \cdot \prod_{v+1}^k \cdot \prod_1'^{v-2}. \quad (53a)$$

In one of the products $\prod_m^k$, if the lower limit m be greater than the upper limit k (as in the case of a **single** prism, where $k = 2$, the surface index being $v = 2$), we may suppose a $(k+1)^{\text{th}}$, $(k+2)^{\text{th}}$ etc. refracting plane added, all being in air and at right angles to the ray under consideration. All angles of incidence and refraction at these planes, viz. i_{k+1}, i'_{k+1}, etc., are equal to zero, so that their cosines are unity. Each product $\prod_m^k$ in which $m > k$ has accordingly unit value. Similarly, we shall find that $\prod'^{v-2}_1$ has unit value so long as $1 > v - 2$, or as $v < 3$.

Equation (53) may be expressed verbally as follows: If a monochromatic ray of wave-length λ entering the system of prisms, and another incident ray of wave-length $\lambda + d\lambda$, before entrance at the first prism face, include an angle δi_1, the two rays will comprise an angle $\delta i'_k$ after emergence from the last face of the prisms.

When, as is generally the case, the dispersion of the pencil of rays entering the system of prisms is $\delta i_1 = 0$ the pencil acquires by equation (53) the dispersion:

$$\delta i'_k = -\sum_{v=2}^{k} \frac{\prod_{v+1}^{k}}{\prod'^{k}_{v-1}} \cdot \sin a_{v-1} \cdot \delta n_v . \qquad \dots \quad (54)$$

By reason of the convention adopted in § 227 the sign of $\delta i'_k$ in equation (54) gives the position of the refracting edge of the prism by which we may imagine the system of prisms to be replaced for the two colours under consideration.

When the angles of the prisms and the angles of refraction are small the dispersion may be expressed directly in terms of the change of the deviation. The deviation ε to which such a system gives rise is in the case of a prism

$$\varepsilon = (n-1)\, a ;$$

whilst in the case of several prisms it is

$$\varepsilon^{(k)} = \sum_{v=2}^{k} (n_v - 1)\, a_{v-1} . \qquad \dots \qquad \dots \quad (55)$$

A system consisting of two cemented prisms in air gives rise accordingly to a deviation expressed by the formula

$$\varepsilon^{(3)} = (n_2 - 1)\, a_1 + (n_3 - 1)\, a_2 ;$$

and, if we proceed to an adjacent colour,

$$\delta\varepsilon^{(3)} = \delta n_2\, a_1 + \delta n_3\, a_2$$

expresses the dispersion of the prism system for the two colours under consideration.

It will be seen that the equations are of the same form as those expressing the condition under which it is possible to attain the achromatisation of systems of lenses (§ 195, vii). In fact, the history of the subject shows that the possibility of the achromatisation of combinations of lenses derived its inception from the corresponding investigations of systems of prisms (cf. Klingenstierna (***1.***), Dollond (***2.***), Clairaut (***1.***), Boscovich (***1. 2.***), etc.).

251. Width of a Monochromatic Slit Image.—As we have already seen above, the purity of a spectrum is governed not only by the difference of the angular displacement which the dispersion imparts to the chromatic images of the slit but also by the width of the component slit-images.

To determine the magnitude of the latter it is not sufficient to apply the formulæ derived solely in terms of geometrical optics. On the contrary, this should be done with due regard to the fundamental principles of the theory of diffraction, which shows that the width of the image of the slit depends in a great measure upon the width of the pencils bringing about the formation of this image.

This is not the place to study the subject in detail, and we must content ourselves with reminding the reader that the image of a self-luminous line due to pencils whose width at right angles to the direction of the line is q and whose wave-length is λ, by whatever optical means it may be produced, can never be a straight line, and must always present the appearance of a band whose intensity falls off gradually towards the edges. Reverting to the object, as we have done previously in the investigation of the aberration, the length of the image, *i.e.* the height of the band, is approximately equal to the primary line, whilst the spreading of the light at right angles to it, that is the width of the image, is only a function of q and λ, which, other things being equal, diminishes with increasing values of q, whilst it increases as λ increases. Of two adjacent luminous lines, whether real initial objects or virtual configurations derived from a single linear object, for example by dispersion, the optical system forms two image bands which partly overlap when q is too small or λ insufficiently small. The intensity of the image in these overlapping parts is equal to the sum of the intensities of the component images at these places.

A more critical investigation of the varying conditions of the intensity of the component images with the aid of the theory of diffraction has been made by Rayleigh (***2.*** *266*) and has furnished the following result:

Supposing two overlapping slit-images to be of equal intensity and to differ infinitesimally in colour, a dark band between the two maxima of intensity corresponding to the centre-line of the images becomes distinguishable only when the dispersion di'

of the two spectrum lines is greater than the angle which the wave-length λ of the active light subtends at a distance equal to the diameter of the cross-section q'; that is, the two slit images will just be distinguishable as separate lines when

$$di' > \frac{\lambda}{q'}.$$

Now, the refraction of a pencil of parallel rays through a system of prisms is, in general, associated with a change of its cross-section in the principal plane, and this should be taken into consideration together with the width of the pencil entering the system.

We may readily see this if we consider that every refraction at a plane causes the cross-section q of the pencil to undergo a change in the ratio of the cosines of the angles of refraction, thus

$$\frac{q'_v}{q_v} = \frac{\cos i'_v}{\cos i_v}. \quad \dots \quad \dots \quad \dots \quad (56)$$

In any system of prisms $q_{v+1} = q'_v$. Hence the variation in the width of the pencil due to refraction through such a system is

$$\frac{q'_v}{q_1} = \frac{q'}{q} = \frac{\prod_1^k \cos i'_v}{\prod_1^k \cos i_v} = \frac{\prod'_1}{\prod_1^k} = \frac{\delta i}{\delta i'}.$$

This, as a matter of fact, follows immediately from the Smith-Helmholtz theorem.

It should be noted that q is not identical with the cross-section of the entire pencil which enters the prism system, viz.:—the quantity $b_1 \cos i_1$, where b_1 is the length of the surface intercepted by the pencil at the first prism face. It stands for that part only which does not experience any subsequent restriction by any of the succeeding surfaces.

For the determination of this effective cross-section in a triple symmetrical set of prisms, the so-called direct-vision dispersion prism of Amici, the reader is referred to § 253.

252. The Intensity of the Spectrum.—The intensity of a spectrum at any given point is governed by the number of colours which contribute to the formation of the slit-image at this point, if only by overlapping from either side. At any given width of the slit δi its image $\delta i'$ receives the light of those wave-lengths extending from λ to $\lambda + d\lambda$ whose dispersion di' is equal to the width $\delta i'$ of the slit-image. Since the height of the slit is not affected by the refraction through the prism, the intensity h' of the slit-image for any homogeneous colour is inversely proportional to the change in the width of the slit due to refraction, so that, if h be the original intensity of illumination of the slit itself,

$$h' / h = \delta i / \delta i',$$

The intensity H of the spectrum for the wave-length λ, assuming the intensity over the slit to be constant within the interval λ to $\lambda + \delta\lambda$, is

$$H = h'. d\lambda = h \frac{\delta i}{\delta i'} d\lambda, \quad \dots \quad \dots \quad (57)$$

where $d\lambda$ is subject to the condition that the angular value di' of the dispersion is equal to the width of the slit $\delta i'$. Equation (57) may accordingly be written in the form :—

$$H = h . \frac{\delta i}{\delta i'} . \frac{d\lambda . \delta i'}{di'} .$$

Under the conditions of minimum deviation $\delta i = \delta i'$, so that

$$H = h . \frac{d\lambda . \delta i'}{di'} = \frac{h}{\boldsymbol{R}}, \quad \dots \quad \dots \quad (58)$$

where $\boldsymbol{R}$ denotes the **purity of the spectrum,** as defined by Helmholtz (*1*, *261*).

The intensity of the spectrum, disregarding the losses of light caused by reflection and absorption within the system of prisms, is under these conditions proportional to the intensity of the light which enters the slit, and inversely proportional to the purity of the spectrum (Helmholtz).

The intensity of the spectrum, besides undergoing changes under the conditions already referred to, is likewise modified by losses of light which are inseparable from any form of reflection and refraction as well as those due to absorption within the prisms. The amount of light which is lost to the spectrum-image by partial reflection may be determined by Fresnel's intensity formulæ from the angles of reflection and the refractive indices of the absorbing media. The amount of the loss of light by absorption is governed by the length of the path within the medium in question and its power of absorption. From these premises general rules applicable to simple cases have been deduced by Pickering (*1.*), Lippich (*4.*), Robinson (*1.*), H. Kruess (*2.*).

In the whole of the above investigation of the purity of the spectrum and the resolving power of a system of prisms no attention has been paid to the losses due to reflection and absorption. In spectroscopes comprising several prisms, or of the kind in which a prism is traversed several times, these losses are particularly pronounced in flint glasses of high refractive power. The majority of the relations established above remain unaffected when a telescope is employed for the observation of spectra, especially those concerning the purity and the intensity of the spectrum and the resolving power of a system of prisms. In all these cases it is naturally

assumed that the aperture of the telescope is greater than that of the system of prisms, otherwise the determination of the intensity and the resolving power should be made subject to the aperture of the telescope instead of that of the prism. In other respects the spectrum, as seen through the telescope, appears as little modified as any other object. Spectra formed by prisms may, however, also be observed through the agency of other than telecentrical pencils with the aid of a magnifying lens or other instruments. For the reasons already stated the prisms should in this case be used in the position of minimum deviation.

C. Direct Vision and Achromatic Systems of Prisms.

253. For many practical purposes it is desirable to so arrange a dispersing system that it will cause no deviation of a particular colour. When it is required that the emergent ray of any particular colour shall be in the continuation of the incident ray the system should consist of at least three prisms. In a direct-vision system of two prisms the emergent ray of a given colour, though parallel to the incident ray, is displaced in the plane of incidence.

We shall first investigate the case of a system consisting of two prisms. Let the light meet the first surface at right angles to it, in which case $i_1 = i'_1 = 0$, and let it enter the second surface where the two prisms are cemented together, and afterwards emerge through the third surface.

If the refracting angle a_1 of the first prism, as well as n_2 and n_3, the refractive indices of the materials of which the two prisms consist, are assumed to be known, it is now our object to determine the refracting angle of the second prism. We then have

$$i_1 = i'_1 = 0$$

$$i_2 = -\,a_1$$

$$\sin i'_2 = -\,\frac{n_2}{n_3}\sin a_1\,,$$

$$i_2 - i'_2 = \varepsilon_2$$

$$\tan i'_3 = \frac{n_3 \sin \varepsilon_2}{n_3 \cos \varepsilon_2 - 1}\,,$$

and, finally,

$$a_2 = -\,(a_1 + i'_3) \qquad \ldots \qquad \ldots \qquad (59)$$

It will be found much more convenient to determine the dispersion by calculating the path of the individual coloured rays than by the differentiation of the deviation.

The above system of two prisms may be regarded as one half of a symmetrical system of prisms consisting of two similar external prisms having a refracting angle a_2 and a refractive index n_3, and an inner prism having a refracting angle $2a_1$ and a refractive index n_2.

Fig. 103 shows a direct-vision so-called triple Amici dispersion prism consisting of a flint glass prism having a large refracting angle bounded on either side by a crown glass prism cemented to it. The width q of the effective cross-section measured at right angles to the refracting edge is less than the height h of the prism, their ratio being

$$\frac{q}{h} = \frac{\cos i_1 \cos i_2}{\cos i_1 \cos i_2 + \sin a_1 \sin \varepsilon_1},$$

from which it will be seen that the unused portion of the height, *i.e.* $h - q$, is related to the effective portion q in the ratio

$$\frac{h-q}{q} = \frac{\sin a_1 \sin \varepsilon_1}{\cos i_1 \cos i_2}. \quad \ldots \quad \ldots \quad (60)$$

It will thus be seen that the prism may be reduced in size by cutting it at right angles to the direction of incidence of the pencils in such a way as to remove the unused portion. The length l by which the multiple prism may be shortened at either end of its broad base is related to the unreduced length of the base in the ratio

$$\frac{l}{L} = n_2 \tan i'_1 \sin \varepsilon_1, \quad \ldots \quad \ldots \quad (61)$$

where n_2 denotes the refractive index of the outer crown prisms.

Extensive investigations on triple and multiple Amici dispersion prisms, including the computation of the displacement by a triple system of the point of common intersection of convergent pencils of rays, have been carried out by Hepperger (*1.*).

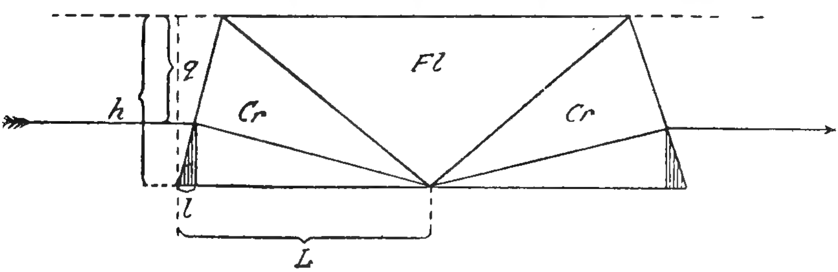

FIG. 103.

Direct-vision Amici dispersion prism. q is the width of the pencil which traverses the prism without being cut off or deviated.

254. Achromatic Systems of Prisms.—The condition that a prism may not give rise to an increase in the dispersion, or, which is the same thing, that it may be achromatic, results in general terms if we equate to zero the right side of equation (53) in § 250. Where, as is generally the case, the incident pencil is free from dispersion, in which case $\delta i_1 = 0$, the condition of achromatism according to equation (53) is

$$\sum_{v=2}^{k} \delta n_v \sin a_{v-1} \cdot \prod\nolimits_{v+1}^{k} \cdot \prod\nolimits'^{\,v-2}_{1} = 0 . \quad \ldots \quad (62)$$

An achromatic system of prisms imparts an unvarying deviation ε to the principal rays of all wave-lengths for which the condition expressed by equation (62) is satisfied. This deviation can be calculated with the aid of equations (14) and (15) in § 234, by introducing the refractive indices for one of the two wave-lengths considered in equation (62).

As an example, we may derive from equation (62) the formulæ for the computation of an **achromatic prism consisting of two cemented prisms.**

In our case since, $k = 3$, equation (62) assumes the form:

$$\delta n_2 \sin a_1 \cos i_3 + \delta n_3 \sin a_2 \cos i'_1 = 0.$$

If in this example we replace i_3 by $i'_2 - a_2$ and expand, we obtain the following

$$\delta n_2 \sin a_1 \cos i'_2 \cot a_2 + \delta n_2 \sin a_1 \sin i'_2 + \delta n_3 \cos i'_1 = 0$$

$$\delta n_1 = \delta n'_3 = 0 . \quad \ldots \quad \ldots \quad (63)$$

This is the general condition that a pair of prisms consisting of a prism of refracting angle a_1 and dispersion δn_2, and a prism of refracting angle a_2 and dispersion δn_3 cemented to the first prism, may be achromatic for the two selected colours at any given angle of incidence.

The equation should be suitably transformed in accordance with the above-mentioned requirements. Supposing a_1 to be given and that it is required that the ray should enter the first prism under the conditions of minimum deviation for one of the two colours, the refracting angle a_2 of the second prism will be obtained from the formula:

$$\cot a_2 = -\frac{\delta n_3}{2\,\delta n_2 \sin \dfrac{a_1}{2} \cos i'_2} - \tan i'_2,$$

where

$$\sin i'_2 = -\frac{n_2}{n_3} \sin \frac{a_1}{2} .$$

A case of greater practical importance arises when the incident ray is at right angles to the first prism surface and is made to deviate, generally by a small amount. The appropriate formulæ follow at once from equation (63).

When a system of prisms of small refracting angle and transmitting rays at small angles of incidence is to be achromatised, the dispersion should be

$$\delta\varepsilon = \delta n_2\, a_1 + \delta n_3\, a_2 = 0\,,$$

i.e. the prism angles a_1 and a_2 should satisfy the condition

$$a_2 \,/\, a_1 = -\,\delta n_2 \,/\, \delta n_3\,.$$

When the system is required to produce any given deviation ε the two prism angles a_1 and a_2 may be determined in the same manner as k_1 and k_2 were found previously (§ 195) in the case of a system of lenses of power ϕ, thus :

$$a_1 = \frac{\varepsilon}{\delta n_2\,(\nu_2 - \nu_3)}\,; \quad a_2 = -\,\frac{\varepsilon}{\delta n_3\,(\nu_2 - \nu_3)}\,. \qquad (64)$$

Conversely, if it be required to produce a dispersion $\delta\varepsilon$ without deviation the corresponding conditions take the form :

$$a_1 = -\,\frac{\delta\varepsilon}{\delta n_2}\,\frac{\nu_3}{\nu_2 - \nu_3}\,; \quad a_2 = +\,\frac{\delta\varepsilon}{\delta n_3}\,\frac{\nu_2}{\nu_2 - \nu_3} \quad \ldots \qquad (65)$$

where

$$\nu_i = \frac{n_i - 1}{\delta n_i}\,.$$

The secondary spectrum affects the dispersion to which these thin prisms give rise, or the achromatism attained by their means, in an exactly similar manner as in the case of lenses, the focal difference in a lens being analogous to an angular deviation in a prism. It will accordingly not be necessary to consider the problem again in detail.

Frequently the results derived from the investigation of the conditions which are applicable to the case of thin prisms of infinitely small angles are applied to prisms having finite angles. This is quite inadmissible. The magnitude and order of the dispersion, in particular, are in the course of a single refraction markedly affected by the magnitude of the angle of incidence ; and, in the succeeding refractions, as we have seen, they are further affected by the preceding values. From a more exhaustive investigation it follows in the first place that the spectra formed by two prisms of similar material but having different prism angles are not proportional, even if both occupy the

same position, for example, that corresponding to minimum deviation; and, in the second place, that the spectra due to prisms of dissimilar materials are not necessarily dissimilar, though the true dispersions of these materials are not disposed proportionally in their spectra. On the contrary, these relations are largely governed by the amount and the order of the refractions. It is therefore possible to produce prism systems of optically identical materials which are exclusively of the direct-vision type with finite dispersion, or which are exclusively achromatic with finite deviation. It is even possible to form by two similar prisms a direct-vision system of prisms by so opposing the second prism with its refracting edge reversed with respect to the first, that the ray may enter the second prism at the same angle as that at which it emerges from the first prism. Under these conditions it cannot be said that there shall be no secondary spectrum. When two prisms identical in composition and angles, are so arranged that their inner and outer surfaces are parallel, and that the pair of prisms forms a plane parallel plate, in this case only do the deviation, the dispersion and the secondary spectrum necessarily vanish simultaneously. In a more or less complete form these conditions were investigated by Brewster (***1.***).

Chapter IX.

THE LIMITATION OF RAYS IN OPTICAL SYSTEMS (THEORY OF STOPS).

(M. v. Rohr.)

1. PROJECTION SYSTEMS FOR OBJECTS EMITTING RAYS IN ALL DIRECTIONS.

255. Disregarding for the moment the spherical and chromatic aberrations treated in preceding sections of this work, we may consider an optical instrument as a device whereby an image of objects is formed by a process which ensures that corresponding to every object-point there is a single image-point. When the objects are at different distances s along the axis their images will likewise occupy positions at different distances s' along the axis, and hence a configuration of solid objects in the object-space gives rise to a relief-image which is conjugate point for point to the solid configuration in the object-space.

Now, whatever the nature of an optical instrument, whether it be designed for visual observation or for projection on a receiving screen, the relief-image never comes into play as such. In both classes of optical instruments there is a given surface upon which the images are received. In instruments designed for visual observation this surface is the retina of the eye, and in projection systems it is the screen or the sensitive plate. The process by which an image is thus thrown on a given surface by an optical instrument is, however, in no way to be confounded with the optical formation of the image itself, which it was the object of our investigation of the aberrations to render as perfect as possible. In attempting to receive an image on a screen we must realise that such a screen cuts the pencils which tend to form the relief-image referred to. What we perceive on this screen is a true image for those object-points only which are conjugate to the screen surface, whilst for all other points situated in front of or behind these points the place of

an actual image-point is occupied by a **figure of confusion** due to the intercepted cone of rays which traverse the screen. The association of true image-points with the substitutive figures of confusion, which arise on the retina or the screen, constitutes the relief-image and is interpreted by us as such. From these considerations it is clear that the formation of images of object-points which are not conjugate to any point on the image-receiving surface may be regarded as being brought about by a projective process, in that the intersection of the image-forming pencil by the receiving screen, *i.e.* the figure of indistinctness, may be looked upon as the projection of the base of the pencil through the image-point upon the receiving screen.

In the following investigations we shall make certain assumptions which conform to conditions as they exist in the great majority of the cases that arise in practice. We shall assume that we are dealing with centred systems of spherical surfaces, and that the boundary of the pencils of rays is determined by circular stops (diaphragms) arranged concentrically about the axis. We shall, at the outset, confine our attention to projection systems; and as a concrete case we shall choose the photographic objective as the best developed representative. We shall further suppose the receiving screen to be at right angles to the optical axis, so that we may rightly speak of **circles of confusion.**

It should also be recalled that for the purposes of this investigation we have supposed the system to be free from aberrations. On this assumption all data relating to the position of the image and its size, and linear and angular magnification, as determined by the usual approximate formulæ, will likewise hold for finite apertures and inclinations of the principal rays. In the following investigation we shall, accordingly, extend further the limits within which the quantities previously ascertained with respect to paraxial rays can be rightly applied; and, where nothing is said to the contrary, we shall assume that the images are flat and free from distortion. When it is necessary to take image-defects into consideration we shall expressly qualify our assumptions.

256. Determination of the Entrance and Exit Pupils. —Of all rays which are emitted by an object-point those only go to form the image that are transmitted through the system. To determine these effective rays we shall make use of the principle of the reversibility of the path of the rays, which holds for every optical instrument. In Fig. 104 let the eye be situated at O, which is the position of the object on the axis. At the conjugate image-point O' let a ground glass focussing screen be illuminated from the back, so that it may radiate light in all directions, whilst no light is allowed to enter from the direction of the object. Then the eye, which we will suppose to have an unlimited power of accommodation, will see from

its position at O on the axis a bright circle on the object side, and this circle furnishes a measure of the base of the pencil which reaches the eye and which, conversely, proceeding from the object-point, which is supposed to emit rays in all directions, is transmitted through the system. Now, the boundary of this system is obviously furnished by a circular concentric stop at the objective, which may be the mounting of the lens itself, or a diaphragm specially introduced. The only stop which can be seen directly is the margin of the mount or the diaphragm situated in front of the first lens surface, whilst all others are furnished by the stop-images (mostly virtual) which are formed on the side facing the object by the components of the system situated between the eye and the respective stops. Each of these stop-images, including the physical stop situated in front of the system, would determine the boundary at

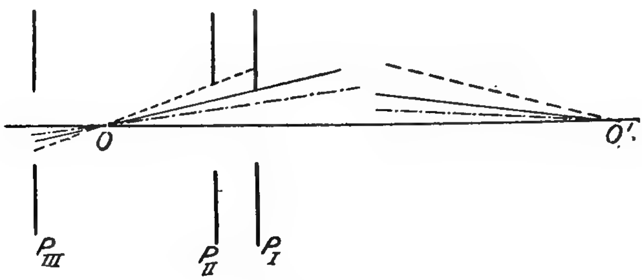

FIG. 104.

O : object-point ; O' : image-point. P_I, P_{II}, P_{III} : stop images on the object side.

— — — — — — extreme ray admitted by P_{II},

———————— extreme ray admitted by P_I and hence transmitted through the system.

—.—.—.— an intermediate ray of smaller angular aperture.

Magnitude of the entrance pupil.

the base of the pencil of rays reaching the eye if it were not associated with other stop-images. Since, however, the various images exist simultaneously, the duty of limiting the rays on the object-side is performed by the stop-image which subtends at the eye the smallest angle. In our case this is P_I. A similar restriction of the aperture might also be effected by a stop-image such as P_{III} situated on the left of O ; for, as a means of excluding any ray it is immaterial where the intercepting obstacle is placed in the path of the pencil traversing the optical system.

We may now conceive a case such as that represented in Fig. 105, in which there are two stop-images P_{I} and P_{II} subtending equal angles as seen from O_{I}. In this case the lines drawn crosswise so

as to join the boundaries of the two stops furnish another point O_{II}, where the stop-images subtend equal angles. The two points O_I and O_{II} divide the axis into two segments $\overline{O_I\, O_{II}}$, and $\overline{O_{II}\, \infty\, O_I}$ in such a manner that P_{II} appears at a smaller angle than P_I as seen from any point on the segment $\overline{O_I\, O_{II}}$, whilst P_I appears at a smaller angle than P_{II} as seen from any point on the segment $\overline{O_{II}\, \infty\, O_I}$. In this case the positions of the limiting stop may vary with the distance of the object, and it will be readily seen that the addition of other stops may lead to a further subdivision of the axis into segments, each of which has corresponding to it its own particular effective stop.

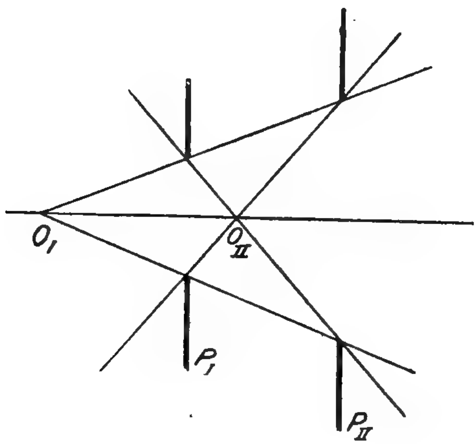

FIG. 105.

P_I, P_{II} ; Stop-images on the object-side ;
O_I, O_{II} ; Extreme positions of axial object-points.
The case of two entrance pupils.

In the preceding section we have already shown the important part which the effective stop plays in the correction of a system, and it will therefore be realised at once that an abrupt change in the size of the effective stop, such as would occur at O_I or O_{II}, may seriously affect the state of correction of the system. We shall therefore exclude these cases from our investigation, and we shall restrict the range of displacement of the object on the axis in such a way that the boundary of the pencil of rays is formed by a single stop-image. The physical stop, whatever its position, is then called the **aperture stop**, and to its image which is formed by the components of the system preceding it on the side facing the object, we give the name **entrance-pupil**, whilst the image formed by the succeeding parts of the system is called the

exit-pupil. Then the entrance-pupil and exit-pupil are related with respect to the entire system in the manner of an object to its image. From equation (21) in § (55),

$$\tan u' = -\frac{x_s}{f'} \tan u,$$

it follows that the exit-pupil determines the aperture of the pencils on the image-side. For in any given system the factor of $\tan u$ depends solely on the position of the object and not on that of the stop. Hence also, in any given system, the image of the aperture stop on the image-side, as seen from the image-point O', appears under a smaller angle than any other stop-image.

A. Narrow Aperture Stops.

257. The Field Stops and the Windows (Entrance-window and Exit-window). In the case of a photographic objective, which we are now considering, the possible occurrence of several different entrance-pupils does not arise, and in this case the aperture is limited by the image of a stop of variable diameter, which for the present we will suppose to be so small that it may be regarded as a mere point. Since under all circumstances the stop is central we shall only consider its centre. This point-like entrance pupil fixes the boundary of the aperture of the system within the entire compass of the object in that it transmits none but single rays intersecting the axis at the centre of the aperture stop. In conformity with the notation to which we have adhered in the investigation of the aberrations, we shall describe those rays which proceed from the object-space to the entrance-pupil as the **principal rays.**

In Fig. 106, let planes be described at right angles to the axis at the object-point O and image-point O', and let the ground-glass focussing screen at O' be illuminated from the back. Then it is clear that of all the rays proceeding from every luminous point A' on it those only which proceed to the exit-pupil pass through the system. After traversing the system they emerge through the entrance-pupil and intersect the object-plane at a point A conjugate to the point A'. The entrance-pupil and the exit-pupil are, however, not the only stops present in the system. There are other diaphragms whose images on the object-side we will assume to be situated at W_I, W_{II}. Hence, if we cause the selected point on the ground glass screen to move farther away from the axis towards B', the part of the principal ray proceeding from the object will ultimately graze the boundary of the diaphragm W_I. Now, if we consider image-points at a still greater distance from the axis there will be no emergent ray corresponding to them, and hence in the object-space the indefinitely large ground glass focussing screen

illuminates only a circle of radius OB. The diaphragm images W_I, W_{II} . . . . affect the principal rays proceeding from the entrance pupil and those proceeding towards it in precisely the same manner as physical diaphragms of similar size and placed in a similar position. To determine the size of the cone of principal rays which is allowed to pass through all stops we must ascertain the size and position of the diaphragm-image which appears to subtend the smallest angle as viewed from the entrance-pupil. We shall call this diaphragm-image the **entrance-window.** The cone of principal rays having its apex at the centre of the entrance-

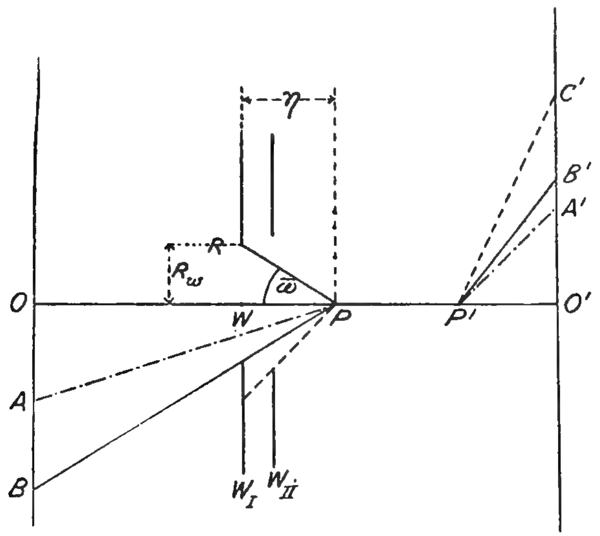

FIG. 106.

O : Object-point ; O' : Image-point ; W_I, W_{II} : Diaphragm-images on the object-side.

— — — — — : Extreme ray passing through W_{II} ;

———————— : Extreme ray passing through W_I and accordingly through the system ;

—. — —. — : An intermediate ray of smaller angular aperture.

The entrance window and the field of view on the object-side.

pupil and whose base is the entrance-window divides the object-space into two parts, one containing all the points whence rays can proceed through the system and the other containing all those points whence no such rays are transmitted through the system. The section in which this cone cuts a plane at right angles to the axis on the object-side is a circle at right angles to the axis and concentric with it, and which we will call the **field of view on the object-side.** To obtain an expression for ϖ, the solid angle of the cone, *i.e.*, the extreme angle of inclination of the principal ray on the object-side,

let $\eta = PW$ be the distance between the entrance-pupil and the entrance-window, and let R_w be the radius of the entrance-window. Also, let the investigation be confined to the space above the axis at O between the entrance-pupil and the normal plane at O. The magnitude of the angle ϖ will then be

$$\tan \varpi = \frac{R_w}{-\eta} \text{ or } \tan \varpi = \frac{R_w}{\eta},$$

according as the entrance-window or the entrance-pupil is the first stop to be traversed by the light coming from the object. The stop whose image constitutes the entrance-window and which determines the limiting angle of the inclination of the principal ray on the object-side is accordingly the **field-stop.** If now we suppose the images of all stops to be formed on the image-side, the image corresponding to the field-stop will constitute the **exit-window** and limit the field of view on the image-side, since, together with the exit-pupil, it determines the **limiting angle of inclination** of the principal ray on the image-side. By a notation analogous to that adopted above its magnitude may be expressed by

$$\tan \varpi' = \frac{R'_w}{-\eta'} \text{ or } \tan \varpi' = \frac{R'_w}{\eta'}.$$

In general, this differs from that corresponding to the object-side. That the exit-window, which constitutes the image of the entrance-window, actually limits the field of view on the image-side follows analogously from the fact that the exit-pupil on the image-side determines the aperture.

Now, in all instruments having a considerable field of view and having the aperture stop situated between the components of the system, it is a desirable condition that the apertures of the mounts of these parts, as seen from the centre of the aperture stop, should subtend equal angles. When this case arises their images on the object-side and on the image-side subtend equal angles at the entrance and exit-pupils. We then have the case of two entrance-windows, which is of considerable importance when the aperture stop is of finite size. We are therefore justified in formulating the following statement:

The stop which subtends the smallest angle at the object-point O, *i.e.*, the aperture diaphragm, having been defined as regards position and then narrowed down to a pin-hole, its image on the object-side together with the stop-image on the object-side subtending the smallest angle at O, viz., the entrance-window, determines the limiting angle ϖ of the inclination of the principal ray on the object-side; whilst the image of the aperture stop on the image-side together with the exit-window, *i.e.*, the image of the stop subtending the smallest angle at O' on the image-side, gives a measure of the limiting angle ϖ' on the image-side.

258. Magnification due to a Projection System.—Let y_s be the height of a given object at right angles to the axis at a distance A_p from the entrance-pupil* and let y'_s be the height of the corresponding image at a distance A'_p from the exit-pupil. We then have the following relations between these quantities and the angles which the object and image subtend at the entrance-pupil P and the exit-pupil P'.

$$y'_s = -A'_p \tan w'; \qquad y_s = -A_p \tan w.$$

In conformity with the convention applied to paraxial rays the linear magnification β is represented by the expression

$$\beta = \frac{y'_s}{y_s} = \frac{A'_p \tan w'}{A_p \tan w} = \frac{A'_p}{A_p}\Gamma.$$

In the case of projection-systems the magnification β is frequently a common fraction and is then usually spoken of as the **scale of reduction.**

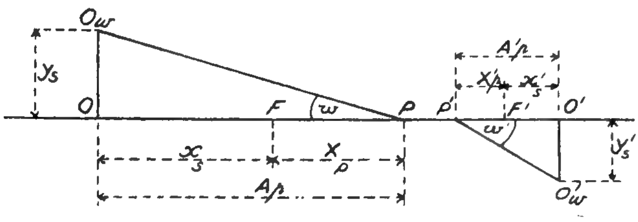

FIG. 107.

$$FO = x_s;\ F'O' = x'_s;\ FP = X_p;\ F'P' = X'_p;\ PO = A_p;\ P'O' = A'_p.$$
$$OO_w = y_s;\ O'O'_w = y'_s.$$

Diagram showing the introduction of the focal distances in the expression for the magnification.

From the formula given for β it will be seen that the magnification is proportional to the distance A'_p between the image-plane and the exit-pupil. If for any reason (for example owing to the smallness of the convergence of the image-forming pencils) this distance is determined with insufficient accuracy, the inaccuracy will affect the value of β.

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* As in the succeeding articles we shall have frequent occasion to measure the intercepts on the axis with reference to conjugate points we shall do well to express this in the notation. In our special case, in which the conjugate points coincide with the pupils we shall replace the general symbols A, A' adopted in § 60 by A_p, A'_p.

For this reason the expression may be given in a modified form, denoting, in accordance with Fig. 107, the distances of the pupils from the principal foci by X_p, X'_p, and those of the object and image by x_s, x'_s respectively. Then since

$$P'O' = P'F' + F'O'\,; \quad PO = PF + FO\,,$$

$$A'_p = x'_s - X'_p\,; \; A_p = x_s - X_p\,.$$

By equation (21) § 55

$$\Gamma = -\frac{f}{X'_p} = -\frac{X_p}{f'}\,,$$

and hence we may write

$$\beta = f\frac{1-\dfrac{x'_s}{X'_p}}{x_s - X_p} \quad \text{or} \quad \beta = \frac{1}{f'}\frac{x'_s - X'_p}{1-\dfrac{x_s}{X_p}}\,.$$

If now, in the design of an instrument the entrance-pupil is chosen coincident with the front principal focus, we shall have the equations

$$X_p = 0\,; \quad X'_p = \infty,$$

and hence

$$\beta = \frac{f}{x_s} = \frac{f}{A_p}\,.$$

In this case the determination of the value of the magnification is not affected by changes in the position of the image (A'_p, x'_s), since the principal rays are parallel to the axis within the image-space. The configuration of the rays has been described by Abbe (*4.*) as a system of rays which is **telecentrical on the image-side.**

An analogous case results when the exit-pupil coincides with the back focus. In this case the principal rays are parallel to the axis in the object-space; the system of rays is **telecentrical on the object-side,** and the magnification is then expressed by the second formula obtained for β, viz.:

$$\beta = \frac{A'_p}{f'}\,.$$

There now remains the case in which the object to be projected is situated at infinity. It is then necessary to compare the linear magnitude of the image y'_s with the angular magnitude $\tan w$ of the object. Forming the appropriate expression, we obtain

$$-\frac{y'_s}{\tan w} = A'_p\,\Gamma = f\left(1 - \frac{x'_s}{X'_p}\right) = f\left(1 - \frac{X_p}{x_s}\right).$$

In the limit when $x_s = \infty$ we obtain the magnitude of the image formed by a projection system of focal length f of an infinitely distant object subtending an angle w. The equation then becomes

$$-\frac{y'_s}{\tan w} = f.$$

259. Planes in Focal Adjustment.—In our introductory remarks we pointed out that true images were formed on the projection screen corresponding only with certain definite points of the solid object. If now we determine that point O of the object on the axis of the system which is conjugate to the centre of the image-receiving screen at O', and at O let a plane be described at right angles to the axis, the latter is the focussed plane of the object, *i.e.* **the field plane F.P.**, and is conjugate to the plane in which it

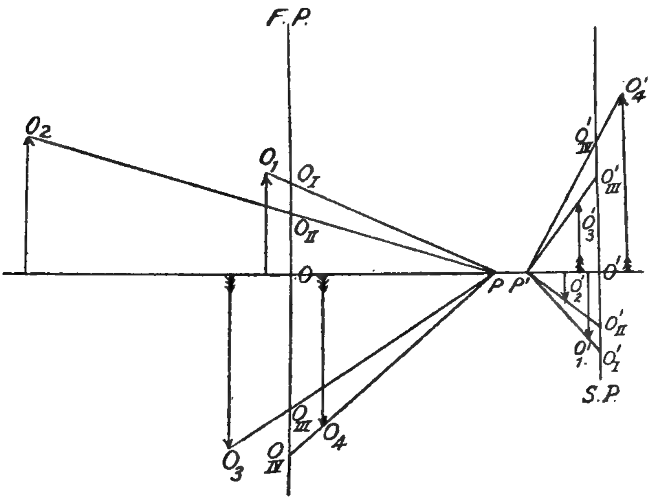

FIG. 108.

$F.P.$ = Field plane at O; $S.P.$ = Screen-plane at O'.
P = entrance-pupil; P' = exit pupil.
O_I, O_{II}, O_{III}, O_{IV} = representative points in the object-space (the field).
O'_I, O'_{II}, O'_{III}, O'_{IV} = their conjugate points on the screen and likewise representative points in the image space.
The projected field-picture and its reproduction on the screen.

appears sharply defined, *i.e.* the **screen plane S.P.** Any point not contained in the field plane (Fig. 108), together with the entrance-pupil, determines a principal ray which, after traversing the system,

emerges from the exit pupil and cuts the screen-plane at a certain point which is not conjugate to the selected object-point. In fact, to find the conjugate object-point in the object-space we must assume the principal ray to be produced backwards to where it cuts the field-plane. Since this applies to every point of the screen plane we arrive at the statement:

The cone of principal rays determined by a pin-hole entrance-pupil and by the aggregate of object-points maps out on a field-plane, which for the present we will suppose to be arbitrarily chosen, a projection figure, the **projected field picture**, of which the system forms a scale image on the screen-plane.

It will thus be seen that the entire process by which an image is formed by means of a system stopped down to a pinhole, and by which external objects of three dimensions appear represented on the focussing screen, is reduced in essence to the reproduction of a single plane, at right angles to the axis, viz., the field-plane; and the only share which the optical system has in the formation of the projected field picture on the focussed field-plane is that it determines the position of the entrance-pupil.

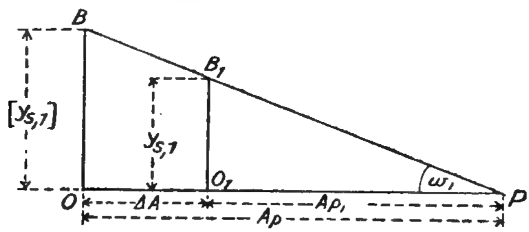

FIG. 109.

$PO = A_p$; $PO_1 = A_{p,1}$; $\Delta A_p = OP + PO_1 = OO_1$; $O_1B_1 = y_{s,1}$; $OB = [y_{s,1}]$.

The perspective longitudinal variation of objects which are out of focus.

We shall now proceed to study more closely the genesis of the projected field picture. From Fig. 109 it will be seen that

$$A_{p,1} - A_p = \Delta A_p,$$

where A_p is the distance of the field-plane from the entrance-pupil, and $A_{p,1}$ the distance of another object of magnitude $y_{s,1}$ from the latter. ΔA_p will then be negative or positive according as the object-point is situated in front of or behind the focussed field-plane as measured in the direction of the incidence. In what follows we shall always regard it as positive and employ the sign $\pm$ accordingly.

Projecting $y_{s,1}$ from the centre of the entrance pupil into the focussed field-plane and denoting this apparent magnitude by $[y_{s,1}]$, we obtain the equation

$$\frac{[y_{s,1}]}{y_{s,1}} = \frac{A_p}{A_{p,1}}$$

and hence

$$[y_{s,1}] = \frac{y_{s,1}A_p}{A_p \pm \Delta A_p},$$

and therefore

$$[y_{s,1}] = y_{s,1}\left(1 \mp \frac{\Delta A_p}{A_p}\right), \quad \text{approximately,}$$

so long as the distance of the second object from the focussed field-plane is small in comparison with that from the entrance-pupil, as is generally the case in projected systems.

260. The Perspective of Projected Pictures. — The quantity

$$\frac{[y_{s,1}] - y_{s,1}}{y_{s,1}} = \mp \frac{\Delta A_p}{A_p \pm \Delta A_p}$$

is a **measure of the relative perspective shortening** or **lengthening.** This quantity acquires practical significance when two objects of equal magnitude $y_{s,1}$ and separated by a distance ΔA_p, are viewed from a point at a distance A_p from one of these objects. If we write the expression in the form

$$\frac{[y_{s,1}] - y_{s,1}}{\mp \Delta A_p} = \frac{y_{s,1}}{A_p \pm \Delta A_p} = \frac{[y_{s,1}]}{A_p} = -\tan w_1,$$

we see that the ratio of the distance between the objects of equal size and the perspective shortening itself does not depend upon the distance A_p but solely upon the angle of view w_1 subtended by the object upon which the system is focussed. Hence the impression which the retina receives does not change when the eye is made to view at a similar angle w_1 a reproduction of the projected field-picture magnified or reduced in the scale of ε. We may under these conditions speak of the reduced or magnified **reproduction of the projected field-picture** as **having the same perspective** as the projected field-picture itself.

When viewing such a reproduced field-picture of relative size ε, where the objects are of a known kind, we instinctively judge them from common experience of similar objects.

Thus it may be that we know something of the transverse dimensions of the various objects which are not in focal adjustment. In this case we draw our conclusions respecting these dimensions in the depth of an object represented by the reproduced field-picture in conformity with the equation

$$\pm \varepsilon \, \Delta A_p = \varepsilon \, ([y_{s,1}] - y_{s,1}) \cot w_1 \, .$$

In the alternative case we may have some data by which we can estimate relative positions in a direction towards or away from us. In this case we can form an idea regarding relative transverse dimensions of objects situated in front of or behind the focussed field-plane in conformity with the equation

$$\varepsilon y_{s,1} = \varepsilon \, [y_{s,1}] \mp \varepsilon \, \Delta A_p \tan w_1 \, .$$

When there is nothing to guide us in our appreciation of the natural space relations the impressions which we receive enable us, generally subconsciously, to re-construct mentally an object every part of which is ε times greater or smaller than the corresponding part of the natural object, and this mentally re-constructed object, when viewed from a position of the eye at a distance εA_p (and from this position only) and disregarding the effects of changes in the accommodation of the eye, produces the same impression as the object itself when viewed from the centre of the entrance-pupil.

The corresponding image on the screen-plane, which we shall call the **projected screen-picture,** is strictly conjugate to the projected field-picture with respect to the given optical system, no matter in what manner the inclination of the principal rays is modified by the optical system. As a rule, it is required that the screen-picture shall be **similar** to the projected field-picture; that is, it is required to be free from distortion, in which case the two pictures are capable of being brought into perspective relation. In this case they are conjugate point to point by virtue of rays proceeding from the entrance-pupil, and this holds also when there is no image-forming system between the two picture-planes.

Such a geometrically similar projected screen-picture is subject to the same relations as those deduced previously with respect to a reproduction in the scale of ε, if in the former case we replace β by ε.

When the objective is free from astigmatism but shows a certain amount of curvature of the field, the screen-plane no longer corresponds to the field-plane, but rather to a field-surface on which it will be necessary to ascertain the projected field-picture. The perspective is not thereby affected in any way; for if the projected

screen-picture be correctly placed in front of the entrance-pupil it will coincide, point for point, with the configuration on the sharply focussed surface.

The limitation of the field of view is generally determined by the projection of the entrance-window from the entrance-pupil upon the field-plane. When the focussed object comprises only a portion of the available field the projection of the boundary of the object itself into the field-plane determines the boundary of the field of view.

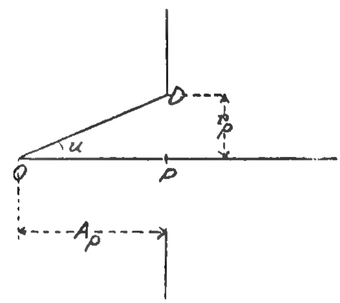

FIG. 110.

$PO = A_p$; $PD = r_p$.

Determination of the field-aperture in terms of the radius of the entrance-pupil and the distance of the object.

B. Finite Aperture Stops.

261. In our discussion of the spherical aberration in § 122 we introduced a quantity defined by Abbe as the numerical aperture of a system, viz.

$$[NA] = n \sin u .$$

Now, if the radius of the entrance-pupil is a given finite quantity r_p, and if the object-point is at a distance A_p from the entrance-pupil, we have the relation

$$\tan u = \frac{r_p}{A_p}$$

and also

$$[NA] = \frac{n r_p}{\sqrt{A_p^2 + r_p^2}} .$$

This equation furnishes a measure of the change which the aperture experiences in consequence of any variation of the aperture of the entrance-pupil or by reason of any displacement of the object-

point. In the latter case we must consider whether $A_{\prime\prime}$ is positive or negative, that is whether the entrance-pupil lies in front of or behind the object-point. It follows that the aperture diminishes in the first case when the object-point is displaced in the direction of the incident light, and the nearer the object-point is initially to the entrance-pupil the greater will be the effect produced by a given displacement.

262. Mutual Independence of Perspective and Aperture.—If now we consider the process by which an optical picture is formed of a solid object by the intervention of a system having a finite entrance-pupil we can no longer confine our attention to the principal rays. The presence of a finite entrance-pupil will then have the effect that every object-point which is not at too great a distance from the axis, the exact limits of which will be discussed later, will send through the entrance-pupil of the system not merely principal rays but entire cones of rays having their bases at the entrance-pupil and their vertices at the object-point. On the image-side the base common to this two-fold system of cones is furnished by the exit-pupil. Since the screen-plane cuts these cones in front of, at, or behind their vertices, it follows that in systems of finite apertures the projected screen-picture will necessarily consist of image-points and substitutive circles of confusion, according as the corresponding object-points are situated within or without the field-plane. The object corresponding to this screen-picture is obviously the field-picture which arises when the finite entrance-pupil is projected through all object-points into the field-plane, whilst every circle of confusion is concentric with the point where the principal ray cuts both the field-plane and the screen-plane, as shown in Fig. 111.

This conception of a field-picture replacing the solid object is particularly useful when systems of very wide apertures enter into consideration. As was seen in § 165, it is in such cases no longer permissible to speak of point-to-point representation of points in front of or behind the field-plane. On the other hand, we may regard the various points of the circles of confusion as points of the aplanatic field-plane, such that they emit rays in a single direction (determined by the external object-point) and that by reason of their singular position they are represented in the aplanatic image-plane with constant magnification.

It will thus be seen that even under such extreme conditions as we have here indicated, the point where the principal ray cuts the plane of reference persists as the centre of the circle of confusion, and the perspective does not undergo any modification since unconsciously we seek the image-point in the centre of the circle of confusion.

Now, pictures composed of points and substitutive circles of confusion, when projected upon a given surface, cannot be expected to result in optical pictures giving a true impression of nature, since a projection system lacks the power of accommodation possessed by the eye, which enables us to perceive objects distinctly at widely different distances in rapid succession. The defect would be quite noticeable and most disturbing if the eye were endowed with absolute acuity of vision. From the construction of the eye this is, however, by no means the case, so that we are not able to distinguish any lack of precision which does not exceed a certain

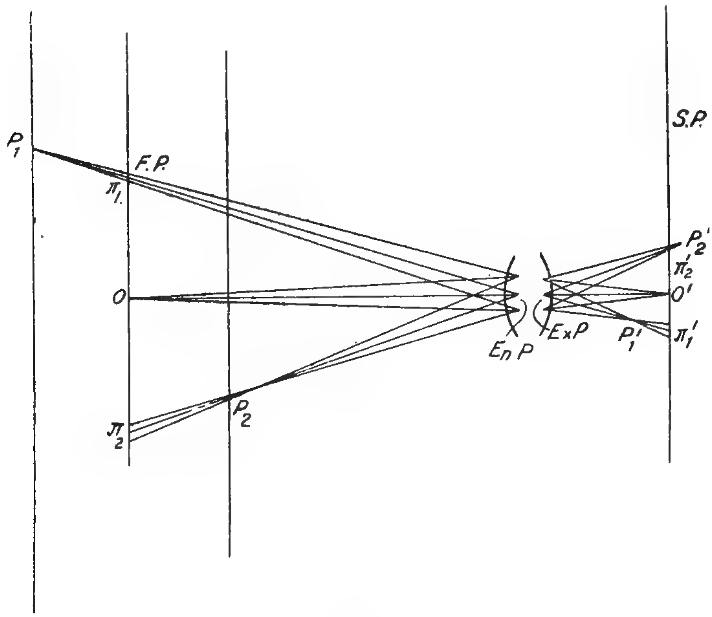

FIG. 111.

Diagram showing the field-picture on the field-plane at O, consisting of points O and corresponding circles of confusion π_1, π_2, and the corresponding screen-picture consisting of points O' and circles of confusion π'_1, π'_2.

Process by which a picture of a solid object (P_1OP_2) is projected by a system of finite aperture.

limit. Objects which do not attain this limit appear to our vision as points. This applies likewise to the circles of confusion in the field-plane. So long as they do not attain this angular measure of indistinctness when viewed from the entrance-pupil they figure as points and hence appear to the eye sharply defined. This consequence of the limited acuity of vision of the eye is known as the **depth of focus** of projection lenses.

263. Depth of Focus in terms of absolute measure.—We now proceed to investigate the magnitudes of the circles of confusion from data relating to the field by which they are governed, viz., the size and position of the entrance-pupil and the distances of the object-points from the field-plane.

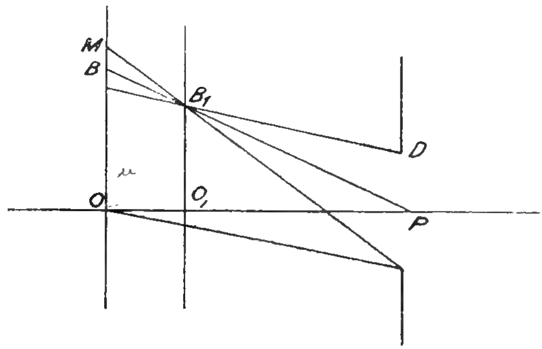

FIG. 112.

$PO = A_p$; $PO_1 = A_{p,1}$; $OO_1 = \Delta A_p$; $PD = r_p$; $BM = \delta$.
Diagram illustrating Depth of Focus.

Let A_p be the distance between the field-plane and the entrance-pupil, r_p the radius of the entrance-pupil, and $\pm \Delta A_p$ the distance of the object-point from the field-plane behind or in front of it. The magnitude of the radius δ of the circles of confusion in the field-plane then follows from the formula

$$- \delta = \frac{r_p \Delta A_p}{A_p \pm \Delta A_p} = \frac{r_p}{A_p} \frac{A_p \Delta A_p}{A_p \pm \Delta A_p} = \frac{A_p \Delta A_p}{A_p \pm \Delta A_p} \tan u$$

and when ΔA_p is relatively very small this expression reduces to

$$- \delta = \Delta A_p \tan u, \text{ approximately.}$$

From the more rigorous formula for δ we can calculate the distances $\Delta_1 A_p$ in front of, and $\Delta_2 A_p$ behind the field-plane (reckoning in the direction of the light) which the object-points may attain without exceeding the radius of indistinctness conforming to the angular sharpness of vision ζ, viz.

$$\delta = - \frac{1}{2} A_p \tan \zeta .$$

Then

$$\Delta_1 A_p = - \frac{A_p \delta}{r_p - \delta} ; \quad \Delta_2 A_p = - \frac{A_p \delta}{r_p + \delta} .$$

It will be seen that if we keep A_p constant $\Delta_2 A_p$ is always less than $\Delta_1 A_p$. In other words: The absolute indistinctness increases more rapidly when the object-point moves away from the field-plane in the direction of the incident light than when it moves away in the opposite direction. The entire range of depth, or **depth of field,** is the sum of $\Delta_1 A_p$ and $\Delta_2 A_p$, viz.

$$\Delta A_p = \Delta_1 A_p + \Delta_2 A_p = -\frac{2\, A_p\, r_p\, \delta}{r^2_p - \delta^2}.$$

The reciprocal of the depth of the field $1 / \Delta A_p$ can be regarded as a measure of the accuracy of the adjustment and, as a matter of fact, the precision with which the position of the screen-plane can be found depends upon the smallness of the depth of the field.

The formulæ for $\Delta_1 A_p$ and $\Delta_2 A_p$ hold quite generally for objects of any vertical dimensions y_s. Hence the results of our investigation can be stated in the following terms:

The diameter of the permissible circle of confusion having been ascertained, the space limits bounding the object-points whose quality of indistinctness is required not to exceed a given amount are determined by two planes parallel to the field-plane, the one in front being at the greater distance from the field-plane.

The connection between the quantities which we have defined in the preceding paragraphs may be expressed in a form which is more convenient for numerical calculation by the introduction of an auxiliary quantity a, viz.

$$\delta = r_p \tan \frac{a}{2};$$

thus

$$\Delta A_p = - A_p \tan a.$$

We shall discuss the various relations by which these quantities are connected when we come to deal with the theory of the most important projection system, viz. the photographic lens.* At this stage it will be sufficient to note that all these formulæ involve quantities relating exclusively to the entrance-pupil and its position with respect to the object-point, whereas the focal length of the transforming system does not enter into them.

264. Depth of Focus in terms of relative measure. —The counterpart of the absolute indistinctness in the projected field-picture, as represented in magnitude by the radius δ of the circle of confusion, is the **relative** indistinctness which accounts for the blurring of details in the projected field-picture. To obtain an expression for the relative indistinctness we must

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* Not included in the present work.

establish a relation connecting the apparent (*i.e.* perspectively changed) magnitude of the object $[y_{s,1}]$, as determined by the path of the principal rays, and the diameter of the circle of confusion due to the finite aperture of the entrance-pupil. Since according to § 259

$$\frac{[y_{s,1}]}{y_{s,1}} = \frac{A_p}{A_p \pm \Delta A_p},$$

it follows that

$$-2\delta \bigg/ \frac{[y_{s,1}]}{y_{s,1}} = \frac{2r_p\,\Delta A_p}{A_p} = 2\Delta A_p \tan u.$$

This signifies that the indistinctness caused by the formation of the projected field-picture, or the relative indistinctness, is determined by the angular aperture of the system, and that incidentally it is no longer governed by the sign but solely by the absolute distance between the object and the focussed field-plane. The space limits conforming to a certain specified relative indistinctness $\frac{\Theta}{y_{s,1}} = \frac{2\delta}{[y_{s,1}]}$ are determined by two planes at right angles to the axis and situated at equal distances $\Delta A_p = \frac{\Theta}{2\tan u}$ from the focussed field-plane, Θ being a specified fraction of $y_{s,1}$.

265. Secondary Limitation of the Aperture by the Windows.—In § 262, when showing that the perspective is not affected by the aperture of the projection-system, we restricted the transmission of complete pencils filling the entire pupil to object-points at a moderate distance from the axis, thereby suggesting that beyond certain limits the pencils might not find an unimpeded path.

We now proceed to determine these limits in precise terms. It is conceivable that rays issuing from points occupying particular positions may reach the centre of the entrance-pupil without meeting an obstacle, whereas marginal portions of the entrance-pupil may be covered by parts of the object. For the present we shall disregard such objects, and shall reserve their discussion for a chapter which will deal exclusively with photographic lenses.* By disregarding this contingency we are merely assuming that the distance of the entrance-pupil from the object is great in comparison with its diameter. We shall first consider the effect of the presence of a single entrance-window.

266. Secondary Limitation of the Aperture by a Single Entrance-Window.—We shall first consider the case in which the entrance-window is situated behind the entrance-pupil reckoned in

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* Not included in the present work.

the direction of the incident light. This case arises in photographic lenses fitted with a diaphragm in front, that is to say in all single landscape lenses.

Through the entrance-window let a plane be described at right angles to the axis. Then the projection of the entrance-pupil with respect to the object-point O on the axis, is a circle which is concentric with the entrance-window. Retaining the notation which in our previous investigation we applied to the radii of these stop-apertures, we obtain the following expression for the limiting angle of inclination ϖ of the principal object-ray corresponding to the positive distance $y_{s,II}$ from the axis, viz.:

$$\tan \varpi = \frac{R_w}{\eta}.$$

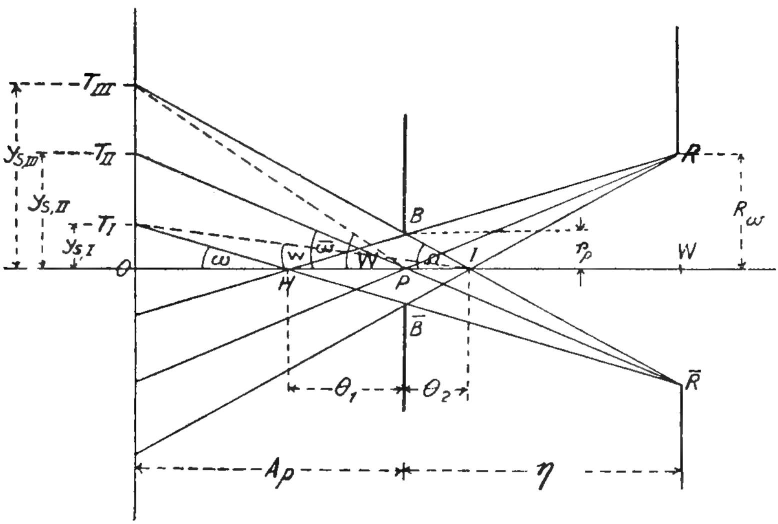

FIG. 113.

$PO = A_p$; $PW = \eta$; $PH = \theta_1$; $PI = \theta_2$; $OT_I = y_{s,I}$; $OT_{II} = y_{s,II}$; $OT_{III} = y_{s,III}$; $PB = r_p$; $WR = R_w$.

Division of the object-space by the entrance-pupil and an entrance-window into a full aperture region, a vignetted region, and an excluding region.

Now, in the meridian plane let the object-point O assume various positions at increasing distances from the axis, or in a plane normal to the axis at O let points be considered having their principal rays inclined at an angle w to the axis. The projection of the entrance-pupil on the entrance-window will then be an eccentric circle. This circle will ultimately be in internal contact with the periphery of the entrance-window, and we obtain the tangent ray if we join the

terminal points $\bar{B}$ and $\bar{R}$ of the two diameters on the same side of the axis. This straight line forms with the latter an angle ω, the magnitude of which follows from the relation

$$\tan \omega = \frac{R_w - r_p}{\eta} = -\frac{r_p}{\theta_1}.$$

We call this angle ω the **vignetting angle.**

The point of intersection H on the axis is situated at a distance θ_1, viz.:

$$\theta_1 = -\frac{r_p\,\eta}{R_w - r_p},$$

and by means of this quantity we obtain the radius $y_{s,I}$ of the field projected at full aperture at a distance A_p along the axis, viz:

$$y_{s,I} = \frac{r_p}{\theta_1} A_p - r_p,$$

and the angular field of view w of this **region of undiminished aperture** is

$$\frac{y_{sI}}{-A_p} = \tan \mathrm{w} = \tan \omega + \tan u.$$

Accordingly, when the vignetting angle ω is known in any given system having the entrance-pupil in front, the tangent of the angular field of view with respect to the region of undiminished aperture is equal to the sum of the tangents of the vignetting angle and the aperture angle.

Any increase in the inclination of the principal ray represented by $w > \mathrm{w}$ causes the aperture to be encroached upon by the entrance-window, and the open circle will then be replaced by the segment of a circle. The centre of the entrance-pupil is then projected upon the margin of the entrance-window for all object-points subtending at the centre of the entrance-pupil the angle which is equal to the limiting angle of the principal rays proceeding from the object.

The limit of the rays which can be transmitted at all can be found by joining $B\bar{R}$ and $\bar{B}R$, *i.e.*, by the crosswise connection of the terminal points of the two diameters. The **angle of exclusion** Ω follows from the equation

$$\tan \Omega = \frac{R_w + r_p}{\eta} = \frac{r_p}{\theta_2}$$

whilst the distance θ_2 of the point of intersection I follows from the equation

$$\theta_2 = \frac{r_p \eta}{R_w + r_p}.$$

Proceeding on similar lines as before, we may find the angle of view W of the limit of exclusion for the distance A_p on the axis, viz:

$$\frac{y_{e,III}}{-A_p} = \tan \mathrm{W} = \tan \Omega - \tan u.$$

The angle of exclusion having been thus determined in a system in which the entrance-pupil is in front, the tangent of the angle of view at the limit of exclusion is equal to the difference of the angle of exclusion and the angular aperture.

In these two rules the "sum" becomes a "difference," and conversely, when the entrance-pupil is at the back, *i.e.*, when the light proceeding from the object first traverses the entrance-window and then the entrance-pupil. Instances of this kind are furnished by the Galilean type of telescope of small and moderate magnifying power, and tele-photographic lenses having a single converging system and a diaphragm between the lenses. The quantitative investigation is analogous to that given above.

It will thus be seen that the entire field-space is divided up by these three solid angles, viz., the vignetting angle, the limiting angle of the inclination of the principal ray and the angle of exclusion, into three regions which we may define, respectively, as the region of undiminished aperture, the region of undiminished-to-half aperture, and thirdly the region of half-to-vanishing aperture. In dividing the field up in this way we should, however, keep in mind that it is typical of a system with one entrance-window only and that the principal ray serves as the centre-line of figure of a pencil of finite aperture so long as $w \leqq \mathrm{W}$. As soon as w exceeds this limit the centre of figure of the area of confusion, which will now be a segment of a circle, ceases to coincide with the point where the principal ray intersects the entrance-window, since this is the centre of a single boundary circle. Whenever the object-points are situated outside the focussed field-plane, the positions of the representative points due to pencils of finite aperture within the region for which $w \geqq \mathrm{W}$ may be very different from those prevailing within narrow pencils.

267. Secondary Limitation of the Aperture by Two Entrance-Windows.—When there are two entrance-windows let the diagram Fig. 114, in which the object-field is supposed to be on the left, represent a typical case. We may now distinguish a front entrance-window with the radius $R_{w\,I}$ and a back entrance-

window with the radius $R_{w\,II}$, and we shall denote their distances from the entrance-pupil by η_I and η_{II}. The portions of the space between ϖ and Ω will then vanish owing to the presence of a two-fold stop, and we thus obtain the boundaries of the region of diminishing apertures by the following method, which is again essentially an application of the principles of shadow construction.

To this end, if we join the terminal points of the diameter of the entrance-pupil with every marginal point of either entrance-window on the same side of the axes, we shall have, in the place of the lost region of rays of inclinations exceeding the angle ϖ, a further

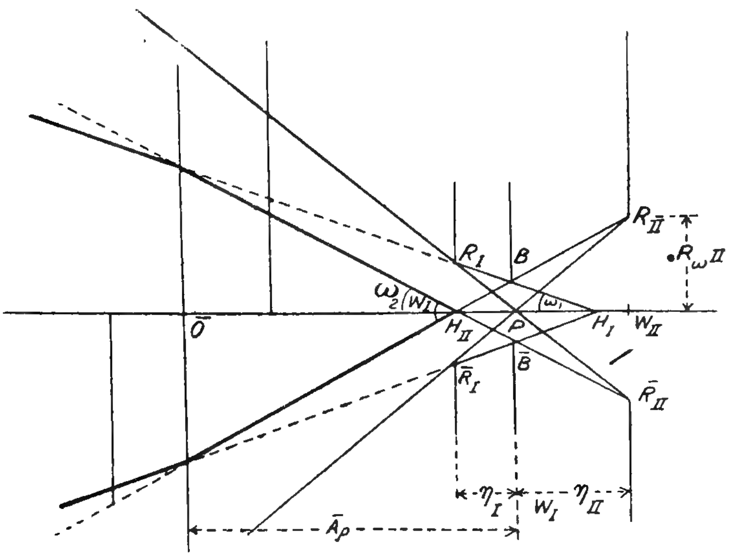

FIG. 114.

$PB = r_p$; $W_I R_I = R_{wI}$; $W_{II} R_{II} = R_{wII}$; $PW_I = \eta_I$; $PW_{II} = \eta_{II}$;
$P\bar{O} = \bar{A}$.

The region of undiminished aperture lies within the heavy lines. The adjacent vertical spaces between the heavy and dotted lines are the regions of unilateral stopping. Within the spaces between the dotted lines and the lightly drawn straight lines proceeding from R_I, $\bar{R}_I$ the aperture of the system, being bounded by both windows, gradually diminishes to zero, the light lines representing the boundary lines of the dark region.

Diagram showing the effect of two entrance-windows.

distinction in the remaining regions. It will be seen at once that the region of full aperture is now bounded by two pairs of straight lines, one of which is correlated to the front entrance-window, the other to the back entrance-window. If now we determine the

abscissa $\bar{A}_p$ of the point of intersection of the two pairs of straight lines conforming to the arbitrarily chosen value of r_p, we obtain the following expression for the distance from the entrance pupil:

$$\bar{A}_p = \frac{2\,\eta_I\,\eta_{II}}{\eta_I + \eta_{II}},$$

a quantity, therefore, which is independent of r_p. In fact, more exactly, the required point is the fourth point in a harmonic range comprising the centres of the entrance-pupil and those of the two entrance-windows.

The ordinate of the point of intersection, it will be readily seen, diminishes as the aperture r_p increases.

The regions of undiminished aperture are now determined by two vignetting angles ω_1 and ω_2, whose magnitudes can be calculated with the aid of the preceding considerations from the expressions

$$\tan\omega_1 = \frac{R_{wI} - r_p}{-\eta_I}\ ;\ \tan\omega_2 = \frac{R_{wII} - r_p}{\eta_{II}}\ .$$

The angles of view (w_I, w_{II}) embraced by the regions of undiminished aperture for any (usually negative) distance of a field-plane at right angles to the axis may be computed from the equations

$$\left.\begin{aligned}\tan\mathrm{w}_I &= \tan\omega_1 - \tan u\\ \tan\mathrm{w}_{II} &= \tan\omega_2 + \tan u\end{aligned}\right\}\quad\left\{\begin{aligned}&A_p \leqq \bar{A}_p\\ &A_p \geqq \bar{A}_p\ .\end{aligned}\right.$$

Accordingly, in any system in which the position of the entrance-pupil and the magnitude and position of the two entrance-windows subtending similar angles at the entrance pupil are given in the manner indicated in the diagram, a plane at right angles to the axis at the fourth point of a harmonic range comprising the centres of the three stops, will have the property that, as the object-point moves away from the axis, the front entrance-window will be the first to confine the aperture with respect to the space in front of that plane, whilst the back entrance-window will be the first to confine the aperture with respect to the space behind it.

The figures of confusion may be formed by projecting the free portion of the entrance-pupil from each object-point, considered as an apex, into the field-plane. It will then be seen that we must confine ourselves to the region of undiminished aperture when making any statements which are to be generally applicable to narrow and wide apertures of any given system.

This leads us to indicate that in practice it will frequently be advisable to confine our attention to those parts of the field of

view within which the pencils proceeding from the object-points operate with undiminished aperture. If the objects which are to be projected are contained, or nearly so, within a plane, this can be accomplished by the simple means of a centred diaphragm having a radius

$$y_s = - A_p \tan \mathrm{w}\,,$$

or

$$\left.\begin{aligned} y_s &= - A_p \tan \mathrm{w}_I \\ y_s &= - A_p \tan \mathrm{w}_{II} \end{aligned}\right\} \left\{\begin{aligned} A_p &\leqq \bar{A}_p \\ A_p &\geqq \bar{A}_p . \end{aligned}\right.$$

When placed in the plane of the object itself this diaphragm will give a free path to that region only which is within the angle of view w or w_I (w_{II}), as defined above. Having reached this limit we shall have a discontinuous or stepwise transition from full-aperture to no-aperture. The diaphragm may also with equal advantage be placed in the plane of the image projected by this system, and this is particularly advantageous when the system is exclusively designed for use at a fixed distance of the object, as in the case of a telescope or a microscope objective of high power and accordingly corrected for an aplanatic pair of points.

On the other hand, if the object has a finite extension in depth such a diaphragm does not exist. For, in whatever position in the field, symmetrically to the axis, we may arrange a circular diaphragm, it will, in conjunction with the entrance-pupil, always determine two angular spaces such that, owing to the effect of the new entrance-window, the object-points contained within these spaces are projected through the agency of pencils of decreasing aperture. These angular spaces may, however, have smaller apertures than those determined in conformity with the entrance-window of the system itself.

268. Dimensional Relations between the Pupils and the Images.—When considering the linear magnification β in systems stopped down to a narrow aperture (§ 258) we established the relation

$$\beta = \frac{A'_p}{A}\,\Gamma.$$

Since we are now concerned with finite pupils we may with the aid of the fundamental equation

$$B\Gamma = \frac{n}{n'},$$

replace the angular magnification of the pupils by the lateral magnification referred to the same points, thus

$$\frac{A'_p}{A_p} = \frac{n'}{n}\;\beta B.$$

This is an equation which we might also have derived directly from the equations (28) in § 60.

It will thus be seen that if in two different instruments we maintain unchanged the ratio of magnification for the object and image, any change in the ratio of magnification of the pupils in one instrument is necessarily attended by a change in the ratio of the distances and conversely.

Examples of this conclusion will be given later in the discussion of the theory of various instruments.*

When the object and the image lie in an aplanatic pair of points

$$\frac{\sin u}{\sin u'} = \frac{n'}{n} \beta,$$

and hence the first equation takes the form

$$\frac{[\mathrm{NA}]}{[\mathrm{NA}]'} = \frac{A'_p}{A_p} \Gamma.$$

Hence this relation, which is based upon the definition of the sine condition for finite angles, enables us to express the apertures on the object-side and image-side in terms of the constants of the instrument. The expressions assume particularly simple forms when the transforming pencils have a vanishing aperture either on the image-side or on the object-side. This case will also be discussed more fully when dealing with the theory of various instruments.

2. PROJECTION SYSTEMS IN THE CASE OF OBJECTS WHICH DO NOT RADIATE LIGHT IN ALL DIRECTIONS.

269. Objects are not necessarily always self-luminous or such as to reflect diffuse light in all directions. Frequently, as in practical microscopy and photography, objects are illuminated by transmitted light, and generally objects of this kind are more or less of the nature of a surface.

270. The Source of Light operating as an Aperture or Field Stop.—To enable us to form definite conceptions we will suppose that the source of light is a movable flame placed at a finite distance, and for the sake of simplicity let it be bounded by a circular stop. On this assumption the object-point O on the axis together with the apparent contour $SQ\bar{S}$ of the flame of finite dimensions, determines a solid angle of a certain magnitude in a direction opposite to that of the light. If now we place in front of the object the transforming instrument, which we

* Not included in the present work.

shall assume to have a single entrance-window and the entrance-pupil in front, the latter together with the object-point on the axis will determine another solid angle which is necessarily smaller than the flame-angle if the entrance-pupil retains its function as the boundary of the effective pencils. When the angular aperture of the system is larger, as when the flame occupies the position $S_I Q_I \bar{S}_I$, in that case the flame becomes the entrance-pupil, and the entrance-pupil of the instrument performs the function of the entrance-window. From what we have said above concerning the importance of the position of the entrance-pupil with respect to the state of correction of the instrument, it will be realised that in the majority of cases it becomes a necessary condition that the entrance-pupil of the instrument should retain its function.

Let us suppose, in the first place, that by moving the flame towards the object into the position $SQ\bar{S}$ it is possible to so increase the magnitude of the solid angle that it becomes greater than that subtended at O by the entrance-pupil. In this case the entrance-pupil remains the element which determines the aperture with respect to O itself.

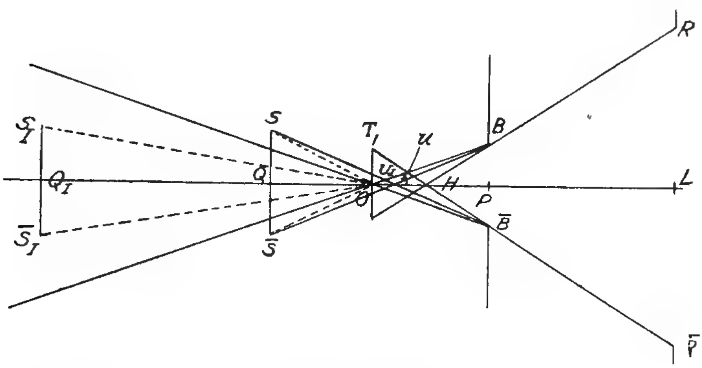

FIG. 115.

$S_IQ_I\bar{S}_I$, $SQ\bar{S}$ = two positions of the movable source of light;
$BP\bar{B}$ = the entrance-pupil;
$RL\bar{R}$ = the entrance-window of an optical system transforming the plane T_1U_1O.

Determination of the region of undiminished aperture corresponding to a given position of the source of light.

Now, to determine on the plane at right angles to the axis at O, the region containing those points which form images through the undiminished aperture at O, in the case which we are now considering, let the terminal points of the diameters of the flame and the entrance-pupil be joined crosswise. In this way a new region is circumscribed in the object-plane (of radius OU_I), which

may be smaller than, equal to, or larger than that corresponding to the combination of the entrance-pupil and the entrance-window and having the radius OT_I. The smaller of these two regions contains all points which are sufficiently illuminated by the flame $SQ\bar{S}$ with respect to the aperture of the system, and whose rays pass without restriction through the entrance-window. In our case this is, accordingly, the region whose radius is OU_I. Within this region of the plane of the object the conditions again are precisely the same as those which obtained when the object was supposed to emit rays in all directions.

271. The Condenser as a Means of Diminishing the Distance of the Source of Light.—Reverting to the case in which the source of light subtended an insufficient angle at the point O, it is not always practicable to increase this angle by moving the source of light up to the object, as, for example, by the source of light being inaccessible, or by excessive radiation of heat. Under these circumstances the expedient may be adopted of forming an image of the flame in such a way that it may subtend an increased angle at the object-point.

A **condenser** is an optical device of this kind. Fig. 116 represents it in its simplest form as a single lens $C\bar{C}$, and shows how a source of light $O\bar{O}$ subtending too small an angle may be projected into a plane at $O'\bar{O}'$ by means of pencils of rays bounded by the aperture $C\bar{C}$ of the condenser. Every point of the projected image of the source of light at $O'\bar{O}'$ receives light from its entire surface $C\bar{C}$ and sends it onwards at an angle $2u'$. If now we join the marginal points of the image $O'\bar{O}'$ directly and crosswise with those of the aperture $C\bar{C}$, as represented in Fig. 116, the space will divide itself up into regions of different intensities precisely in a similar manner to that explained in the case of the entrance-pupil and an entrance-window. This follows from the fact that, in order that any point may receive light, the projection of the source of light through the point and upon the aperture of the lens must fall within its boundary, as indicated in the lower diagram by the different shading of the respective regions. In the space behind the image $O'\bar{O}'$ this subdivision of the entire space has precisely the effect of a material diaphragm, and between the image and the aperture $C\bar{C}$ the effect is precisely the same. This statement can be proved by reasoning similar to that employed in § 259 as a means of explaining the fact that in the case of an object radiating in all directions, it is possible to limit the aperture by a virtual entrance-pupil disposed in front of the field-plane.

The space $O'_I\ O'\ O'_{II}\ \bar{O}'$ embraces all points which receive light through the condenser with undiminished aperture. Geometrically,

it is of the form of a double cone, since we are here concerned exclusively with meridian sections. It should be noted that the aperture of the illuminating pencils within the double cone diminishes a little from O'_I towards O'_{II}.

The manner in which such a condenser should be employed in connection with an optical instrument depends upon the purpose the instrument is designed to serve; for in the condenser and projection system the like as well as the unlike stops can be made to coincide. This furnishes the two limiting types of possible combinations. We shall confine our attention to these two cases.

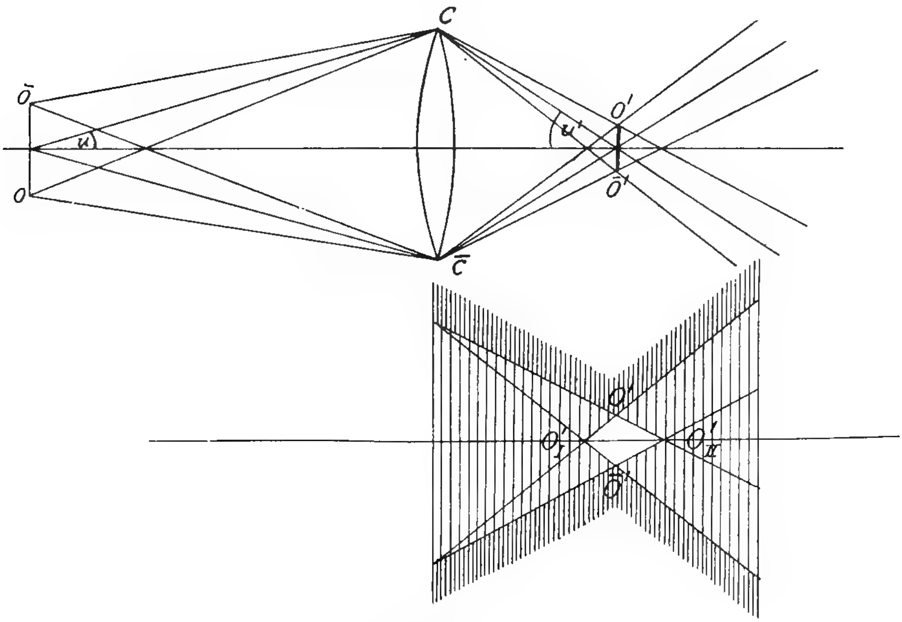

FIG. 116.

The condenser employed as a means of diminishing the distance of the source of light.

When the instrument is required to form images of object-points by means of pencils of large aperture, the object should be made to coincide with the image of the flame itself, or it should be placed in its immediate neighbourhood and, given a system of sufficient angular aperture, use should be made of the entire aperture of the condenser. The image of the flame will then provide the boundary of the field of view in the manner already explained in the case of the movable flame, unless the field is already determined by the entrance-pupil of the instrument.

On the other hand, if the instrument be required to transmit pencils of principal rays of a great angle of inclination, the source of light should be projected into its entrance-pupil The aperture of the condenser will then agree with the angular aperture of the entrance-window of the transforming system, whereas the function of the entrance-pupil remains unchanged, provided the image of the flame is large enough to fill it with light. If it is smaller it will in its turn serve as an aperture stop. From the point of view with which we are regarding this problem this amounts to the same thing as if the entrance-pupil, whilst retaining its position, had acquired a smaller aperture.

272. The Converging Lens.—A simple problem presents itself, when the image formed by an optical instrument becomes, with respect to another instrument, a luminous object emitting rays in a certain direction (as determined by the exit-pupil, for example). In this case the usual course is to let the stops of like kind coincide, viz. the exit-pupil of the first with the entrance-pupil of the second system, and the boundary of the first image will then naturally determine the limitation of the field of view for the second instrument. The diminished aperture and the smaller inclination of the principal ray will then determine the aperture and the inclination of the principal ray with respect to the combination.

As a rule, the coincidence of the two pupils cannot be achieved without the intervention of some special device, such as a lens of suitable focal length placed in the neighbourhood of the image formed by the first system for the purpose of forming two mutually coincident images of the two pupils. Such a lens constitutes a **field-lens.** It should be so large that it may not have the effect of a field-stop. It is not usual to place it actually in the position of the image formed by the first system, as this would cause a sharp image to be formed of any blemishes on the lens surface.

3. THE EYE IN CONJUNCTION WITH AN OPTICAL INSTRUMENT.

273. The Field of View of the Eye in Indirect and Direct Vision.—We shall first consider the case of indirect vision. If we suppose the eye to be associated with the projection-system of the kind just considered, we must realise that the eye in itself is a projection-system in which the retina is the receiving surface, taking the place of the screen surface. The pupil of the eye constitutes the entrance-pupil of the system, and the entrance-window of the eye at rest subtends a very large solid angle at the centre of the entrance-pupil, the boundary of which is not, however, clearly defined. Theoretically we may proceed in the following manner:—

The entrance-pupil of the eye together with the image of the lens in front of the eye, determines the three conical regions which are governed by the diameter of the pupil, as described above in our discussion of the combination of the entrance-pupil and an entrance-window. Now, these angular spaces are subject to limitations both within the eye and in front of it. Within the eye this arises from the fact that the sensitive layer does not completely cover the portions of the fundus of the eye corresponding to those regions ; in front of the eye the field of view is encroached upon by the apparent contours of the cheek and nose.

It should, however, be noted that indirect vision affords a means of rough orientation only. As soon as we proceed to make observations in the lateral portions of the field of view we do so exclusively by direct vision, *i.e.* by fixing a point and concentrating our attention upon it.

The principal rays which come into operation in direct vision meet at the point of rotation of the eye, which is about 10·5 mm. behind the pupil of the eye and 15 mm. behind the vertex of the cornea. We may therefore imagine the natural movable eye to be replaced by a hypothetical rigid eye whose entrance-pupil lies in the position of the point of rotation of the eye. From the nature of things this hypothetical pupil in direct vision is under all circumstances the entrance-pupil of the eye and the exit-pupil of the instrument by which the vision is aided.

The field of view of the movable eye, within which it is possible to speak of sharply defined sensations of vision, may be ascertained in the following manner :

If the eye-ball be caused to roll to the greatest possible extent within its socket, the centre of the pupil maps out on a sphere described with radius $r = 10{\cdot}5$ mm. about the point of rotation, a certain region which determines the solid angle of the movable eye. The projection of this boundary from the centre of rotation of the eye upon any surface may then be regarded as the entrance-window.

A. The Eye in conjunction with a Projection System stopped down to a Small Aperture.

274. When employing an instrument designed for visual observation, the obvious procedure, if possible, is to place the pupil of the immovable eye in the exit-pupil of the instrument, for under these conditions we can generally avoid any encroachment upon the field of view of the instrument by the pupil of the eye. Where however, the construction of the instrument is such that the position of the exit-pupil renders it impossible to bring it into coincidence with the pupil of the eye, it may happen that the latter plays the part of a field-stop. In this connection it should, however,

be borne in mind that the pupil of the eye varies in size with the intensity of the light, and that the latter affects accordingly the magnitude of the angular field of view.

An expedient by means of which the field of view can be increased, consists in moving the eye from side to side so as to fix successively different portions of the field of view of the instrument through its exit-pupil as through a key-hole. This expedient is frequently adopted in drawing appliances.

In the case of a narrowly stopped projection system, to maintain the head in a stationary position, the exit-pupil of the instrument should coincide with the entrance-pupil of the moving eye, that is with its centre of rotation, if it be required to see sharply more than a single point in the field of view.

275. Magnification *N* in Instruments designed for Visual Observation.—The magnification obtaining in instruments designed for visual observation differs from that of projection systems in that the image formed on the retina is no longer measurable, so that the expression previously established in our investigation of objects situated at a finite distance, viz.:

$$\beta = \frac{y'_s}{y_s}$$

ceases to be applicable.

In this case it will therefore be necessary to determine the magnitude of an object, as it appears to the eye, by the tangent of the angle subtended at the eye.

At an early stage in the history of optical instruments the expedient was resorted to of expressing the magnification due to an optical instrument, of an object situated at a finite distance, in such a manner as to introduce the influence of the observer's distance of distinct vision upon the resulting effect.

Let this distance of distinct vision be denoted by l. The angle at which an object y_s is seen with the unaided eye will then be such that

$$\tan \mathrm{w} = -\frac{y_s}{l} .$$

The angle of view which the image y'_s subtends at the eye, as seen through an instrument focussed for the distance A'_p will then be

$$\tan w' = -\frac{y'_s}{A'_p}$$

when the pupil of the eye coincides with the exit-pupil of the instrument, and hence the **magnification** N follows from the formula

$$N = \frac{\tan w'}{\tan \mathrm{w}} = \frac{y_s' l}{A'_p\, y_s} .$$

276. The Magnifying Power *M* of an Instrument.—The definition just given of the magnification N has the advantage of being self-explanatory. On the other hand, it is open to the objection that the distance of distinct vision l which it involves, is an arbitrary element and entirely independent of the optical system; for if we eliminate the quantity tan w, which is entirely independent of the optical system, the above formula becomes

$$N = -\frac{\tan w'}{y_s} l$$

and we thus see that l occurs as a factor. We shall presently show that in the majority of instruments employed for visual magnification the other factor in the expression for the magnification, viz.

$$-\frac{N}{l} = M = \frac{\tan w'}{y_s},$$

though involving the focussing distance A'_p, is nevertheless essentially governed by the arrangement of the instrument only.

Since

$$M = -\frac{1}{A'_p}\frac{y'_s}{y_s},$$

it follows by substitution of the value of y'_s/y_s from equation (20) in §54 that

$$M = -\frac{1}{A'_p}\frac{x'_s}{f'}.$$

From this expression it will be seen that the value of the right side depends upon the focussing distance A'_p of the instrument in so far only as

$$x'_s = A'_p + X'_p$$

is affected thereby, thus we have

$$M = -\frac{1}{f'}\left(1 + \frac{X'_p}{A'_p}\right).$$

Now, in the case of instruments designed for visual observation, with which we are here concerned, however we may adjust them, A'_p is large in comparison with X'_p, the distance of the pupil of the eye and the exit-pupil of the instrument from the back focus, thus ultimately we may write

$$-\frac{N}{l} = M = \frac{\tan w'}{y_s} \text{ (approx.)} = -\frac{1}{f'}.$$

The expression for $\frac{\tan w'}{y_s}$ may, in accordance with Abbe (*6.*), be defined as the magnifying power of an instrument designed for

ocular observation. This quantity, which is essentially governed only by the nature of the optical instrument, is accordingly the ratio of the tangent of the inclination w' of the principal ray on the image side which the image subtends at the centre of the exit-pupil, and the linear distance y_s of the object-point from the axis.

We obtain this quantity by dividing the amount of the magnification referred to the distance of distinct vision, viz., $l = -25$ cm. by this same distance of distinct vision.

The magnifying power, or strength, especially in the case of spectacles, is expressed in dioptres D, a dioptre being the reciprocal of the focal length of a lens of $f = 1$ metre, so that, if m is the focal length in metres,

$$D = \frac{1}{m}.$$

The magnification $N = -l\,M$ is then capable of being split up into one part which depends upon the instrument, and another part which is governed by the condition of the eye. The latter defines accordingly the personal advantage which an instrument affords the observer. This advantage increases with the distance of the near-point from the observer's eye, so that it is more pronounced in the case of long-sighted than with short-sighted individuals.

The expression for the magnifying power M, as defined by Abbe, on the contrary, contains essentially in a detached form the portion which is due to the instrument itself and is represented by the reciprocal of the back focus, *i.e.*, the power, of the system. The small quantity $\frac{X'_p}{A'_p}$ which occurs in the formula relating to instruments designed for ocular magnification vanishes entirely for an eye accommodated to infinity, when $A'_p = \infty$, or when $X'_p = 0$ in instruments in which the centre of the exit-pupil coincides with the back focus. Now, in the instruments belonging to this category the value of this quotient is invariably negligibly small as compared with unity.

From the formula given at the end of § 258 for projection-systems employed for producing an image of an object at a great distance, it will be seen that the above definition is wholly analogous to that arising in the case of the projection-system.

277. Perspective due to an Optical Instrument employed for Ocular Observation.—When a certain object-point, say on the axis of the instrument, is sharply defined by the system, the instrument is said to be sharply focussed for this point. In this case the object-point is conjugate to the retina of the eye through the intermediary of the optical system, and a focussed field-plane, together with the entire projected field-picture thereon, is represented on the retina of the hypothetical eye. Since

the centre of rotation of the eye is supposed to coincide with the exit-pupil of the system, the field-picture subtends an angular field of view w', whereas at the entrance-pupil of the instrument it subtends an angle w. If $w' = w$, the perspective remains unaffected; at most the object is reproduced in a scale ratio β of the retinal picture, but in any case it is similar in all three dimensions.

On the other hand, if $w' \gtrless w$ (as a rule, $w' > w$), we shall see the screen-picture at a different angle to the one subtended at the entrance-pupil by the field-picture.

If, by experience, we are acquainted with the vertical dimensions of the object we may determine the extension in depth by reference to the equation

$$\Delta A'_p = ([y'_{s1}] - y'_{s1}) \cot w'_1 ;$$

hence, by § 260,

$$\varepsilon \Delta \bar{A}_p = \frac{\varepsilon}{\Gamma} ([y_{s1}] - y_{s1}) \cot w_1 = \frac{\varepsilon}{\Gamma} \Delta A_p .$$

This expression justifies the following statement: If we view a solid object of known vertical dimensions with the aid of an instrument whose angular magnification at the pupils is Γ its relative dimensions in depth will appear changed in the ratio of 1 to Γ.

It may also happen that we are better informed with respect to the dimensions in depth (ΔA_p) of the object. In this case we revert to the equation

$$\begin{aligned} \bar{y}'_{s1} &= [y'_{s1}] - \Delta A'_p \tan w'_1 \\ \varepsilon \bar{y}_{s1} &= \varepsilon [y_{s1}] - \varepsilon \Delta A_p \Gamma \tan w_1 \\ &= \varepsilon [y_{s1}] - \varepsilon \Gamma ([y_{s1}] - y_{s1}). \end{aligned}$$

Translated into words this signifies: If a solid object of known extension in depth be viewed with the aid of an instrument having an angular magnification Γ in the pupils, the perspective shortening or lengthening appears exaggerated Γ times. In this connection it should be noted that these discrepancies are not very pronounced so long as we are concerned with comparatively small differences of depth and with small angles w. Moreover, when viewing known objects we correct by a subconscious process defects due to changes in the inclination of the principal rays.

B. The Eye in Conjunction with an Instrument of Finite Aperture.

278. Indirect Vision.—The investigation of the combined effect of the eye and an instrument of finite aperture is greatly simplified if we first consider the behaviour of the eye under the conditions of indirect vision.

279. The Pupil of the Eye Operating as an Aperture Stop.—When the diameter of the pupil of the eye is greater than, or equal to, the exit-pupil of the projection-system placed in the same position along the axis, nothing is changed in the previously determined boundary of the rays. On the other hand, if the diameter of the pupil of the eye is smaller than that of the system an image of the pupil of the eye should be formed on the field-side, and the object corresponding to it will then be a circle in the plane of the entrance-pupil concentric with, and smaller than, the boundary of the entrance-pupil. By our original definition the pupil of the eye operates under these circumstances as an aperture stop, and the diameter of the object conjugate to it in front of the instrument should be taken into account when computing the effective aperture of the system.

280. The Pupil of the Eye Operating as a Field-stop and Determining the Regions of Decreasing Apertures.—In the case in which the pupil of the eye acts as a field-stop the new entrance-window should be determined by ascertaining the object conjugate to it in front of the optical system, after which we may proceed, as explained above, to determine the region whose points are reproduced at full aperture. Since in any case the object which is conjugate to the pupil of the eye does not coincide with the object perceived by the eye, the existing conditions will give rise to regions of diminished aperture, as explained above. If we take into consideration the mobility of the eye under the conditions of indirect vision the simplest expedient is to imagine the eye to be endowed with an enlarged pupil. These regions will then be widened as though the new entrance-window of the instrument had been enlarged in proportion.

As in the case of the projection system with a narrow stop we can also in this case enlarge the field of view by movements of the head, so as to fix different points of the field in succession.

281. Direct Vision.—In the case of direct vision with the head held in a stationary position the investigation becomes a little more difficult in that the centre of the aperture stop in the eye no longer coincides with the crossing point of the principal rays. In fact, in direct vision it is immaterial whether, or in what manner, the optical system of the eye brings about the transmission of the rays from external points to the retina. On the contrary, we are now concerned with the formation of **sharp** images through the agency of a **movable** system. The crossing point of the principal rays is in this case given at the outset, being the centre of rotation C_r of the eye, and the aperture stop may be regarded as situated at a fixed distance ($r = 10.5$ mm.) from the crossing point on the principal ray and movable with it. As in the case of the optical system at rest, let the aperture stop be supposed to be situated in

the immediate vicinity of the crossing point, and about the centre of the entrance-pupil of the hypothetical eye let a sphere be described. Its radius is equal to that of the pupil of the eye or greater, according as the object viewed is supposed to be situated at a finite or at an infinite distance in front of the eye. In our diagrams this substitutive entrance-pupil appears as a circle which is constant for points at equal distances from C_r.

If now we cause the eye to approach the exit-pupil of the instrument until the latter coincides with C_r the diameter of the smaller of the two pupils will determine the aperture. In general, the field of view is determined by the instrument. Yet cases may arise in which the boundary is determined by the mobility of the eye.

When it is not possible to render the exit-pupil of the instrument coincident with C_r it will be necessary, in accordance with the definition of direct vision, to determine the angular field of view at the centre of rotation C_r of the eye, even though this

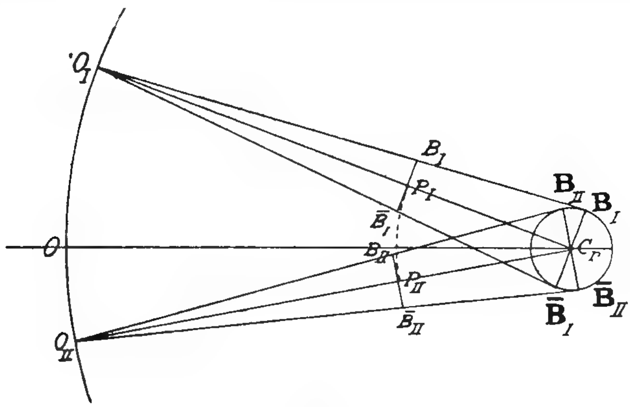

FIG. 117.

$O_I O O_{II}$ = object at a constant finite distance from C_r ;

$B_I P_I \bar{B}_I$, $B_{II} P_{II} \bar{B}_{II}$ = two different positions of the pupil of the movable eye corresponding to $O_I O_{II}$;

$\mathsf{B}_I C_r \bar{\mathsf{B}}_I$, $\mathsf{B}_{II} C_r \bar{\mathsf{B}}_{II}$ = two different positions of the equivalent entrance-pupil of the movable eye corresponding to O_I, O_{II}.

Diagram showing the positions of the equivalent entrance-pupil of the movable eye.

should involve a displacement of the crossing-point of the principal rays. We may determine accordingly the three regions due to a projection-system by deciding upon the exit-pupil and the exit-window, and then consider the case in which the exit-pupil of the system is larger than the pupil of the eye. From Figures 118 and 119 it will be seen that under these conditions the exit-window of the instrument acts as the boundary of the field of view, when C_r can be

advanced sufficiently far into the bright double cone of Fig. 116, whilst the exit-pupil serves this purpose when C_r cannot be made to enter this region of greatest intensity. The new regions of diminishing aperture are determined by tangents to the circle of the hypothetical entrance-pupil, and they have been taken into account in so far only as the principal rays which enter into the process of direct vision actually enter the eye.

When the exit-pupil of the instrument is smaller than the pupil of the eye the case does not present any difference as compared with the preceding case so long as we adhere to the condition that the principal rays by which direct vision is obtained shall enter the eye. The field of view in this case is determined by the angle $2\,w'$ which the exit-pupil subtends at C_r. Within the field of view which we are now considering, if for the sake of simplicity we assume that the emerging rays constitute a parallel pencil, the entrance-pupil moves gradually across the pupil of the eye when first the one and then the opposite edge of the field of view is regarded (Fig. 120).

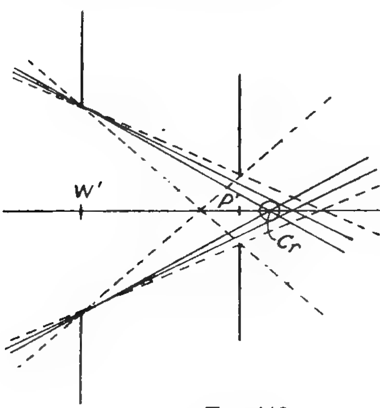

FIG. 118.

The centre of rotation C_r of the eye is supposed to be situated within the bright rectangular space. The exit-window of the instrument determines the field of view in ocular observation.

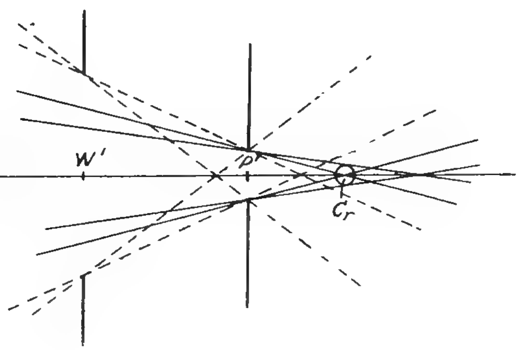

FIG. 119.

The centre of rotation C_r of the eye is supposed to be situated without the bright rectangular space. The exit-pupil of the instrument determines the field of view in ocular observation.

Boundary of the rays in direct vision, the pupil of the eye being smaller than the exit-pupil of the instrument.

It is only by assuming that we are concerned with direct vision, even when the principal rays themselves fail to enter the eye, that we may employ the entrance-pupil of the hypothetical eye together with the centre of the exit-pupil as a means of determining the field of view. In Fig. 121, where for the sake of simplicity

the rays emerging from the system are again supposed to form a parallel pencil, the two tangents to the circle about C_r are drawn from the centre of the exit-pupil. The principal rays corresponding to these directions are shown in dotted lines, and it will be seen that they fail to enter the eye, being stopped down by the exit-pupil. The eye is only reached by a narrow pencil which falls upon the edge of the pupil whose area is smaller than one-half of the exit-pupil. In Fig. 121 two pencils of this kind (inclined at angles W' and $-W'$) are indicated by the pair of lightly-drawn parallel lines. To ascertain the conditions under which all light is intercepted we must draw the inner pair of tangents to the circle about C_r from the marginal points of the exit-pupil.

It is, however, unlikely that these marginal pencils will intersect on the retina exactly at the point where it is traversed by the principal ray transmitted through the stop. In the case just considered, in which the aperture is restricted by the optical system, it is therefore necessary to proceed on the principle that the field of view obtained in direct vision is determined by the angle subtended by the exit-pupil at C_r.

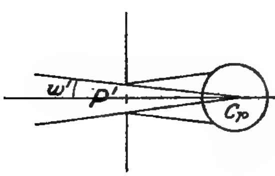

FIG 120.

The angle of vision w' due to the principal rays.

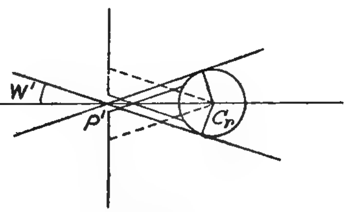

FIG. 121.

The angle of vision W' due to the impression of light.

The boundaries of the rays in direct vision when the pupil of the eye is larger than the exit-pupil of the instrument.

282. Depth of Focus.—We now consider an optical instrument whose exit-pupil has been made to coincide with the pupil of the eye at rest. In this case the distance from the pupil of the image presented to the eye is equal to the distance A'_p from the exit-pupil of the system in which the images are formed. The eye views, accordingly, as if it were the object, the whole image-picture reproduced by the system in scale size β at a distance A'_p.

The absolute magnitude $2\,\delta'$ of the circle of confusion due to points out of focus with respect to this image-picture follows from the equation § 263, namely

$$2\,\delta' = 2\,\beta\delta \text{ (approx.)} = 2\,\beta\,\Delta\,A_p\,\tan u,$$

assuming the instrument to have a comparatively high magnifying power, and the depth of definition $\Delta\,A_p$ to be small in comparison with the distance A_p of the object.

The tangent of the angle of vision ζ, which the circle of confusion subtends at the eye, is then

$$\tan \zeta = -\frac{2\,\delta'}{A'_p} = -\frac{2\,\beta}{A'_p}\,\Delta\,A_p\,\tan u.$$

The radius δ' of the small circle of confusion having corresponding to it a principal ray inclined at an angle w' is obtained from the equation

$$\delta' = \beta\delta = -\,A'_p\,\tan w' = -\,\delta\,M\,A'_p,$$

from which it follows that

$$-\frac{\beta}{A'_p} = M,$$

and hence

$$\tan \zeta = -\frac{2\,\delta'}{A'_p} = 2\,M\,\Delta\,A_p\,\tan u.$$

Conversely, if we assume the angular amount of the permissible indistinctness, as expressed by $\tan \zeta = \zeta$, to be given, we shall obtain for the magnitude of the depth of focus $2\,\Delta\,A_p$, namely,

$$2\,\Delta\,A_p = \frac{\zeta}{M\tan u}\text{ (approx.) } = -\frac{\zeta f'}{\tan u}\,.$$

The value to be assumed for ζ is governed by various physical and physiological conditions. Under average conditions the permissible degree of indistinctness ranges from 1 to 5 minutes of arc, so that ζ varies from 0.0003 to 0.0015.

The permissible amount of indistinctness having been determined, the depth of focus $2\,\Delta\,A_p$ depends then only upon the magnifying power and the angular aperture subtended by the object.

The reciprocal value of the depth of focus may also be employed in instruments designed for ocular observation as a measure of the precision of the focussing adjustment.

283. Depth of Accommodation.—In instruments designed for ocular observation the image is not projected on a screen but is presented to the eye in the form of an aerial image. We are therefore able to accommodate the eyes for all the points of the solid object in succession, at least in so far as the object-points at different distances are sharply defined by the system. In aplanatic systems of large aperture this is not the case, as we know from § 165, whilst in the case of aplanatic systems of small aperture we may speak of sharply defined images in reference to solid objects extending in depth from A_{p1} to A_{p2}. These can be seen distinctly, wholly or partly, owing to the **depth of accommodation of the eye.** This space, by reason of the depth of focus, is continued in front as a region of sufficiently sharp definition of depth $\Delta_1 A_{p1}$, and at the back by a region of depth $\Delta_2 A_{p2}$, so that it may be said that the **total depth of vision**, or the **power of penetration,** is made up of the depth of focus and the depth of accommodation.

We shall assume the accommodating power of the eye to be given by an expression of the form

$$A = \frac{1}{A_{p\,N}} - \frac{1}{A_{p\,F}},$$

where N and F refer respectively to the near and far points. In view of the coincidence of the pupil of the eye with the exit-pupil of the system we may thus also refer the distances to the exit-pupil, and accordingly we may write

$$A = \frac{1}{A'_{p\,N}} - \frac{1}{A'_{p\,F}};$$

or if, to obviate misconceptions, we introduce the symbol D,

$$D\,A'_p = A'_{p\,F} - A'_{p\,N} = A\,A'_{p\,F}\,A'_{p\,N}.$$

To ascertain the value of $D\,A_p$, which in the object-space corresponds to $D\,A'_p$, we have the general relations:

$$D\,A'_p = Dx'_s;\; D\,A_p = Dx_s,$$

and these, in conjunction with the equations established in §§ 52, 54, 84, give us the following connection between DA_p and DA'_p:

$$DA'_p = Dx'_s = -ff'\,\frac{DA_p}{x_{s\,N}\,x_{s\,F}} = \frac{n'}{n}\,\beta_N\,\beta_F\,D\,A_p,$$

$$\frac{DA'_p}{DA_p} = \frac{n'}{n}\,\beta_N\,\beta_F,$$

where β_N and β_F are the linear magnifications corresponding to the distances A_N and A_F. Substituting this value, we obtain

$$DA_p = \frac{n}{n'}\,A\,\frac{A'_{p\,F}}{\beta_F}\,\frac{A'_{p\,N}}{\beta_N} = \frac{n}{n'}\,A\left(\frac{A'_p}{\beta}\right)^2_M,$$

where

$$A'_{p\,M} = \sqrt{A'_{p\,F}\,A'_{p\,N}} \text{ and } \beta_M = \sqrt{\beta_F\,\beta_N}$$

are the geometrical means of the corresponding values at the projected limits of accommodation in the field, for which we may substitute the arithmetical means when $\Delta\,A_p$ has a very small value.

By our previous investigations it follows from the last equation that

$$DA_p = \frac{n}{n'}\,\frac{A_p}{M^2_M},$$

and when the pupil is situated at the back principal focus we may write

$$DA_p = \frac{n}{n'}\,Af'^2,$$

4. HISTORICAL NOTES.

284. It remains now to add a few notes on the development of the aspects discussed in the preceding paragraphs. In the first place, it should be mentioned that before Abbe approached the subject, various papers had already been published relating to the theory of the limitation of pencils by stops.

Petzval (*3.* 57), dealing with the subject in a more or less popular manner, treated quite correctly the principle of vignetting in connection with his double objective. He made a distinction, in fact, between the aperture stop and the field-stop, but only referred to this matter incidentally.

In England Th. Grubb (*1.*) appears to have been the first who investigated the subject more fully. He examined more precisely the path of the rays in a photographic lens and thereby had his attention directed to the difference subsisting in a composite photographic objective as regards the magnitudes of the diameters of the aperture stop and the entrance-pupil. He also drew attention to the dissimilarity of the angle of view subtended by the object and the image.

With respect to the eye, Helmholtz pointed out the important influence which the pupil exercises upon the boundary of the pencil of rays, seeing that the lines of sight converge towards its centre. He defined (*2.* 679) also the field of view of the eye as the external projection of the retina with all its features. This is probably the most detailed and best investigation relating to a section of the theory of circumscribed pencils of rays prior to the publication of Abbe's theory.

Abbe's (*1.*) theory was from the first conceived on general principles and was accordingly immediately applicable to any optical instrument.

Supplementary investigations of considerable importance with respect to the eye, and hence also to all optical instruments designed for ocular observation, have been furnished by Gullstrand, who emphasised the significant bearing of the centre of rotation of the eye on the perspective obtaining in direct vision, which had already been known to Listing, more especially with reference to the important case in which an image-picture is viewed by direct vision.

Chapter X.

INTENSITY OF RAYS TRANSMITTED THROUGH OPTICAL SYSTEMS.

(PHOTOMETRY OF OPTICAL INSTRUMENTS.)

(M. v. Rohr.)

285. When a luminous body illuminates other bodies in its neighbourhood we imagine the process to consist of the interception by the intrinsically non-luminous body, of rays emitted by the surface of the luminous body, and we make the further supposition that the intensity of the illumination is governed by the *number* of rays which are received by a unit area of the illuminated surface.

Moreover, the illumination of any portion of a surface is looked upon as a summation of the component illuminating effects due to each of the luminous surface-elements, and in this way the entire problem is reduced to the simple case of the effect of radiation from a luminous surface element upon a receiving surface-element.

In estimating the illuminating effect a distinction has to be made regarding the relative positions of the two surfaces or according to the intensity of the radiation. We may do so by considering every ray to be the carrier of an intrinsic intensity which is inherent in the luminous element and which may change with it.

1. RADIATION DUE TO SELF-LUMINOUS BODIES.

A. Radiation from Element to Element.

286. Fundamental Law of Photometry. — The law respecting the quantity of light dL emitted from one element da to another dA, as first enunciated by Lambert, can be expressed in the form

$$dL = \frac{I\, da\, dA \cos\theta \cos\Theta}{r^2},$$

Let the two elements be defined in position by the points p and P, and let θ and Θ be the angles comprised between the normals to da and dA at p, P and the radius vector $r = p\,P$. The factor I is then independent of the geometrical conditions and represents accordingly the **intrinsic intensity** referred to in our introductory remarks. Its physical significance will be considered later.

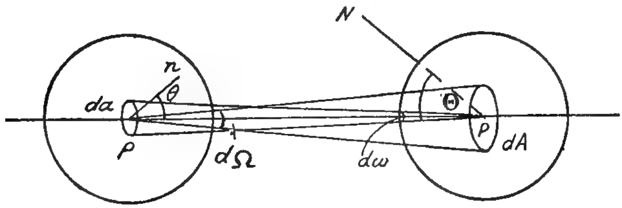

FIG. 122.

Diagram showing the mutual radiations due to two surface elements da at p and dA at P.

The principal feature of this mathematical expression of the law of radiation is the symmetry between the radiating and the irradiated elements. This relation may also be expressed in the following terms: The quantity dL which is transmitted from the element da to dA is equal to that which dA would transmit to da if it possessed the same intrinsic intensity I. This aspect of the law of radiation is occasionally applied to the theory of optical instruments.

About the point P let a sphere of unit radius be described. Any portion of its surface of area ω will then subtend at its centre P a solid angle ω. If now from p vectors be drawn to the boundary of the irradiated surface element dA, the area $d\Omega$ on the unit sphere about p bounded by these vectors will subtend at p a solid angle $d\Omega$. This solid angle clearly satisfies the equation

$$d\Omega = \frac{dA \cos \Theta}{r^2},$$

and we may define it as the **apparent size of the illuminated element** dA **as seen from** p. Clearly, the apparent size of the radiating element da as seen from P, can be defined by the expression

$$d\omega = \frac{da \cos \theta}{r^2}.$$

Introducing these expressions, we may write the fundamental equation in the form

$$dL = I\, da \cos \theta\, d\Omega = I\, dA \cos \Theta\, d\omega,$$

in which the symmetrical form referred to above is preserved. The quantity of light transmitted between two elements, apart from the apparent size of one element, is governed by the actual size of the other and its inclination with respect to the radius vector.

In recent photometrical investigations by Brodhun (*1. 450*), Lummer (*2. 24*), and Drude (*3. 72*), we note the introduction of another quantity, which is thus defined: The quantity of light which a luminous body radiates upon a surface of unit area at right angles to the vector r and at unit distance from the luminous body determines the luminosity dK in the direction r. It is represented by the equation

$$dK = \boldsymbol{I}\, da \cos\theta\,.$$

The unit of the luminosity, so far as optical sources of light are concerned, is furnished by standard candles and lamps which are so devised as to emit at all times light of an approximately unvarying intensity.

287. Intensity of Illumination.—In accordance with our general theorems we may measure the intensity of illumination dI_P at P due to da in terms of the quantity of light distributed over unit area, or

$$dI_P = \frac{dL}{dA} = \boldsymbol{I} d\omega \cos\Theta$$

and, similarly, at p due to dA,

$$di_p = \frac{dL}{da} = \boldsymbol{I} d\Omega \cos\theta\,.$$

In the former of these two equations, if we suppose the surface element dA to be at right angles to the radius vector, Θ becomes zero and hence $\cos\Theta = 1$, so that we have the expression

$$\frac{dI_P}{d\omega} = \boldsymbol{I}\,.$$

From this relation we are therefore able to determine by measurement the intrinsic intensity $\boldsymbol{I}$ in terms of the quotient of the intensity dI_P of an illuminated element at right angles to the vector and the solid angle $d\omega$ which the radiating element da subtends at a point in the illuminated element. This indicates the physical significance of the factor $\boldsymbol{I}$.

From the fundamental equation of photometry and the definition of dK, we can now express the intensity of illumination by the equation

$$dI_p = \frac{dL}{dA} = \frac{dK \cos \Theta}{r^2},$$

from which it will be seen that the intensity of illumination can be calculated from the geometrical conditions when the intrinsic luminosity is given.

The unit of the intensity of illumination, or the foot-candle, may be defined as the intensity produced at the centre of a screen by a standard candle placed normal to the direction of the rays and at a distance of one foot.

288. Significance of the "Grain" of Illuminated Screens.—As an expression for the intensity of the illumination we obtained in the preceding article

$$dI_p = \frac{dL}{dA} = I d\omega \cos \Theta .$$

In the derivation of this formula it was tacitly assumed that dA differs from zero, as implied by the fact that dA occurred as a divisor.

Now, all screens employed in connection with optical projection, such as the focussing screen or the sensitive coating of the photographic plate, are made up of solid elements of a certain size, or, to use the customary expression, of a certain **size of grain.** In such cases the ordinary surface measurements cease to be applicable since we are no longer dealing with a surface which is uniformly composed of strictly similar particles. Instead of actually measuring the surface we are therefore constrained to count the component elements, and from the nature of things the resulting number must necessarily be an integral positive number. Let Q be the integral positive number of elements contained in the surface dA, and ε the mean effective area of the projection of a single element. Then

$$dA = \varepsilon Q .$$

We shall suppose this mean effective area ε to be determinable by describing the apparent contour of each participating grain on the surface of the screen by means of tangent rays proceeding from the radiating element. If we assume the presence of a sufficiently thick stratum below the surface we shall obtain a

reticulated design, the meshes of which are formed by the contours of the grains. We are thus justified in considering the projection of the elements on the screen surface as a continuous pavement.

The intensity of illumination will then be

$$dI_P = \frac{dL}{\varepsilon Q}.$$

If we suppose the surface element dA to diminish in size, this will cause Q in the expression $dA = \varepsilon Q$ to become smaller and smaller. From our supposition that dA diminishes continuously in size it follows that the resulting expression, from the nature of its origin, must necessarily diminish abruptly after the manner of integral positive numbers.

When finally the area of dA reduces to a very small number of elementary grains or a simple grain, the markedly discontinuous change of Q will render the expression $dA = \varepsilon Q$ entirely inapplicable as a means of representing dA, so that the equation

$$dI_P = \frac{dL}{dA} = \lim_{Q=1} \left[\frac{dL}{\varepsilon Q}\right]$$

becomes meaningless. For, if dA diminishes still further εQ cannot follow as a function, and hence the expression involves two elements which may lead to a fallacy. On the one hand, if dL remains constant, though dA diminishes, dI_P ought to increase, whereas in the expression on the right, viz.

$$\lim_{Q=1} \left[\frac{dL}{\varepsilon Q}\right]$$

nothing is changed, since Q has already assumed its least possible value, viz., $Q = 1$. On the other hand, if dL diminishes with dA in such a way that $dI_P = \frac{dL}{dA}$ remains constant, then the value on the right, viz., $\lim_{Q=1} \left[\frac{dL}{\varepsilon Q}\right]$ diminishes, and accordingly yields a fictitiously small result.

As soon as the illuminated element becomes so small as to embrace only a single or very few structural elements of the receiving screen, the calculation of the intensity of illumination becomes fictitious, and in such a case the intensity of the illumination should be measured by reference to the quantity of light dL which is transmitted to the element.

With respect to the eye Abbe (*1*. *269*) has evolved the reasons for the necessary departure from the usual method, and Lummer (*1*.), in enunciating his "point law," has drawn attention to this aspect in relation to astrophotographic lenses.

289. The Equivalent Distribution of Light. — The expression which we obtained above for the amount of light transmitted by radiation between two elements was

$$dL = \boldsymbol{I}\, d\,\omega\, dA \cos \Theta = \boldsymbol{I}\, d\Omega\, da \cos \theta\,.$$

According to this expression the quantity of light radiated upon the element dA is not modified in any way if we replace the radiating element da by another element $d\bar{a}$ of a different form and occupying a different position, provided that the intrinsic intensity of the radiation $\boldsymbol{I}$ remains the same, and provided also that the radiating element subtends the same solid angle at P. In fact the condition which should be maintained is

$$\frac{da \cos \theta}{r^2} = d\,\omega = \frac{d\bar{a} \cos \bar{\theta}}{\bar{r}^2}\,.$$

A substituted element $d\bar{a}$ of this kind may be described as **equivalent** to the original element da with respect to P. Given a radiating surface made up of m elements da_v radiating with intrinsic intensities $\boldsymbol{I}_v$, we may accordingly replace it by a surface occupying any other position and furnishing an **equivalent distribution** of the light with respect to P, if every element of the substituted surface satisfies the condition

$$d\,\omega_v = \frac{d\bar{a}_v \cos \bar{\theta}_v}{\bar{r}_v^{\,2}}\,.$$

From this it will be seen that the radiation due to a single surface $a = \sum_v^m a_v$, the component elements of which radiate with an intrinsic intensity $\boldsymbol{I}_v$, may in all cases be completely replaced with respect to dA at P by any surface $\bar{a}$, by projecting upon $\bar{a}$ every element a_v from P as centre of projection, and ascribing to it there the intrinsic intensity $\boldsymbol{I}_v$. According to Abbe (*1*. *268*) we may also enunciate this statement as follows:

"Conversely, two sources of light of dissimilar size, shape and position will produce precisely similar effects of radiation at any point whence they can be projected upon one another in such a manner that any radius vector proceeding from the illuminated point and meeting them will do so at points of equal intrinsic intensity."

B. Radiation upon Surfaces of Finite Extension.

290. Let a surface be made up of m elements dA_v, and let it be illuminated by an element da radiating with the intrinsic intensity $\boldsymbol{I}$. The quantity of light which falls upon a surface element dA_v will then be

$$dL_v = \frac{\boldsymbol{I}\, da \cos\theta_v\, dA_v \cos\Theta_v}{r^2_v},$$

and therefore the total quantity of light which falls on the surface is

$$d\boldsymbol{L} = \sum_v^m dL_v = \boldsymbol{I}\, da \sum_v^m \frac{\cos\theta_v\, dA_v \cos\Theta_v}{r_v^{\,2}}.$$

This expression ultimately assumes the form of an integral if we suppose that the elements become indefinitely small, so that in the limit $m = \infty$. We may at once note that this expression may also be regarded as the quantity of light which the element da receives when it is subjected to the radiation of the same surface, assuming its elements to radiate with the constant intrinsic intensity $\boldsymbol{I}$.

In certain cases the integration indicated by the expression can be performed. We shall illustrate the procedure by a few cases of practical importance.

291. Illumination of a Circular Surface Element which is Symmetrical about the Axis.—Let the radiant element da be at a perpendicular distance A from the centre of the circle of radius r arranged parallel to it, and let the illuminated element da be an annular strip bounded by two consecutive circles subtending angles θ and $\theta + d\theta$ at p (Fig. 123).

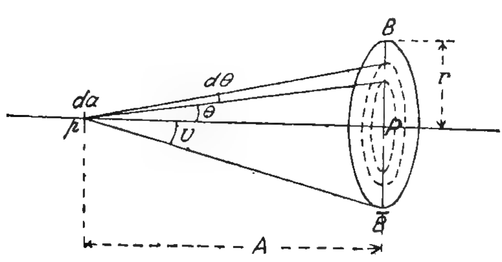

FIG. 123.

Illumination of a Circular Surface Element by a Parallel and Axially Symmetrical Surface Element.

Then

$$dA = 2\pi \mathsf{A} \tan \theta . d[\mathsf{A} \tan \theta] = 2\pi \mathsf{A}^2 \frac{\sin \theta}{\cos^3 \theta} d\theta$$

and since

$$r = \frac{\mathsf{A}}{\cos \theta} \quad \text{and} \quad \theta = \Theta$$

$$dL = 2\pi \boldsymbol{I} da \sin \theta \cos \theta d\theta ;$$

hence

$$d\boldsymbol{L} = 2\pi \boldsymbol{I}\, da \int_0^{\mathsf{r}} \sin \theta \cos \theta d\theta = \pi \boldsymbol{I}\, da \left[\sin {}^2\theta\right]_0^{\mathsf{r}} .$$

If now we introduce the limiting angle U in accordance with the equation

$$\tan U = \frac{\mathsf{r}}{\mathsf{A}}$$

it follows that the integral will extend to U as the upper limit, and accordingly we shall obtain the final expression

$$d\boldsymbol{L} = \pi \boldsymbol{I} da \sin {}^2 U .$$

This signifies that the quantity of light radiating from an axially symmetrical element at right angles to the axis upon an axially symmetrical circular surface parallel to it, is proportional to the square of the sine of the angular aperture U. This result was already known to Lambert.

Regarding this quantity of light as radiated upon the element da from a circular surface supposed to shine with a uniform intrinsic intensity $\boldsymbol{I}$, the intensity of illumination of this element will be given by the expression

$$I_p = \frac{d\boldsymbol{L}}{da} = \pi \boldsymbol{I} \sin {}^2 U.$$

292. Illumination of a Circular Surface by a Point on the Axis.—If we suppose the source of light to be a point, it will not experience any apparent diminution if we assume the rays to be inclined at finite angles. Hence the resulting integral is

$$d\boldsymbol{L} = 2\pi \boldsymbol{I} da \int_0^U \sin \theta d\theta = 2\pi \boldsymbol{I} da (1 - \cos U) = 4\pi \boldsymbol{I} da \sin^2 \frac{U}{2} .$$

The result is greater in the proportion of $1 / \cos^2 \dfrac{U}{2}$ than the quantity of light received from a surface element at right angles to the axis.

293. Illumination of a Circular Surface by a Parallel Surface Element at a Distance from the Axis. —The integrations have already been worked out in a number of cases, respecting which the reader is referred to the writings of Lambert and Beer. Particular interest attaches to the following general case, as investigated by Beer (*1.57*), of which we shall make further use. The radiant element is assumed to be parallel to the illuminated circle and at a perpendicular distance A from it, but at a distance $y_s = OK = PL$ from the axis. The elegant

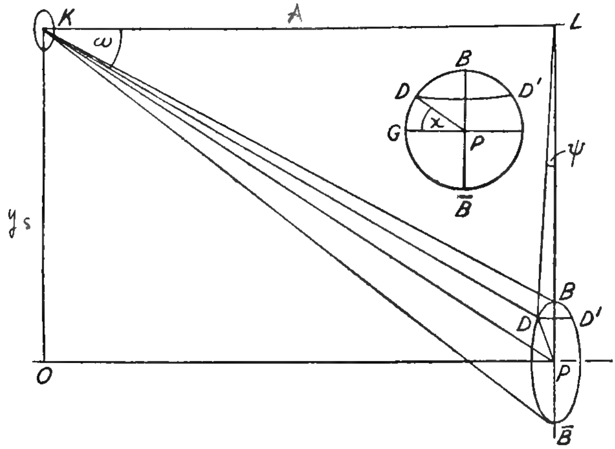

FIG. 124.

$PO = \mathsf{A}$; $OK = y_s$; $PB = \mathsf{r}$.

Illumination of a circular surface by a parallel surface element at a distance from the axis.

method of integration devised by Lambert may be consulted in the original. In our present investigation we shall confine ourselves to a discussion of the results. Let

$$p = \frac{\mathsf{A}^2 + y^2_s - \mathsf{r}^2}{2\, y_s \mathsf{r}} \; ; \; q = \frac{\mathsf{A}^2 + y^2_s + \mathsf{r}^2}{2\, y_s \mathsf{r}} \, .$$

Next consider the segment $DD'B$ cut off from the circle about P by a circle described about L with radius LD and having the angles $DPG = \chi$, $DLP = \psi$ and $DKL = \omega$ corresponding to it. Then the quantity of light which falls on $DD'B$ is

$$dL_{\chi,\psi} = Ida \left\{ \psi \sin^2\omega + \frac{1}{2}\left(\frac{\pi}{2} - \chi\right) \right.$$

$$\left. + \frac{p}{\sqrt{q^2 - 1}} \left(\tan^{-1} \frac{q \tan \frac{\chi}{2} - 1}{\sqrt{q^2 - 1}} - \tan^{-1} \frac{q - 1}{\sqrt{q^2 - 1}} \right) \right\} .$$

To ascertain the quantity of light which falls upon the whole circle we must put $\chi = -\frac{\pi}{2}$ and $\psi = 0$. Then in the equation

$$dL = Ida\left\{\frac{\pi}{2} - \frac{p}{\sqrt{q^2-1}}\left(\tan^{-1}\frac{q+1}{\sqrt{q^2-1}} + \tan^{-1}\frac{q-1}{\sqrt{q^2-1}}\right)\right\}$$

the products of the two tangents are equal to unity; hence the sum of their arcs is $\frac{\pi}{2}$ and thus we have

$$dL = \frac{\pi}{2}Ida\left(1 - \frac{p}{\sqrt{q^2-1}}\right).$$

When $y_s = 0$ this expression assumes the form which we have already established independently, viz.

$$dL = \pi Ida \sin^2 U.$$

If now we introduce the symbols $KP = med.$, $K\bar{B} = max.$, $KB = min.$, which will be readily understood from the figure, the above expression may be presented in an elegant form, viz.

$$dL = \frac{\pi}{2} Ida\left(1 - \frac{(med. + \underline{\underline{r}})\,(med. - \underline{\underline{r}})}{max. \times min.}\right).$$

Problems relating to radiation have also been investigated in a few cases involving radiation from one finite surface to another. The subjoined formula applies to two circles arranged symmetrically and at right angles to the same axis, having radii r and R respectively, and separated by a distance A along the axis, the intrinsic intensity on one of the circles being I throughout its surface. The quantity of light received by the other will then be

$$L = \frac{1}{2}I\pi^2\left\{\mathsf{A}^2 + \mathsf{r}^2 + \mathsf{R}^2 - \sqrt{[\mathsf{A}^2 + (\mathsf{R}+\mathsf{r})^2]\,[\mathsf{A}^2 + (\mathsf{R}-\mathsf{r})^2]}\right\}.$$

Substituting the symbols

$$Max = \sqrt{\mathsf{A}^2 + (\mathsf{R}+\mathsf{r})^2}$$

$$Min = \sqrt{\mathsf{A}^2 + (\mathsf{R}-\mathsf{r})^2},$$

we then obtain the elegant expression

$$L = \frac{1}{4}I\pi^2\left\{Max - Min\right\}^2.$$

C. Brightness.

294. The general photometric theorems which we have just considered hold good when the illuminated point P is situated on the retina of the eye and if by dA we understand the illuminated portion of the retina.

Illumination, as appreciated by the eye, is termed **brightness.** In particular, the intrinsic intensity I of the radiant element in the eye constitutes the **absolute brightness,** whilst the **apparent brightness** is the sensation due to the intensity of illumination which a radiant body produces on a diffusely illuminating surface. It is a difficult matter to establish identity of intensity of illumination due to sources of light differing in colour. Under a feeble illumination Purkinje's phenomenon will then cause similar differences in the intensity of the illumination to be appreciated as dissimilar differences of brightness.

In the case of the eye, as with any illuminated surface, we can calculate the intensity of illumination only if the retinal elements dA are not excessively small. The limit is fixed by the "grain" of the retina, viz. the cross-section of the cones. When the case arises that the radiant element comprises a single grain it will again become necessary to determine the quantity of light dI by direct measurement, since the measurement of the intensity of illumination dI_p on the retina furnishes a fictitious result.

The theorems established with respect to the equivalent distribution of the light are applicable also to the eye, since in their derivation all modifications were restricted to the elements of the source of light, whereas the quantities dA and Θ, which now refer to the eye, remain unchanged.

2. INDIRECT RADIATION.

A. Radiation due to Diffusely Reflecting Surfaces.

295. In the introduction it was pointed out that self-luminous bodies are capable of imparting the quality of being luminous to non-luminous bodies in their vicinity. We must now consider this radiation from the point of view of the geometrical relations and of the intrinsic intensity of the directly radiating body, which we will call the **borrowed intensity.**

We shall in the first instance consider bodies having a rough surface and capable of reflecting in all directions the light received by them. Geometrically considered, they behave precisely in the same manner as self-luminous bodies, and the problem then is to derive the factor which determines the magnitude of the borrowed intensity from the data respecting the illumination of the diffusely reflecting surface.

Consider a point on the reflecting surface of the rough body, and at this point let a tangent plane to it be described. This plane defines the boundary within which rays can be reflected in any direction towards the body. Given a surface which actually reflects light in all directions, the entire hemisphere of unit radius and surface 2π, having its centre at the point considered, will be occupied by rays.

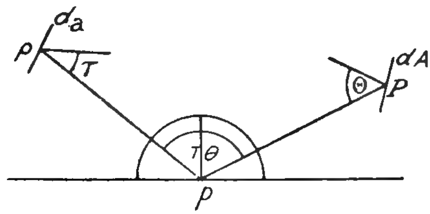

FIG. 125.

$\mathsf{p}p = R$; $pP = r$
Geometrical aspect of the borrowed intensity.

If I is the intrinsic luminosity of the primary radiant, the surface element da receives from the element $d\mathsf{a}$ of the primary radiant a quantity of light expressed by the equation

$$d\mathsf{L} = \frac{I\, d\mathsf{a}\, da \cos \tau \cos T}{R^2},$$

and if we denote provisionally its borrowed intensity by $\bar{I}$, it transmits to the element dA situated at a distance r a quantity of light which is expressed by the formula

$$dL = \frac{\bar{I}\, da\, dA \cos \theta \cos \Theta}{r^2},$$

where θ is the **angle of emanation.**

In accordance with the above assumptions, the diffusely reflecting body reflects the quantity of light $d\mathsf{L}$ received by it in all directions which are comprised within the hemisphere, and it does so uniformly and with diminished intensity in the ratio $1/\varepsilon$. The quantity of light reflected in every direction is accordingly determined by the expression

$$\frac{\varepsilon\, d\, \mathsf{L}}{2\pi}.$$

Since the surface element da of the diffusely reflecting surface which receives the quantity of light $d\mathsf{L}$ reflects light upon the surface corresponding to da in a direction inclined at an angle θ to the normal, the last expression should be multiplied by $\cos \theta$.

The resulting quantity $\frac{\varepsilon\, d\, L \cos\theta}{2\pi}$ replaces accordingly, in the expression for dL, the factors relating to the size, inclination and intensity of radiation at da, viz.

$$\bar{I}\, da \cos\theta.$$

Hence, assuming that we are dealing with completely diffuse reflection, we obtain by equating these two quantities

$$\frac{\varepsilon\, d\, L \cos\theta}{2\,\pi} = \bar{I} \cos\theta\, d\, a\ ,$$

whence we derive the equation

$$\bar{I} = \frac{\varepsilon}{2\pi} \frac{I\, d\mathrm{a} \cos\tau \cos T}{R^2}$$

$$= \frac{\varepsilon}{2\pi}\, di_p\ ,$$

that is to say:

The magnitude of the borrowed intensity $\bar{I}$ due to a surface element which diffusely reflects ε per cent. of the incident light, is derived by multiplying its intensity of illumination di_p by $\frac{\varepsilon}{2\pi}$.

The factor ε, which indicates the proportion in which the intensity of illumination is radiated onwards by the surface, was termed by Lambert its **coefficient of whiteness** or **albedo,** and bodies for which $\varepsilon = 1$ might accordingly be described as **absolutely white** and those for which $\varepsilon = 0$ as **absolutely black.** The substances which occur in nature have intermediate values of ε; for example, bodies ordinarily described as white have a coefficient of whiteness of $\varepsilon = 0 \cdot 4$. In one and the same body ε assumes widely differing values according to the different wave-lengths.

The intensity of illumination di_p of an element dA at P, due to a diffusely luminous element da at p, follows from the fundamental law, if we assume this to hold good under the conditions of diffuse reflection,

$$di_p = \frac{dL}{dA} = \bar{I}\, d\omega \cos\Theta = \frac{\varepsilon di_p}{2\pi}\, d\,\omega \cos\Theta,$$

so that nothing is changed, except that $\bar{I}$ has now taken the place of I.

The conclusions previously arrived at respecting the equivalent distribution of light over any given surface are accordingly applicable to the circumstances as they arise in indirect illumination, and it is only necessary to consider that the borrowed intensity is made up of the albedo ε and the intensity of illumination which the diffusely reflecting surface happens to experience.

In reality, we are not justified in assuming that the law of photometry holds good for diffusely reflecting surfaces, and therefore the investigation is much more involved. From the most recent researches on this subject by Thaler (*I.*). who has extended Wiener's investigations, it appears that at small and intermediate angles of incidence and emission the magnitude of the reflected radiation is almost invariably less than is demanded by Lambert's law.

Let the angle ν intermediate between 0° and 180° comprised between the planes defined by the directions of incidence and emission with respect to the normal to the diffusely reflecting element be called the **azimuth of radiation.** Wiener (*I.*) has shown that this angle has a significant influence upon the amount of reflected light.

A similar effect has also become apparent in the course of Thaler's experiments and was conspicuous in the majority of surfaces subjected to tests, as in the cases of dull opal glass, deposits of magnesium oxide, and smoothed and roughened plaster-of-Paris, and it was generally found that the reflected radiation increased with the azimuth. In agreement with what was formerly regarded as a universal rule, it attained extraordinarily high values when very large angles of incidence and emission ($i = 80° = \theta$) were associated with an azimuth $\nu = 180°$. For example, in the case of dull opal glass the value was found to be 12·6 times greater than Lambert's formula would lead us to expect. In cases of this kind conditions arise which are not far removed from those prevailing in regular reflection. In this respect magnesium oxide forms an exception in that the reflection was a maximum at $\nu = 0°$, and at small and intermediate angles of incidence and emanation this maximum extended even over the greater part of the total radiation, whilst at greater values of i and θ it was exceeded by the contributory radiations reflected along an azimuth of 180°.

An entirely different behaviour was observed in the case of a surface consisting of sifted plaster-of-Paris. In this case the reflecting properties of magnesium oxide at small and intermediate values of i and θ were of a very markedly similar order at large values of i and θ. Thus at $i = 80° = \theta$ the maximum of the reflected radiation is 2·5 times greater than that conforming to Lambert's law at $\nu = 0°$, whilst at $\nu = 180°$ the reflected quantity is

only 1·4 times greater. Instead of an approximation to regular reflection, observed in the case of the first-mentioned substances, we here find that the light is reflected back along the direction of incidence.

B. Radiation transmitted by Polished Surfaces.

296. Reflecting Surfaces. In the case of a metallic reflecting surface, every point p on it has proceeding from it a certain cone of reflected rays, whose surface determines the boundary of all object-points which are to come under the influence of light radiated from the source and transmitted by the reflecting element. As this applies to every point of the surface, it is permissible, in accordance with the law of equivalent distribution, to assume the transmitted intrinsic intensity to be localised in the reflecting surface (see Fig. 126).

The amount of the transmitted radiation is governed by the two constants κ and n of the metal, viz., its coefficient of absorption and the index of refraction.

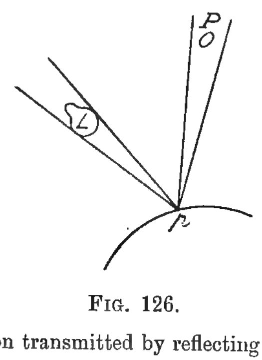

FIG. 126.

Radiation transmitted by reflecting surfaces.

The values adopted in these investigations have been determined by polariscopic measurements. The reflecting power R at normal incidence, as determined by the formula

$$R = \frac{n^2(1 + \kappa^2) + 1 - 2n}{n^2(1 + \kappa^2) + 1 + 2n},$$

agrees satisfactorily with the values obtained from direct determination. In the case of silver and steel the discrepancy in the respective values of R amounts to about 2·5 %.

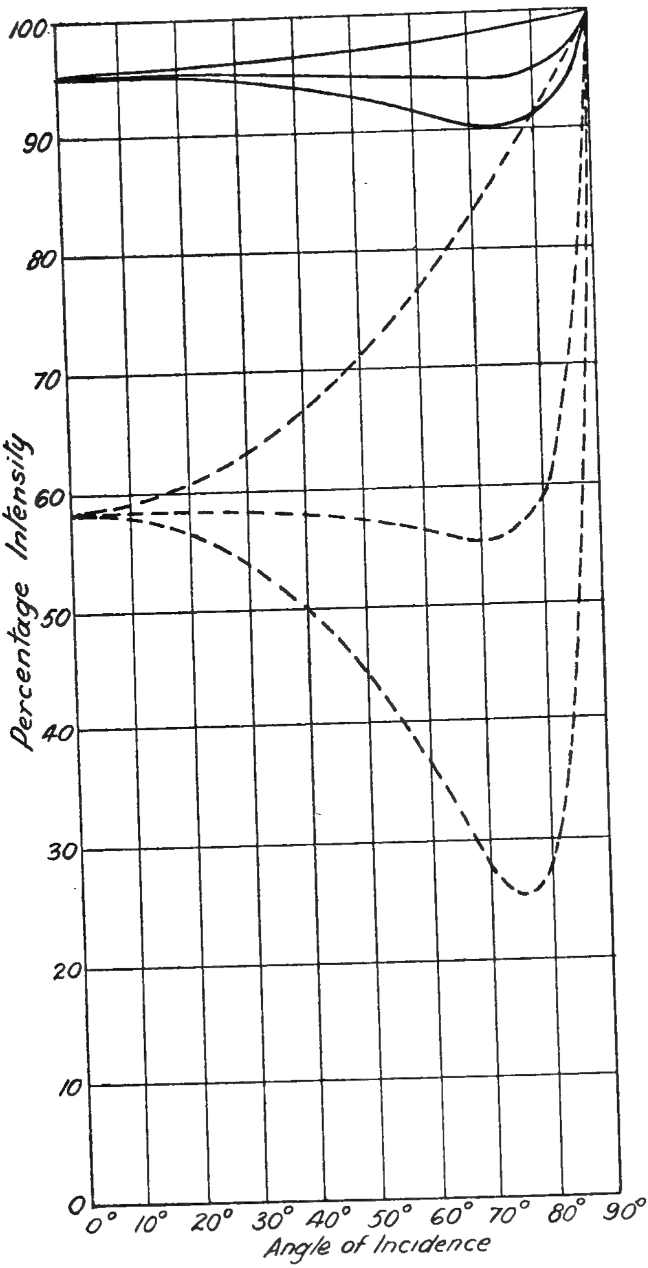

FIG. 127.
Reflecting capacities of silver and steel.

At oblique incidence the light undergoes partial polarisation. The formulæ for calculating the intensities of the two components from the values of the refractive index n, the coefficient of absorption κ and the angle of incidence i, are exceedingly involved. If we express the intensity of the reflected radiation in terms of percentages of the incident radiation we must evaluate the quantities $\tan^2 \psi_p$ and $\tan^2 \psi_s$ in accordance with the procedure given by Drude (*2.824*). We have here computed these angles by the application of the rigorous formulæ given by Drude (*1.520*). In the calculation the incident light has been assumed to be of the nature of natural sunlight; and the two curves shown in Fig. 127 are based upon this assumption.

Silver and steel have been chosen as the characteristic representatives of metals employed in the construction of mirrors. The constants of these metals for yellow light, as given by Drude (*3.338*) are

		n	κ	R
Silver	...	0·18	$\frac{3{\cdot}67}{0{\cdot}18} = 20{\cdot}39$	95·16
Steel	...	2·41	$\frac{3{\cdot}40}{2{\cdot}41} = 1{\cdot}411$	58·43

It is evident that the refractive index of silver is exceptionally low, whilst that of steel is abnormally high; and accordingly, silver has a high and steel a low reflecting power.

From the curves it will be seen that the total intensity of the reflected light $\frac{1}{2}(I_s + I_p)$ remains in both cases unchanged until the angle of incidence reaches a value of about 80°, and increases markedly only at inclinations approaching grazing incidence, which is carefully avoided in optical instruments. In practice, we may accordingly regard the value of R as a characteristic index of the decline of the resultant intensity at any obliquity of incidence.

Very careful observations of this quantity have been made by Hagen and Rubens (*1.* and *2.*), We append a table of the results of these investigations in accordance with the later of the two memoirs, which gives amended values for old silver, steel and copper.

The table shows that in general the reflecting power R is a function of the wave-length and that it decreases with diminishing wave-lengths. This decrease is particularly definite in the case of metals such as copper and gold, which have a pronounced colour in natural light.

Reflecting Powers in Percentages of the Incident Radiation.

Wave-lengths in $\mu\mu$.	251	288	305	316	326	338	357	385	420	450	500	550	600	650	700	800
Fresh Silver	34·1	21·2	9·1	4·2	14·6	55·5	74·5	81·4	86·6	90·5	91·3	92·7	92·6	93·5	94·6	96·3
Old Silver	17·6	14·5	11·2	5·1	8·0	41·1	55·7	65·0	73·0	81·1	83·9	85·0	86·3	88·6	—	91·6
Platinum	33·8	38·8	39·8	—	41·4	—	43·4	45·4	51·8	54·7	58·4	61·1	64·2	66·3	69·0	70·3
Nickel	37·8	42·7	44·2	-	45·2	46·5	48·8	49·6	56·6	59·4	60·8	62·6	64·9	65·9	68·8	69·6
›l (unhardened)	32·9	35·0	37·2	—	40·3	—	45·0	47·8	51·9	54·4	54·8	54·9	55·4	55·9	57·6	58·0
d	38·8	34·0	31·8	—	28·6	—	27·9	27·1	29·3	33·1	47·0	74·0	84·4	88·9	92·3	94·9
'est Commercial Copper ...	25·9	24·3	25·3	—	24·9	—	27·3	28·6	32·7	37·0	43·7	47·7	71·8	80·0	83·4	88·6
›se's (Brashear's) Alloy 68·2 Cu + 31·8 Sn.) ...	29·9	37·7	41·7	—	—	—	51·0	53·1	56·4	60·0	63·2	64·0	64·3	65·6	66·8	71·5
›rœder's Alloy No. 1 66 Cu + 22 Sn + 12 Zn)...	40·1	48·4	49·8	—	54·3	—	56·6	60·0	62·2	62·6	62·5	63·4	64·2	65·1	67·2	71·5
ch's Magnalium 69 Al + 31 Mg.)	67·0	70·6	72·2	—	75·5	—	81·2	83·9	83·3	83·4	83·3	82·7	83·0	82·1	83·3	84·3
andes&Schünemann's Alloy (41 Cu + 26 Ni + 24 Sn + 8 Fe + 1 Sb)	35·8	37·1	37·2	—	39·3	—	43·3	44·3	47·2	49·2	49·3	48·3	47·5	49·7	54·9	63·1

Summarising these observations, we may make the following statement:

Reflection at metallic mirrors causes the resultant intensity of natural incident light to be diminished by different amounts according to the wave-length but uniformly as the angle of incidence is varied, the diminished intensity being

$$RI,$$

where R is the coefficient of the reflecting power at normal incidence.

97. Refracting Surfaces.—Provisionally we must assume that when refraction occurs at the boundary of two transparent media there is generally also reflection. (Fig. 128.)

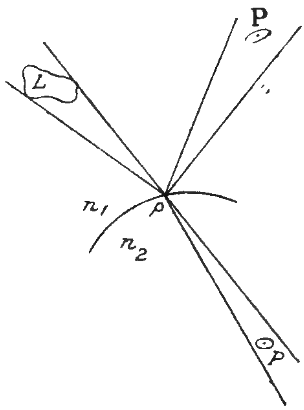

FIG. 128.

Indirect Radiation by Refraction.

Assuming the source of light L to be finite and of the nature of a radiant surface, every element of the refracting surface supposed to be localised at a point p, will, in general, emit two cones of rays obeying the laws of reflection and refraction respectively. Of course, the former is reflected back into the first medium, whilst the latter traverses the second medium after refraction. The surfaces of these cones determine the boundaries, within which the radiation due to reflection and refraction at this element is propagated, from the radiant element. Geometrically considered, points P, P within these boundaries receive the radiation transmitted by the surface element precisely as if the surface element

radiated with an, as yet, undetermined intrinsic intensity. Since this applies to every element of the surface we may for every point of the two spaces separated by the surface provisionally suppose the radiation emitted by the given source of light and transmitted by the surface to proceed from the surface itself. All that is then necessary is to ascribe to the radiation from point to point a certain specific intensity.

In order to calculate these **transmitted** intrinsic intensities, we must derive the corresponding formulæ from the results of the undulatory theory, and thus we find the intrinsic intensity of the reflected ray, viz. $\boldsymbol{I}'$, in terms of that of the incident ray, viz. $\boldsymbol{I}$, in accordance with the formula

$$\boldsymbol{I}' = \eta \boldsymbol{I},$$

in which η is a real fraction dependent upon the angles of incidence and reflection of the ray, viz. i, i', and the participating intensities $\boldsymbol{I}_p$, $\boldsymbol{I}_s$ of the polarised light before refraction, thus

$$\eta = \left[\boldsymbol{I}_p \frac{\sin^2(i-i')}{\sin^2(i+i')} + \boldsymbol{I}_s \frac{\tan^2(i-i')}{\tan^2(i+i')}\right].$$

In the case of unpolarised light $\boldsymbol{I}_s = \boldsymbol{I}_p = \frac{\boldsymbol{I}}{2}$, so that if we substitute m_p for $\frac{\sin^2(i-i')}{\sin^2(i+i')}$ and m_s for $\frac{\tan^2(i-i')}{\tan^2(i+i')}$ the expression for η becomes

$$\eta = \frac{\boldsymbol{I}}{2}(m_p + m_s).$$

In the limit, when $i = i' = 0$, if n denotes the quotient of the absolute indices of refraction

$$\frac{m_p + m_s}{2} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 = \left(\frac{n-1}{n+1}\right)^2.$$

We shall see presently that this formula, which is strictly true for paraxial rays, is suitable as a sufficiently accurate working formula for moderately inclined rays. We append accordingly a curve of η in terms of n, the value of n ranging from 1 to 1·7.

According to Kirchhoff's law the intensities of a source of light within two media are in the proportion of their absolute refractive indices, so that

$$\mathbf{i}_2 = \frac{n_2^2}{n_1^2}\,\mathbf{i}_1 .$$

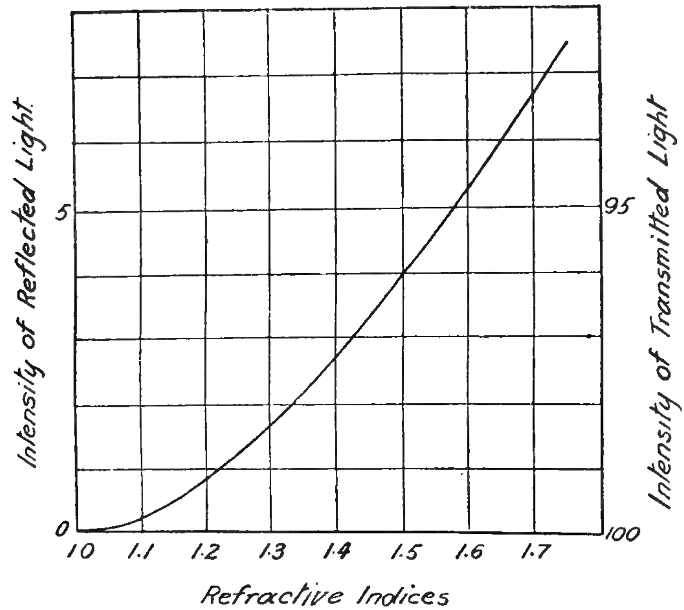

FIG. 129.

Reflection at a refracting surface expressed in percentages of the incident radiation.

Now, we have seen that reflection occurs at the boundary of the refracting medium. Hence, if there is to be no loss of energy it is necessary to introduce in the first medium the difference of the originally existing intensity and the intrinsic intensity due to reflection in the opposite direction, so that

$$\mathbf{i}_1 = \boldsymbol{I} - \boldsymbol{I}' = \boldsymbol{I}\,(1 - \eta).$$

For the calculation of the intrinsic intensity $\boldsymbol{I}''$ due to refraction at the point p on the bounding surface, we obtain accordingly the formula :

$$\boldsymbol{I}'' = \frac{n_2^2}{n_1^2}\,\boldsymbol{I}\,(1 - \eta).$$

From this it will be seen that we can ascertain the intensity I'' transmitted by the surface in the second medium, by subtracting from the specific intensity I of the incident light the loss by reflection ηI and multiplying the resulting difference by the square of the quotient of the indices of refraction. The intrinsic intensity, which is to be attributed to the source of light with respect to the second medium, becomes accordingly greater than the remainder which results after the deduction of the loss by reflection if the second is the denser medium, and it is less if the second medium has a lower refractive index.

In the annexed diagram, Fig. 130, the curve represents, for a surface separating two media of refractive indices $n_1 = 1$ and $n_2 = 1{\cdot}5$, the function connecting the values $1 - m_p$, $1 - m_s$, $1 - \frac{m_p + m_s}{2}$ and that of the angle of incidence i. The intensity of the transmitted light becomes zero at grazing incidence, since $i - i' = \pi - \left(\frac{\pi}{2} + i'\right)$, and consequently $i = \frac{\pi}{2}$ causes both m_p and m_s to become unity, whereas $1 - m_s$ experiences a much smaller decrease when the angle i is acute, and for angles less than $56^\circ{\cdot}3$ $\left(\text{where } i + i' = \frac{\pi}{2}\right)$ there is even a continuous increase relatively to the zero value. From this configuration of the curve representing $1 - m_s$ it follows that at angles of incidence i smaller than 40° to 50°, the quantity $1 - \frac{m_p + m_s}{2}$, experiences small changes from the value which it assumes in the case of paraxial rays. In practical optics we are mostly concerned with angles of incidence of this magnitude, and in all these cases it is permissible to regard the change of the factor $1 - \eta$ due to the reflection of natural light at a surface, as a constant, and as equal to that holding for paraxial rays. This then demonstrates the significance of the expression $\left(\frac{n - 1}{n + 1}\right)^2$ which we have established above.

By applying the theorem of the equivalent distribution of light, as derived above, we may find the respective values of the intensity corresponding to all points of the path of the reflected or refracted ray. Since with respect to all these points the values of i and i' (and likewise those of m_p and m_s) are constant, and on the supposition that the state of polarisation of the incident light is constant, it follows that with respect to them the intensity due to refraction and reflection depends solely upon the value I of the intrinsic intensity at that point where the ray, if traced back through the refracting surface, in accordance with the laws of reflection and refraction, meets the

primary source of light. A different intensity prevails, however, for points which lie on different straight lines proceeding from the given surface element, in that these correspond to other points of the source of light and in any case imply different values of the angles i and i'.

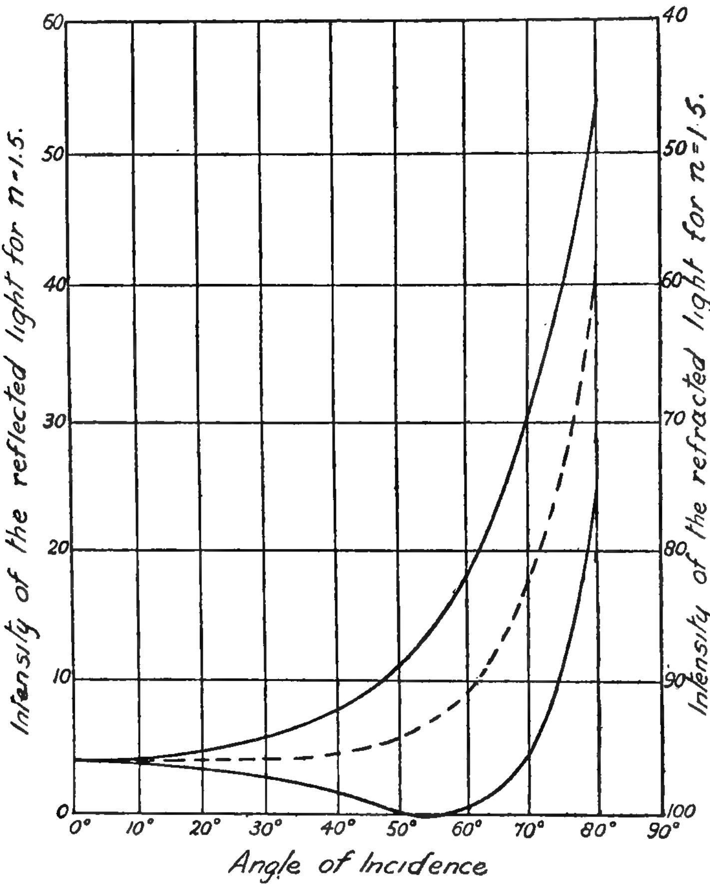

FIG. 130.

Curves of the value of—

$1 - m_s$: The lower full curve ;

$1 - \frac{m_p + m_s}{2}$: The dotted curve ;

$1 - m_p$: The upper full curve ;

For $n = 1{\cdot}5$, from $i = 0°$ to $i = 80°$.

Intensity of transmitted radiation.

Having ascertained the influence of reflection and refraction upon the resultant intensity of the light proceeding from the primary source, we may now combine the cases of reflection and refraction into a single statement. With regard to the resultant effect, it is quite immaterial where we choose to localise the transmitted intrinsic intensity on the rays proceeding from the refracting or reflecting surface.

Under certain circumstances, for instance when the optical device consists of a single refracting or reflecting surface, it is expedient to choose the surface itself for this purpose. In this case the radiant region which corresponds to a given point should be determined by constructing the image of the source of light, by applying to the surface the laws of refraction or reflection, and then projecting it from where it is formed upon the refracting or reflecting surface.

In most cases, however, the better plan is to transfer the seat of the transmitted radiation to the image of the source of light itself. This procedure is to be preferred when there are other refracting or reflecting surfaces. In what follows we shall endeavour to simplify matters by confining our attention, to the effect of refraction on the investigation of refracting surfaces.

This investigation may be comprised in the following statement: Every refracting or reflecting surface subjects the rays proceeding from a source of light and intercepted by that surface to a refraction or reflection of a clearly determinable kind. If now we determine by the Gaussian theory a hypothetical image conjugate to the source of light (producing, if necessary, the rays until they cut the plane of the image), and if then we ascribe to these elements the transmitted intensity computed from the intrinsic intensity of the source of light and angles of refraction or reflection, this image will behave like a new self-luminous source of light with respect to every point which encounters refracted or reflected light, whereas it produces no effect whatever with regard to other points.

The process by which refracting or reflecting surfaces transmit the light of a given source may accordingly be reduced to the formation of optical images of the source of light, which behave like self-luminous sources possessing an intensity which is deducible from the primary intensity. This process is, however, confined within the compass of a certain solid angle embracing the space which may be reached by the refracted or reflected rays.

C. The Effect of Absorption.

298. In the preceding investigations it was assumed that two equivalent sources of light, that is two radiants subtending similar angles of view, exercise similar effects upon a surface element, whatever their distance. This assumption takes no cognizance of

absorption and holds good only so long as we are concerned with absolutely transparent media. Under actual conditions it is necessary to take into account the influence of absorption.

Let a be the **coefficient of transparency** of a medium, and let this represent the fractional part of the incident intrinsic intensity which is transmitted after the ray has traversed unit distance within the medium. Then the intensity reduced by absorption through unit distance will be

$$J_1 = Ia,$$

and in the case of light which has traversed a length d it will be

$$J = Ia^d.$$

The choice of the unit length of the path is governed by the nature of the absorbing substance. In the case of optical glass, which primarily concerns us, it is convenient to choose the centimetre for this purpose.

The coefficients of transparency a of optical glasses are as a rule dependent upon the wave-length. We append in the first place the values tabulated by Müller-Wilsing in the case of crown and flint glasses of older types. Up to wave-lengths $\lambda = 477\ \mu\mu$ the data have been determined optically, whilst those corresponding to shorter wave-lengths were found by a photographic method.

Coefficients of Transparency $a_{(1\text{ cm.})}$ of two glasses of older type.

Wave length in $\mu\mu$.	375	390	400	434	477	535	580	677
	0.203. Ordinary Silicate Crown ; $n_D = 1.5175$; $\nu = 59.0$.							
	0.947	0.947	0.965	0.960	0.985	0.989	0.986	0.990
	0.340. Ordinary Light Flint ; $n_D = 1.5774$; $\nu = 41.4$.							
	0.909	0.923	0.951	0.945	0.986	0.989	0.986	0.992

In the case of more recent glasses a very comprehensive investigation has been made by H. A. Kruess (*1.*) with special regard to the short-wave part of the spectrum. From this paper we have taken the subjoined tables.

Particulars of the Glasses.

No.	Type.	Description.	n_D	ν	*S. G.*	Colour.
3094	*O.* 144	Boro-silicate Crown	1·5100	64·0	2·47	colourless.
2900	*O.* 2388	Telescope Crown ...	1·5254	61·7	2·85	grayish green.
2990	*O.* 60	Lime Silicate Crown	1·5179	60·2	2·49	colourless.
3046	*O.* 1209	Heaviest Baryta Crown.	1·6112	57·2	3·55	yellow brown.
1800	*O.* 722	Baryta Light Flint	1·5797	53·8	3·26	" "
2572	*O.* 846	" " "	1·5525	53·0	3·01	colourless.
3111	*O.* 1266	" " "	1·6042	43·8	3·50	greenish-yellow.
3013	*O.* 748	Baryta Flint ...	1·6235	39·1	3·67	" "
2563	*O.* 919	Ordinary Silicate Flint.	1·6315	35·7	3·73	" "
2625	*O.* 192	Heavy Silicate Flint	1.6734	32·0	4.10	yellowish.

Coefficients of Transparency $a_{(1\ cm)}$ of the above Glasses.

λ in $\mu\mu$	3094	2900	2990	3046	1800	2572	3111	3013	2563	2625
434	—	—	—	—	0·969	—	—	—	—	—
425	0·993	0·970	0·982	0·965	0·961	0·978	0·963	0·952	0·961	0·905
415	0·982	0·968	—	—	0·965	0·973	—	—	—	—
406	—	0·964	—	—	0·974	—	—	—	—	—
396	0·986	0·980	0·981	0·941	0·971	0·987	0·931	0·917	0·944	0·76
384	0·972	0·955	0·975	0·894	0·948	0·968	0·865	0·84	0·86	0·58
361	0·950	0·942	0·949	0·65	0·849	0·952	0·68	0·61	0·66	0·16
347	0·88	0·85	0·91	0·28	0·66	0·88	0·46	0·41	0·30	0·01
330	0·65	0·53	0·77	0·07	0·32	0·66	0·06	0·03	0·05	0
309	0·08	0	0·03	0	0·01	0·02	0	0	0	—

In the case of two media having the coefficients of transparency a_1 and a_2, we must also take into account the distances from the surface to the source of light and to the illuminated point, viz., d_1, before refraction, and d_2 after refraction (or d_1 after reflection). In the case of reflection we obtain accordingly the expression

$$J' = a_1^{d1}\, I'\, a_1^{d1} = a_1^{d1+d1}\, \eta\, I$$

and in the case of refraction

$$J'' = a_1^{d1}\, I''\, a_2^{d2} = a_1^{d1}\, a_2^{d2} \left(\frac{n_2}{n_1}\right)^2 (1 - \eta)\, I.$$

D. Radiation Transmitted through a general Centred System.

299. In a system of refracting and reflecting surfaces at right angles to the same axis the problem presented by any number of centred surfaces can be treated by a repeated application of the preceding case, the image formed by any surface being regarded as the object with respect to the next surface. To simplify the investigation we shall suppose that the first surface receives natural light.

Let

$$I_{1p}'' = a_1^{d1} \frac{I}{2} \left(\frac{n_2}{n_1}\right)^2 (1 - m_{1p})$$

be the intrinsic intensity prevailing in the medium n_2 of one of the components passing through surface 1. Then owing to the absorption a_2 in medium n_2, the intensity which reaches surface 2 over the path of length d_2 will be

$$J''_{1p} = a_2^{d2}\, I''_{1p} = a_1^{d1}\, a_2^{d2} \frac{I}{2} \left(\frac{n_2}{n_1}\right)^2 (1 - m_{1p})$$

and similarly with respect to the other component,

$$J''_{1s} = a_2^{d2}\, I''_{1s} = a_1^{d1}\, a_2^{d2} \frac{I}{2} \left(\frac{n_2}{n_1}\right)^2 (1 - m_{1s}) .$$

The loss by reflection at surface 2 follows from

$$I'_{2p} = J''_{1p}\, m_{2p} = a_1^{d1}\, a_2^{d2} \frac{I}{2} \left(\frac{n_2}{n_1}\right)^2 (1 - m_{1p})\, m_{2p}$$

and

$$I'_{2s} = J''_{1s}\, m_{2s} = a_1^{d1}\, a_2^{d2} \frac{I}{2} \left(\frac{n_2}{n_1}\right)^2 (1 - m_{1s})\, m_{2s} .$$

Hence we can find the intensity transmitted through surface 2 from the equation

$$I''_{2p} = \frac{n_3^2}{n_2^2} (J''_{1p} - I'_{2p}) ,$$

and thus finally we obtain the equations

$$I''_{2p} = \left(\frac{n_3}{n_1}\right)^2 a_1^{d1} \, a_2^{d2} \frac{I}{2} \; (1 - m_{1p}) \; (1 - m_{2p})$$

and

$$J''_{2p} = a_3^{d3} \, I''_{2p} = \left(\frac{n_3}{n_1}\right)^2 a_1^{d1} \, a_2^{d2} \, a_3^{d3} \frac{I}{2} \; (1 - m_{1p}) \, (1 - m_{2p}) .$$

Similar equations are applicable to the other components and therefore it is not difficult to establish a general expression for k surfaces and $(k + 1)$ media. In the $(k + 1)^{\text{th}}$ medium let the distance from the point of emergence at the k^{th} surface to the point of reference be denoted by d_{k+1}. Then the magnitude of the components of the intensity will be given by the equations

$$J''_{kp} = \left(\frac{n_{k+1}}{n_1}\right)^2 \frac{I}{2} \prod_{v=1}^{k+1} a_v^{d_v} \prod_{v=1}^{k} (1 - m_{vp})$$

$$J''_{ks} = \left(\frac{n_{k+1}}{n_1}\right)^2 \frac{I}{2} \prod_{v=1}^{k+1} a_v^{d_v} \prod_{v=1}^{k} (1 - m_{vs})$$

and

$$I''_{k+1} = J''_{kp} + J''_{ks} = \left(\frac{n_{k+1}}{n_1}\right)^2 \frac{I}{2} \prod_{v=1}^{k+1} a_v^{d_v} \left\{ \prod_{v=1}^{k} (1 - m_{vp}) + \prod_{v=1}^{k} (1 - m_{vs}) \right\} .$$

We obtain accordingly the following result for the intensity transmitted by the system, that is the intensity of the principal image: The fractional part of the intensity to which it is reduced by absorption can be separated from the fractional part due to reflection, and the fractional part due to these two combined effects results as the product of both factors. In this connection it is to be noted that the fractional part due to reflection depends solely upon the type of the optical system, but is not affected by any dimensional changes, whereas the fractional part which is due to absorption is governed by the dimensions of the system, inasmuch as the absorption increases exponentially with the linear increase of the dimensions.

The rays reflected at the various surfaces and accordingly having successively diminishing intensities are in their turn subjected to losses by reflection at surfaces situated farther back. The reflected light thus transmitted backwards ultimately emerges

through the first medium in the opposite direction to that of the originally incident light. In general it will be sufficient to treat this reflected light in its effect upon the principal image as a simple loss of light.

The case is different with rays which have been reflected twice (or more correctly any even number of times). These ultimately enter the last medium, where they form, by reflections and refractions *catadioptric secondary images* of the object which do not, as a rule, coincide with the primary image.

These effects have been investigated more fully in relation to photographic objectives.* The resulting intensity of the total light subjected to an even number of reflections has been determined numerically only in the special case in which the surfaces separating the media may be regarded as parallel planes of unlimited extent. A comprehensive paper on the subject has been published by H. Kruess (*3.*).

In the absence of any statement to the contrary, we shall consider radiation transmitted by refraction only. Let κ stand for the fractional part of the total intensity, as formulated above, in which case our last equation will assume the abbreviated form

$$I''_{k+1} = \left(\frac{n_{k+1}}{n_1}\right)^2 \kappa\, I .$$

This expression agrees with Kirchhoff's law, which takes account of the losses due to absorption and reflection. The intensity transmitted by the system experiences accordingly a diminution to the magnitude κI, when the first and last medium are identical. This diminution becomes more pronounced when $n_{k+1} < n_1$. In the reverse case, when $n_{k+1} > n_1$, three alternatives may arise, in that the diminution may become less, or it may just vanish, or it may be converted into an increase according as $\sqrt{\kappa}.n_{k+1}\begin{cases} < n_1 \\ = n_1 \\ > n_1 \end{cases}$.

Having in this way ascertained the transmitted intensity which is to be ascribed to every point in the image of the source of light, we are able to reduce the investigation of the distribution of light within the entire image-space to a problem which has already been discussed. For if we construct the exit-pupil of the instrument, the image of the source of light behaves as if it were a self-luminous surface in relation to those points of the space with respect to which the participating rays are able to pass the exit-pupil either directly or when produced. In Fig. 116 we have indicated the three regions into which the image-space is divided in this way.

* Not included in the present work.

This leads us to **Abbe's first law of radiation** (*I.* 285), which may be enunciated thus :

"The resultant radiating effect within the boundary of the last medium, due to any optical instrument, is completely determined in every respect if we ascribe to every point in the image of the object the specific intensity prevailing at the corresponding point of the object itself (or an intensity proportional to it), and also, if we assume the radiation proceeding from the image to be bounded by the image of the aperture in the same manner as the radiation of a self-luminous surface is restricted by a corresponding diaphragm."

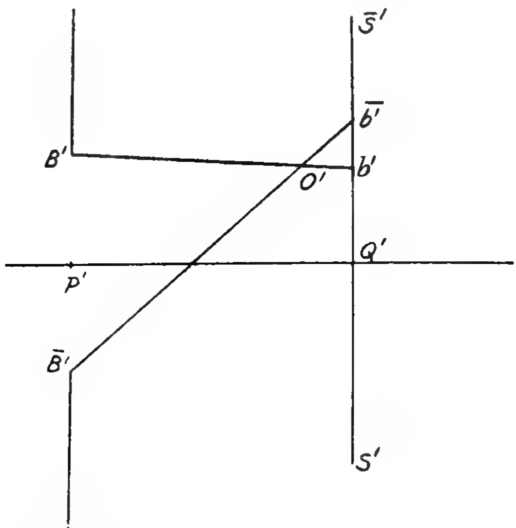

FIG. 131.

Diagram showing the location of the specific intensity in the exit-pupil.

This rule, which is analogous to the investigation of the primary radiation within the object-space, enables us to determine the illuminating effect in the image-space in all cases in which the fundamental rule of the law of photometry can be applied, that is in which the illuminated object is at a finite distance from the image of the source of light. But the rule breaks down if it so happens that the element, for which the illuminating effect is to be determined, coincides with the image of the source of light itself, as may occur, for example, in the case of optical combinations designed for projection on the screen.

We can avoid this difficulty by introducing an equivalent distribution of light of another kind, and we may accordingly suppose the transmitted intensity localised at the corresponding points

of the exit-pupil. To this end we may take a point O', which we suppose in the first instance to be situated outside the boundary of the image $S'Q'\bar{S}'$ of the source of light, and from this point O' let every element of this image comprised between $\bar{b}'$ and b' be projected into the exit-pupil, and at the point between $\bar{B}'$ and B' where the plane of the exit-pupil is traversed, let the transmitted intensity be equal to that of the element. The distribution of light in the exit-pupil which thus results may then be substituted in all respects for the image of the source of light with regard to the point O'. This is expressed in the following terms in Abbe's corollary (*I. 288, 289.*):

"The resultant radiation at any point of the last medium is in every respect identical with a radiation proceeding from the surface of the aperture image, provided we ascribe to every point of the latter an intrinsic intensity equal or proportional to the intensity of that portion of the primary source of light whose image, if projected from that point in the last medium, falls upon the image of the aperture."

Now let the illuminated point O' be moved nearer and nearer to the source of light Q'; then the entire exit-pupil will have corresponding to it a continually diminishing circle in the image of the source of light, and ultimately when the illuminated point O' lies in the image itself this point supplies the whole of the intensity which has to be distributed over the surface of the exit-pupil. The case here considered is expressed in the following terms:

"The luminous effect due to any optical device at any point of the image of a given source of light is always equivalent to that of a radiation proceeding from the surface of the aperture image, supposing every point of the latter to have ascribed to it the intensity of the corresponding object-point, or, if the first and last media differ, an intensity differing from it in the ratio of the squares of the refractive indices."

3. Application of the Laws of Radiation to Optical Instruments.

300. We are now in a position to investigate the transmission of rays through the agency of optical instruments; and, as before, we shall make a distinction between instruments designed for projection and those intended for ocular observation.

A. Projection Systems in Air.

301. Amount of Light which enters the System through an Unrestricted Entrance-pupil.—We shall consider in the first instance a point on the axis at a finite distance A

from the entrance-pupil. Then, by § 261, if the angular aperture of the system is u, we know that

$$\tan u = \frac{\mathsf{r}}{\mathsf{A}},$$

where r is the radius of the entrance-pupil.

As the entrance-pupil always plays the part of the illuminated surface, the quantity of light proceeding into the system from a surface element da at right angles to the axis at O is given by the expression

$$d\boldsymbol{L}_0 = \pi \boldsymbol{I}\, da \sin^2 u,$$

where u is a finite angle.

When the diameter of the pupil is very small, and therefore

$$dA = \pi\, d\mathsf{r}^2,$$

the above expression becomes

$$dL_0 = \boldsymbol{I} \frac{da\, dA}{\mathsf{A}^2},$$

which agrees with Lambert's law.

Proceeding on the assumption that the entrance-pupil is very small we find by the application of this theorem and by regarding the entrance-pupil as being free from aberration that the quantity of light on an element of an object-plane at a distance A about a principal ray subtending an angle w at the entrance-pupil is given by the expression

$$dL_w = \frac{\boldsymbol{I}\, da\, dA \cos^4 w}{\mathsf{A}},$$

hence

$$dL_w = dL_0 \cos^4 w.$$

When the pupil has a finite diameter we find by § 293 the following value for the region within which no portion is cut off by the exit-window

$$d\boldsymbol{L}_w = \frac{\pi}{2} \boldsymbol{I}\, da \left(1 - \frac{p}{\sqrt{q^2 - 1}}\right),$$

where p and q have the meaning defined in § 293, when y_s is given by the relation

$$y_s = -\mathsf{A} \tan w.$$

For the purpose of roughly computing systems devised for use at moderately large apertures facing the object it is very convenient when the quotient

$$\frac{d\boldsymbol{L}_w}{d\boldsymbol{L}_0}$$

corresponds very nearly with the value of $\cos^4 w$,

To demonstrate this statement it may be noted that in an extreme case, in which

$$\mathsf{A} = -100 \text{ mm} ; \; \mathsf{r} = 15 \text{ mm} ; \; y_s = 100 \text{ mm.} \quad (w = 45°),$$

the quotient $\dfrac{d\boldsymbol{L}_{45}}{d\boldsymbol{L}_0}$ agrees within 2·84 % with the value of $\cos^4 w$. The transmitted quantity of light would in this case involve an error of 0·7 % of the quantity of light which obtains at the middle of the field of view. Discrepancies of this small order scarcely enter into consideration even in more exact calculations, since there is no occasion to determine intensities with any very considerable degree of accuracy. Moreover, the above is an extreme case, as in practice it is highly improbable that a system having so large an aperture ([NA] = − 0·148) can be made with an unstopped field of view of 45°.

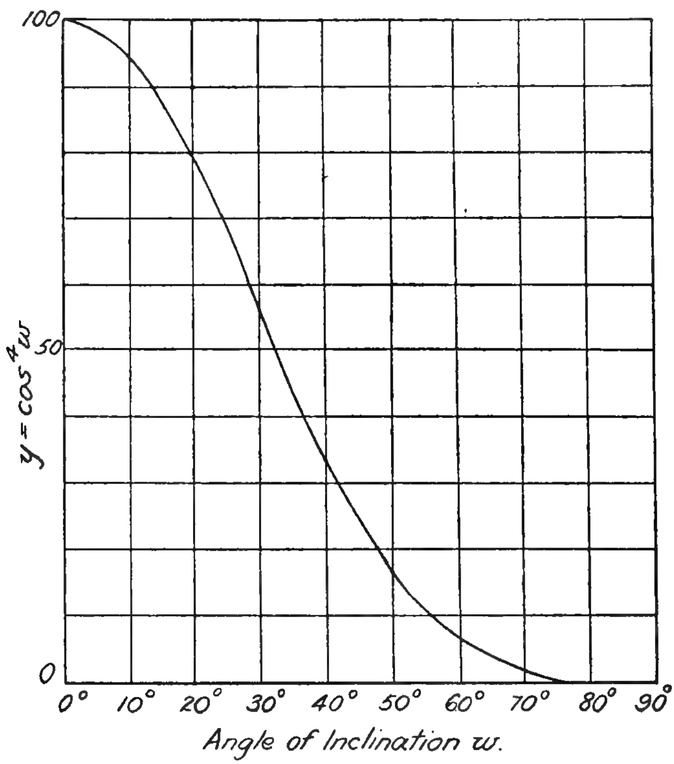

FIG. 132.

Curve of the function $\cos^4 w$ for $w = 0$ to $w = 90°$.

The curve in Fig. 132 of the function $\cos^4 w$ shows fairly correctly the rate at which the quantity of light emitted towards the entrance-pupil decreases with increased distance from the centre of surface elements of the unstopped field of view corresponding to finite values of w.

At $w = 13°\cdot 1$, the quantity of light prevailing at the centre is already reduced to 90 %, and it will be seen that it diminishes at an increasing rate as the angle increases, so that at $w = 32°\cdot 8$ it is only one-half of the original amount.

In the case of an infinitely great distance A in a system which is telecentrical with respect to the object, the surface elements in the unstopped part of the field of view are not subject to any decrease of the quantity of light transmitted through the system as compared with the light at the centre.

302. The Quantity of Light transmitted through a System with Secondary Stopping by Entrance-Windows.—We shall now suppose the object-point O_w to be situated outside the unstopped region. We then have two possible cases, as demonstrated above, in that there may be one or two entrance-windows. We shall confine our consideration to the special case of a window in a finite position, the entrance-pupil being in front and likewise in a finite position, noting that when it is situated at the back of the system the prevailing conditions would need to be determined by entirely similar considerations.

Projecting from O_w the boundary of the entrance-window into the plane of the entrance-pupil, we find for the radius $\bar{\mathsf{R}}$ of this projection

$$\bar{\mathsf{R}} = \frac{-\mathsf{A}}{-\mathsf{A} + \eta}\,\mathsf{R},$$

and the distance $P\bar{L}$ from the axis of the centre $\bar{L}$ in the plane of the paper is

$$m_w = \frac{-\mathsf{A}}{-\mathsf{A} + \eta}\,\eta \tan w\,.$$

The unrestricted area of the entrance-pupil is now represented by the circular segment $DQD'M$, while the boundary is partly made up by that of the entrance-pupil and partly by that of the projection of the entrance-window. From O_w let a perpendicular $O_w\bar{P}$ be let fall upon the plane of the entrance-pupil, and let $\bar{P}D$ and O_wD be drawn. Then, as before, we shall use the symbols

$$\text{angle } D\bar{P}P = \psi\,; \qquad \text{angle } DO_w\bar{P} = \omega.$$

Now, in the same way as we defined the quantities p, q with respect to A, y_s and r, let $\bar{p}$ and $\bar{q}$ be applied to the expression found from the quantities A, $\bar{y}_s = y_s - m_w$, and $\bar{\mathsf{R}}$; also, with the radius $\bar{P}D$ let an arc DND' be described about the centre $\bar{P}$. Then the quantity of light which falls upon the two segments $MDND'$ and $NDQD'$ may be expressed by

$$[MDND'] = d\boldsymbol{L}_{\chi\psi}\Big/\boldsymbol{I}\,da\,;\quad [NDQD'] = (d\bar{\boldsymbol{L}} - d\bar{\boldsymbol{L}}_{\bar{\chi}\psi})\Big/\boldsymbol{I}\,da.$$

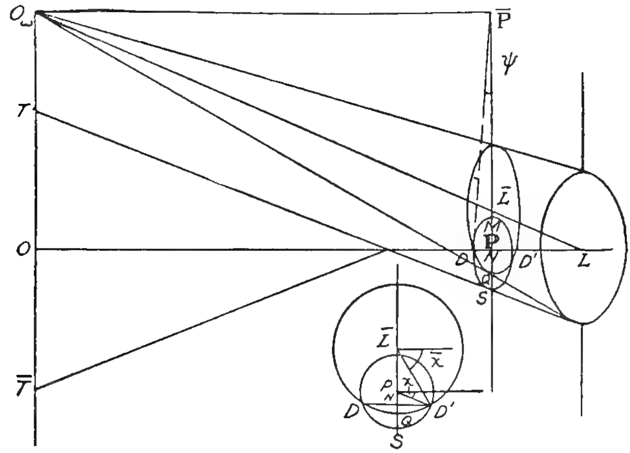

FIG. 133.

$PO = \mathsf{A}\,;\ PL = \eta\,;\ P\bar{L} = m_w.$

Determination of the quantity of light radiated into the system when the latter is stopped down by the entrance-window.

The angles χ and $\bar{\chi}$ are indicated in the front view of the plane of the entrance-pupil. If, finally, we substitute the values of $d\boldsymbol{L}_{\chi\psi}$ and $d\bar{\boldsymbol{L}}_{\bar{\chi}\psi}$ we obtain after a few simplifications

$$[MDQD'] = [MDND'] + [NDQD'] = (\bar{\chi} - \chi)^2$$

$$+\frac{p}{\sqrt{q^2-1}}\left(\tan^{-1}\frac{q\tan\chi_2 - 1}{\sqrt{q^2-1}} - \tan^{-1}\frac{q-1}{\sqrt{q^2-1}}\right) + \frac{\pi}{2}\left(1 - \frac{\bar{p}}{\sqrt{\bar{q}^2-1}}\right)$$

$$-\frac{\bar{p}}{\sqrt{\bar{q}^2-1}}\left(\tan^{-1}\frac{\bar{q}\tan\bar{\chi}_2 - 1}{\sqrt{\bar{q}^2-1}} - \tan^{-1}\frac{\bar{q}-1}{\sqrt{\bar{q}^2-1}}\right);\quad \chi_2 = \frac{1}{2}\,\chi.$$

The case of two entrance-windows leads to a more complicated expression, but it does not involve any special difficulties in principle.

303. The Intensity of the Illumination on the Plane of the Image.—The quantity of light dL which enters the entrance-pupil, owing to the loss due to absorption and reflection which it suffers whilst traversing the system, becomes reduced to κdL, and this amount of light should be ascribed to the exit-pupil, whence it is transmitted to the surface element da' of the image-plane. To ascertain the intensity of the illumination in the image, which we will assume to be sharply defined, we require to know the magnitude of the image da' on the surface element da. Let β^2 be the surface magnification; then

$$da' = \beta^2\, da$$

and hence

$$\frac{dL'}{da'} = \frac{\kappa}{\beta^2}\,\frac{dL}{da}$$

for every element da of the flat object. It is thus seen that the intensity of the illumination of flat objects is affected by the distortion.

This problem has an important practical bearing on the theory of the photographic lens, where it is necessary to deal fully with the factor β^2 which occurs in the above expression as a purely formal quantity.

B. Systems bounded at either end by Media of Different Refractive Indices.

304. In the case in which the first and the last media have different optical properties, the quantity of light transmitted from a surface element da at p at right angles to the axis through an entrance-pupil of a finite aperture u will be

$$dL = \pi I da \sin^2 u$$

and similarly with respect to the conjugate image at p',

$$dL' = \pi I'' da \sin^2 u',$$

assuming that an actual image is formed at that point and that this image is free from spherical aberration as regards the finite angle u'.

Now, by Kirchhoff's law

$$I'' = \left(\frac{n_{k+1}}{n_1}\right)^2 \kappa I,$$

hence

$$\frac{dL'}{\kappa} = \pi I \left(\frac{n_{k+1}}{n_1}\right)^2 da' \sin^2 u'.$$

Now, since $dL = \frac{dL'}{\mathsf{k}}$ and for paraxial rays $\left(\frac{da'}{da}\right)_{u=u'=0} = \beta^2$ it follows that to obtain a distinct image within the finite aperture angles u, u', *i.e.*, in order that the magnification β, as defined with respect to paraxial rays, may also hold for finite angles, we must satisfy the condition

$$\beta n_{k+1} \sin u' = n_1 \sin u.$$

The product of the numerical aperture and the lateral magnification must therefore be constant for conjugate points which are free from aberration, if the image-formation refers to a surface element.

We thus see that the sine condition which we arrived at as a theorem in geometrical optics accords with the theorems of radiation, and it is on these lines that Helmholtz (*4.*) has proved the sine condition. Conversely, Kirchhoff's law of radiation may be established for aplanatic points, as has been done by Czapski (*3. 178*).

305. The Special Case of the Eye.—Disregarding the losses of light occurring within the media of the eye, the intensity of the illumination on the retina, *i.e.*, the **natural brightness** in vision with the unaided eye, is expressed by the formula

$$B_0 = \frac{dL}{da'} = \pi I n'^2 \sin^2 U',$$

where U' denotes the angular aperture within the vitreous humour whose magnitude varies slightly according to the diameter of the pupil and embraces about 5°. From this expression it follows at once that the natural brightness of a uniformly luminous surface is entirely independent of its distance from the eye, so long as it presents the appearance of a surface.

When the eye is associated with an instrument devised as an aid to vision the resultant brightness is expressed by the formula

$$B = \frac{dL}{da'} = \mathsf{k} I n'^2 \sin^2 U',$$

where again U' denotes the angular aperture within the vitreous humour. If now under these conditions of illumination the exit-pupil of the instrument is greater than the pupil of the eye or equal to it, the latter intercepts a portion of the exit-pupil or just encircles it, and the impression of brightness received by the eye is the same as that which is produced when a similarly magnified object endowed with a transmitted intensity $\mathsf{k}I$ is viewed with the unaided eye. It is then usual to say that the **magnified object is seen through the instrument in its natural brightness.**

On the other hand, if the diameter $\bar{\mathsf{p}}$ of the exit-pupil is smaller than that of the pupil of the eye, the corresponding angular aperture will be $\bar{U}'$, and the brightness becomes

$$\bar{B} = \pi \, \mathsf{k} \, I n'^2 \sin^2 \bar{U}'.$$

If now we assume that it is possible with an instrument of similar type to obtain with respect to da' a diminished magnification, such that the angle U' becomes just sufficiently large to correspond to the radius p_0 of the pupil under the same conditions of illumination, it follows from the preceding equation for B that the instrument furnishes the natural brightness

$$B_0 = \pi \, \mathsf{k} \, I \, n'^2 \sin^2 U'$$

and we thus obtain the equation

$$\frac{\bar{B}}{B_0} = \frac{\sin^2 \bar{U}'}{\sin^2 U'}.$$

When the angles are small no sensible error is introduced by interchanging sines and tangents. In this case, if we suppose the instrument to be focussed for distinct vision

$$\frac{\bar{B}}{B_0} = \frac{\mathsf{l}'^2 \tan^2 \bar{U}'}{\mathsf{l}'^2 \tan^2 U'} = \frac{\bar{\mathsf{p}}'^2}{\mathsf{p}'^2_0} = \frac{\bar{\mathsf{p}}^2}{\mathsf{p}^2_0}.$$

The numerator and denominator of the last fraction may also be expressed in terms of the angle of emergence into air of the instrument and the distance of distinct vision, thus

$$\begin{aligned} \bar{\mathsf{p}} &= \mathsf{l} \tan \bar{u}' \\ \mathsf{p}_0 &= \mathsf{l} \tan u'_0 \end{aligned}$$

Further, supposing that the sines and tangents of u' and u'_0 may be interchanged, and that the object-point under observation is an aplanatic point of the instrument; also denoting by $\bar{N}$ and N_0 the linear magnifications of the object-point and by $[\mathsf{NA}]$ the numerical aperture, we obtain the following equations:

$$\bar{\mathsf{p}} = \mathsf{l} \sin \bar{u}' = \frac{\mathsf{l}[\mathsf{NA}]}{\bar{N}}$$

$$\mathsf{p}_0 = \mathsf{l} \sin u'_0 = \frac{\mathsf{l}[\mathsf{NA}]}{N_0}.$$

This latter magnification N_0, which obtains when the diameter of the exit-pupil of the instrument is equal to that of the pupil of the eye, may be called the **standard magnification**, which is accordingly expressed by the formula

$$N_0 = \frac{\mathsf{l}[\mathsf{NA}]}{\mathsf{p}_0},$$

and hence, by § 276,

$$-M_0 = \frac{[\mathsf{NA}]}{\mathsf{p}_0}.$$

With the aid of the expressions for $\bar{\mathsf{p}}$ and p_0 we finally obtain the relation

$$\frac{B}{B_0} = \frac{N_0^2}{N^2} = \frac{M_0^2}{M^2}.$$

Combining these results into one statement we may say :—The brightness transmitted by optical instruments is at most equal to that obtaining in natural vision so long as the magnification is below the normal value, but when the magnification of the instrument exceeds this limit the brightness is inversely proportional to the surface magnification.

4. HISTORICAL NOTES.

306. The fundamental law of photometry and its application to cases which lead to expressions capable of integration have been derived from the works of Lambert (*1. 2.*) and Beer (*1.*), where this part of photometry is dealt with at great length. The application of the theory to optical instruments was not carried very far by these two investigators, though in a few special cases they arrived at correct results respecting the transmission of radiation. They determined the losses by absorption and reflection by empirical methods.

The transmission of diffuse radiation has been investigated by Fr. Thaler (*1.*) in continuation of Wiener's researches. His paper contains numerous references. The reflection at finite angles of incidence in the case of polished surfaces was probably first investigated critically by Maccullagh (*1.*). His curves of the intensity of reflected light were further investigated by Mascart (*1.* ii, *449*). They are quite similar in character to the lower curve of Fig. 127. The loss due to reflection at a refracting surface at any angle of incidence was first determined theoretically by Fresnel (*1.*).

As already stated, a comprehensive paper on the loss of light due to multiple reflection at plane parallel plates was contributed by H. Kruess (*3.*). This paper contains a bibliography including a notable

paper by Stokes (*1.*) as well as data respecting the absorption due to a heavy flint glass as derived from very careful experiments. Accurate determinations of the coefficients of transparency of a series of older kinds of glass have been furnished by Vogel (*1.*) and by Wilsing (*2.*). A very comprehensive bibliography is appended to a paper by H. A. Kruess (*1.*), which gives the coefficients of transparency of recently introduced optical glasses with respect to the short-wave part of the spectrum.

An important contribution to the theory of transmitted radiation through optical instruments has been made by Naegeli and Schwendener (*1. 85.*) incidentally to a discussion of the function of the illuminating apparatus of a microscope. In 1871 Abbe (*1.*) published his general theory of the transmission of rays, in which he incorporated as an essential part the theory of stops discussed in the preceding chapter. Basing his arguments upon Kirchhoff's law he arrived at the laws of radiation which have been embodied in the majority of text-books dealing with this subject. According to these theorems the radiation transmitted through optical instruments may be ascribed in one part to the image of the source of light, and in the other to the exit-pupil of the instrument. Among special applications of the theory we find only the theory of the condenser, whilst Abbe (*2.*) in 1873 stated that the brightness of the image furnished by a microscope can never be greater than that obtaining in the case of unaided vision, being at most equal to it. Very soon thereafter, Helmholtz (*3. 4.*) investigated these problems. The development of the theory of radiation led to the enunciation of the sine theorem and to the definition of the standard magnification.

In more recent times Drude (*3.*) has published a very lucid summary of the general definitions of photometry and of the transmission of radiation through optical instruments.

List of Symbols.

A, A': co-efficients of longitudinal spherical aberration.

Ap, $A'p$: conjugate object and image distances measured from the pupils.

$\mathbf{A}$, $\mathbf{A}'$: abscissæ of conjugate points referred to any given pair of conjugate points.

B, B': co-efficients of the first zonal term of the longitudinal spherical aberration.

$\boldsymbol{c} = \boldsymbol{s} - r$: distance of the object from the centre of the sphere.

c_v: distance between the centres of the v^{th} and $(v+1)^{\text{th}}$ surfaces.

c_x, c_x': direction cosines of a skew ray for the axis of x.

c_y, c_y': direction cosines of a skew ray for the axis of y.

c_z, c_z': direction cosines of a skew ray for the axis of z.

d_v: axial thickness between the v^{th} and $(v+1)^{\text{th}}$ surfaces.

$\boldsymbol{d}_v$: oblique thickness between the v^{th} and $(v+1)^{\text{th}}$ surfaces measured along a principal ray.

$D_{xs} = \dfrac{1}{s_1} - \dfrac{1}{x_1}$: For definition, see page 332.

$\mathbf{D}_{k,v} = \dfrac{y_k^2}{y_v^2}\left(\dfrac{1}{x_h^2} - \dfrac{1}{x_v^2}\right)$: For definition, see page 335.

D_{I}, D_{II}, D_{III}, etc.: SEIDEL's image defects (see page 396).

D: KERBER's equivalent for $\sin i' + \sin u' = \sin i - \sin u$.

$\boldsymbol{e}$, $\boldsymbol{e}'$: normals from the apex of the surface upon the direction of the ray before and after refraction. (Definition, page 78.)

f, f': focal lengths measured from the principal foci to the principal points.

f_t, f': tangential focal lengths of oblique pencils.

f_f, f_f': sagittal focal lengths of oblique pencils.

h: incidence height in general, a finite quantity in most cases.

h_v: paraxial incidence height at the v^{th} spherical surface.

h_v: paraxial incidence height at the v^{th} spherical surface for the second colour.

h_{sf}, h_{vf}': normals from the point of incidence of the principal ray upon the adjoining sagittal rays before and after refraction.

h_{vt}, h_{vt}' : normals from the point of incidence of the principal ray upon the adjoining tangential rays before and after refraction.

Similarly, $\frac{h_{kt}'}{h_{1t}}$ (page 175) and (less precisely) $\frac{h_{kt}}{h_{1t}}$ (page 278).

i, i' : angles of incidence of rays proceeding from points on the axis before and after refraction.

$I = n \sin i = n' \sin i'$: fundamental invariant (page 132); at small apertures $I = h Q_s$.

j, j' : angles of incidence of rays proceeding from the centre of the stop before and after refraction.

$J = n \sin j = n' \sin j'$: fundamental invariant; at small inclinations $J = y Q_x$.

$1/k$: relative aperture of a system when the object is at a great distance.

l, l' : distances from the axis of the point where the rays cut the GAUSS planes, conforming to inclinations w, w' of the principal rays when there is no aberration.

$l_v^{(k)}$ ($v = 0 . 1 . 2 . \mathrm{II} . 3 . \mathrm{III}$) : tangential line of confusion formed in the GAUSS image plane of a system of k surfaces and projected back into the object; v conforms to the power of the angle of aperture.

$L_v^{(k)}$ ($v = 1 . 2 . 3 . \mathrm{III}$) : The corresponding sagittal line of confusion, conforming to the power of the angle of aperture v.

$\boldsymbol{l}$, $\boldsymbol{l}'$: in the tracing of skew rays, the y co-ordinates of the point where the ray meets the vertical plane.

$\boldsymbol{L}$, $\boldsymbol{L}'$: in the tracing of skew rays, the z co-ordinates of the point where the ray meets the horizontal plane.

m_v : vertical co-ordinate of the point where the ray meets the aperture-plane of the v^{th} surface, corresponding to small inclinations u_v ($d\mathrm{u}_v$).

$\boldsymbol{m}$: corresponding to finite inclinations.

M_v : horizontal co-ordinate of the point where the ray meets the aperture-plane of the v^{th} surface, corresponding to small inclinations v_v ($d\mathrm{v}_v$).

$\boldsymbol{M}_v$: corresponding to finite inclinations of skew rays.

n, n' : refractive indices in front of and behind a surface. In a system of k surfaces the notation for the successive refractive indices is : n_1, $n_1' = n_2$, $n_2' = n_3$ . . n_k'.

[N.A.] : numerical aperture $= n \sin u$.

$\boldsymbol{p}$, $\boldsymbol{p}'$: intercepts on aperture-rays proceeding from points on the axis before and after refraction.

$\boldsymbol{q}$, $\boldsymbol{q}'$: intercepts on principal rays before and after refraction.

Q_s, Q_{vs} [Q_x, Q_{vx}] : zero invariant for the object (diaphragm) point of the v^{th} surface :

$$n\left(\frac{1}{r}-\frac{1}{s}\right)\left[n\left(\frac{1}{r}-\frac{1}{x}\right)\right].$$

$\boldsymbol{Q}_s$: invariant for rays of finite aperture $n\,\dfrac{\boldsymbol{s}-r}{\boldsymbol{p}r}$.

Q , Q_{tv} : invariants of the tangential rays : $n\left(\dfrac{\cos j}{r}-\dfrac{\cos^2 j}{t}\right)$.

Q_f , $Q_{f,v}$: invariants of the sagittal rays : $n\left(\dfrac{\cos j}{r}-\dfrac{1}{f}\right)$.

r : radius of a refracting surface.

r_p : radius of the entrance pupil.

R_w : radius of the entrance window.

$\boldsymbol{R}$: HELMHOLTZ' expression for the purity of the spectrum.

$\boldsymbol{r}$: RAYLEIGH's resolving power of a prism.

s, s' : paraxial intercepts, before and after refraction, with respect to points on the axis.

δs : intercept differences of s-values for small aperture angles.

s, s' : paraxial intercepts for points on the axis with respect to the second colour.

$\boldsymbol{s}$, $\boldsymbol{s}'$: (1) Intercepts, before and after refraction, of pencils of small or large aperture u proceeding from points on the axis.

(2) In tracing skew rays, the x co-ordinate of the point where the ray meets the vertical plane.

s_s, s_s' : intercepts of pencils of vanishing aperture on the first secondary axis.

$\boldsymbol{s}_s$, $\boldsymbol{s}_s'$: intercepts of pencils of finite aperture on the first secondary axis.

S_s, S_s' : intercepts of pencils of finite aperture on the second secondary axis.

$\bar{\boldsymbol{s}}_v$, $\bar{\boldsymbol{s}}_v'$: the abscissæ of the projection of the tangential image-points on a principal ray inclined at angles w_v, w_v'.

$\bar{\bar{\boldsymbol{s}}}_v$, $\bar{\bar{\boldsymbol{s}}}_v'$: the abscissæ of the projection of the sagittal image-points on a principal ray inclined at angles w_v, w_v'.

f, f' : sagittal intercepts before and after refraction of rays adjacent to the principal ray.

S, S' : in the computation of skew rays the x co-ordinates of the point where the ray meets the horizontal plane.

t, t' : tangential intercepts, before and after refraction, of rays adjacent to the principal ray.

$\boldsymbol{t}, \boldsymbol{t}'$: tangential intercepts of lesser or greater aperture u before and after refraction.

$\mathrm{t}_v, \mathrm{t}_v'$: intercepts of the auxiliary ray on the principal ray in the investigation of the trough defect.

u, u' : aperture angles at points on the axis before and after refraction, also written du, du'.

u, u' : aperture angles in the tangential section before and after refraction, also written $d\mathrm{u}$, $d\mathrm{u}'$.

u_v : angle between the principal ray and the auxiliary ray, in the investigation of the trough defect, also written $d\mathrm{u}_v$.

$\boldsymbol{U}$: (1) Centre normal for rays proceeding from points on the axis.

(2) Radius vector in the transverse plane from the origin of the system of coordinates to the point of the intersection of the skew ray.

v, v' : aperture angles with respect to the first secondary axis. In the special case of sagittal pencils the aperture angles are denoted in the computation by $d\mathrm{v}$, $d\mathrm{v}'$ and in the result by v, v'.

$\boldsymbol{V}, \boldsymbol{V}'$: aperture angles with respect to the second secondary axis.

V : expression for the chromatic variation, *i.e.*, $Vs = s - \mathrm{s}$.

$\boldsymbol{v}_v$: vertical coordinate of the point of intersection in the transverse plane containing the centre of the sphere, in the computation of skew rays.

$\boldsymbol{V}_v$: horizontal coordinate of the point of intersection in the transverse plane containing the centre of the sphere, in the computation of skew rays.

w, w' : angles of inclination of the principal ray.

$\boldsymbol{W}$: centre normal for principal rays.

x, x' : paraxial intercepts with respect to the centre of the diaphragm before and after refraction.

$\boldsymbol{x}, \boldsymbol{x}'$: intercepts of pencils of lesser or greater inclination w with respect to the centre of the stop before and after refraction.

x_s, x_s' : distances of conjugate object points from the corresponding principal foci.

dx_s, dx_s' : the corresponding variations, for paraxial rays, in the intercepts referred to the principal foci.

dx_s' : the corresponding variations in the intercepts of rays of finite aperture on the image side.

$\boldsymbol{X}_s$, $\boldsymbol{X}_s'$: distances of conjugate centres of pupils from the corresponding principal foci.

y_s, y_s' : conjugate distances from the axis in terms of the Gaussian theory, also written dy_s, dy_s'.

a : longitudinal magnification : $\dfrac{dx_s'}{dx}$.

a, a_v : prism angles.

β : lateral magnification in the image plane : $\dfrac{dy_s'}{dy_s}$, also written $\dfrac{y_s'}{y_s}$.

β_t : lateral magnification in the tangential plane.

β_f : lateral magnification in the sagittal plane.

B : lateral magnification in the pupillary planes.

γ : (1) angle between the axis of the system and the first secondary axis.

(2) angular magnification in the planes of the object and image.

Γ : (1) angle between the axis of the system and the second secondary axis.

(2) angular magnification in the two pupillary planes.

γ_t : angular magnification in the tangential section.

γ : angular magnification in the sagittal section.

$\triangle$: optical interval.

Δ : sign of differences between and after refraction, *e.g.*, $\Delta n = n'-n$.

ε : deviation in prisms.

ε : minimum deviation.

ε : angle between the axis of the system and the projection of the skew ray into the XY-plane.

ζ_v : SEIDEL'S polar angle for U in the transverse plane containing the centre of the sphere C_v.

θ : relative partial dispersion.

Θ : its equivalent value in a combination of lenses.

λ, λ' : SEIDEL'S term for the angle between the skew ray and $\boldsymbol{U}$, $\boldsymbol{U}'$.

ν : reciprocal value of the disperser : $\dfrac{n_v-1}{Vn}$.

N : equivalent ν-value.

ξ, ξ' : reciprocal values of the intercepts relatively to the stops $= \frac{1}{x}, \frac{1}{x'}$.

$\Xi = \frac{\xi}{\phi} = \frac{f}{x}$.

π, π' : SEIDEL'S notation for the directional difference of the projection of the ray into the transverse plane and the axis of y.

$\rho = \frac{1}{r}; \rho_v = \frac{1}{r_v}$: curvature of the v^{th} surface.

P : (1) symbol for $\frac{\rho}{\phi} = \frac{f}{r}$.

(2) symbol for PETZVAL'S condition:

$$P = \frac{1}{R} = \sum_{v=1}^{k} \frac{1}{r_v} \underset{v}{\Delta} \frac{1}{n} = -\sum_{v=1}^{k} \frac{1}{n_{v+1} f_v}.$$

σ, σ' : (1) distances of the principal foci of a composite system from the principal foci of the components.

(2) reciprocal values of the intercepts : $\frac{1}{s}, \frac{1}{s'}$.

$\Sigma = \frac{\sigma}{\phi} = \frac{f}{s}$.

$d\zeta$: displacement of the sagittal image-point corresponding to the variation du.

τ, τ' : SEIDEL'S term for the directional differences between the ray and the axis of the system before and after refraction.

$d\tau$: displacement of the tangential image-point corresponding to a change du.

$\frac{1}{2}\frac{d\tau}{du}$: radius of curvature of the caustic curve in the tangential section.

ϕ : (1) angle subtended by the semi-arc of the aperture at the centre of the sphere for rays proceeding from points on the axis.

(2) power of a lens, being the reciprocal of the focal length.

Φ : angle subtended at the centre of the sphere for rays proceeding from the centre of the stop.

Φ : resulting power of a system of lenses.

$d\psi_v, d\psi_v'$: directional differences, with respect to the principal ray, of the projected trace of the skew ray in the meridian plane (in the investigation of the trough defect).

Ω : secondary spectrum.

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